THE NORM OF A TRUNCATED TOEPLITZ OPERATOR

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Abstract. We prove several lower bounds for the norm of a truncated Toeplitz operator and obtain a curious relationship between the \(H^2\) and \(H^\infty\) norms of functions in model spaces.

1. Introduction

In this paper, we continue the discussion initiated in [6] concerning the norm of a truncated Toeplitz operator. In the following, let \(H^2\) denote the classical Hardy space of the open unit disk \(D\) and \(K_\Theta := H^2 \cap (\Theta H^2)^\perp\), where \(\Theta\) is an inner function, denote one of the so-called Jordan model spaces [2, 4, 7]. If \(H^\infty\) is the set of all bounded analytic functions on \(D\), the space \(K_\Theta^\infty := H^\infty \cap K_\Theta\) is norm dense in \(K_\Theta\) (see [2, p. 83] or [9, Lem. 2.3]). If \(P_\Theta\) is the orthogonal projection from \(L^2 := L^2(\partial D, |d\zeta|/2\pi)\) onto \(K_\Theta\) and \(\varphi \in L^2\), then the operator

\[A_\varphi f := P_\Theta(\varphi f), \quad f \in K_\Theta^\infty,\]

is densely defined on \(K_\Theta\) and is called a truncated Toeplitz operator. Various aspects of these operators were studied in [3, 5, 6, 9, 10].

If \(\|\cdot\|\) is the norm on \(L^2\), we let

\[\|A_\varphi\| := \sup\{\|A_\varphi f\| : f \in K_\Theta^\infty, \|f\| = 1\}\] (1)

and note that this quantity is finite if and only if \(A_\varphi\) extends to a bounded operator on \(K_\Theta\). When \(\varphi \in L^\infty\), the set of bounded measurable functions on \(\partial D\), we have the basic estimates

\[0 \leq \|A_\varphi\| \leq \|\varphi\|_\infty.\]

However, it is known that equality can hold, in nontrivial ways, in either of the inequalities above and hence finding the norm of a truncated Toeplitz operator can be difficult. Furthermore, it turns out that there are many unbounded symbols \(\varphi \in L^2\) which yield bounded operators \(A_\varphi\). Unlike the situation for classical Toeplitz operators on \(H^2\), for a given \(\varphi \in L^2\), there many \(\psi \in L^2\) for which \(A_\varphi = A_\psi\) [9, Thm. 3.1].

For a given symbol \(\varphi \in L^2\) and inner function \(\Theta\), lower bounds on the quantity (1) are useful in answering the following nontrivial questions:

(i) is \(A_\varphi\) unbounded?

(ii) if \(\varphi \in L^\infty\), is \(A_\varphi\) norm-attaining (i.e., is \(\|A_\varphi\| = \|\varphi\|_\infty\))? 

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When $\Theta$ is a finite Blaschke product and $\varphi \in H^\infty$, the paper [6] computes $\|A_\varphi\|$ and gives necessary and sufficient conditions as to when $\|A_\varphi\| = \|\varphi\|_. The purpose of this short note is to give a few lower bounds on $\|A_\varphi\|$ for general inner functions $\Theta$ and general $\varphi \in L^2$. Along the way, we obtain a curious relationship (Corollary 5) between the $H^2$ and $H^\infty$ norms of functions in $K^\infty_\Theta$.

2. Lower bounds derived from Poisson’s formula

For $\varphi \in L^2$, let

$$
(\mathfrak{F}_\varphi)(z) := \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in D,
$$

be the standard Poisson extension of $\varphi$ to $D$. For $\varphi \in C(\partial D)$, the continuous functions on $\partial D$, recall that $\mathfrak{F}_\varphi$ solves the classical Dirichlet problem with boundary data $\varphi$. Also note that

$$
k_\lambda(z) := \frac{1 - \Theta(\lambda)\Theta(z)}{1 - \lambda z}, \quad \lambda, z \in D,
$$

is the reproducing kernel for $K_\Theta$ [9].

Our first result provides a general lower bound for $\|A_\varphi\|$ which yields a number of useful corollaries:

**Theorem 1.** If $\varphi \in L^2$, then

$$
\sup_{\lambda \in D} \left| \int_{\partial D} \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \frac{|dz|}{2\pi} \right| \leq \|A_\varphi\|.
$$

In other words,

$$
\sup_{\lambda \in D} \left| \int_{\partial D} \varphi(z) d\nu_\lambda(z) \right| \leq \|A_\varphi\|
$$

where

$$
d\nu_\lambda(z) := \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi}
$$

is a family of probability measures on $\partial D$ indexed by $\lambda \in D$.

**Proof.** For $\lambda \in D$ we have

$$
\|k_\lambda\| = \sqrt{1 - \frac{|\lambda|^2}{\Theta(\lambda)|^2}},
$$

from which it follows that

$$
\|A_\varphi\| \geq \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle A_\varphi k_\lambda, k_\lambda \rangle\|
$$

$$
= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle P_\Theta \varphi k_\lambda, k_\lambda \rangle\|
$$

$$
= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \|\langle \varphi k_\lambda, k_\lambda \rangle\|
$$

$$
= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial D} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right|.
$$

That the measures $d\nu_\lambda$ are indeed probability measures follows from (4). $\square$
Furthermore, the above is equal to Corollary 3. For an inner function \( \Theta \) has the non-tangential limit \( \Theta \) preceding discussion and Fatou’s lemma yield the following lower estimate for \( H \) then the function in (6) belongs to \( K \). Indeed, if \( \Theta \) is a finite Blaschke product and \( \varphi \parallel \Theta \) has a finite angular derivative at \( \zeta \), then it is known that \( \|A_\varphi\| = \|\varphi\|_\infty \) if and only if \( \varphi \) is the scalar multiple of the inner factor of some function from \( K_\Theta \) [6, Thm. 2].

At the expense of wordiness, the hypothesis of Corollary 2 can be considerably weakened. A cursory examination of the proof indicates that we only need \( \zeta \) to be a limit point of the zeros of \( \Theta \), \( \varphi \in L^\infty \) to be continuous on an open arc containing \( \zeta \), and \( |\varphi(\zeta)| = \|\varphi\|_\infty \).

Theorem 1 yields yet another lower bound for \( \|A_\varphi\| \). Recall that an inner function \( \Theta \) has a finite angular derivative at \( \zeta \in \partial \mathbb{D} \) if \( \Theta \) has a non-tangential limit \( \Theta(\zeta) \) of modulus one at \( \zeta \) and \( \Theta'(\zeta) \) has a finite non-tangential limit \( \Theta'(\zeta) \) at \( \zeta \). This is equivalent to asserting that

\[
\Theta(z) - \Theta(\zeta) = \frac{\Theta(z) - \Theta(r\zeta)}{z - r\zeta} \quad \text{for which} \quad \lim_{r \to 1^-} \int_{\partial \mathbb{D}} \frac{\Theta(z) - \Theta(r\zeta)}{z - r\zeta} \frac{|dz|}{2\pi} = \int_{\partial \mathbb{D}} \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \frac{|dz|}{2\pi}.
\]

Furthermore, the above is equal to

\[
\lim_{r \to 1^-} \frac{1 - |\Theta(r\zeta)|^2}{1 - r^2} = |\Theta'(\zeta)| > 0.
\]

See [1, 8] for further details on angular derivatives. Theorem 1 along with the preceding discussion and Fatou’s lemma yield the following lower estimate for \( \|A_\varphi\| \).

**Corollary 3.** For an inner function \( \Theta \), let \( D_{\Theta} \) be the set of \( \zeta \in \partial \mathbb{D} \) for which \( \Theta \) has a finite angular derivative \( \Theta'(\zeta) \) at \( \zeta \). If \( \varphi \in L^\infty \) or if \( \varphi \in L^2 \) with \( \varphi \geq 0 \), then

\[
\sup_{\zeta \in D_{\Theta}} \frac{1}{|\Theta'(\zeta)|} \left| \int_{\partial \mathbb{D}} \varphi(z) \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \frac{|dz|}{2\pi} \right| \leq \|A_\varphi\|.
\]
In other words,

\[ \sup_{\zeta \in D_\Theta} \left| \int_{\partial D} \varphi(z)d\nu_\lambda(z) \right| \leq \|A_\varphi\|, \]

where

\[ d\nu_\lambda(z) := \frac{1}{|\Theta'(\zeta)|} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right| \frac{|dz|}{2\pi} \]

is a family of probability measures on \( \partial \mathbb{D} \) indexed by \( \zeta \in D_\Theta \).

3. Lower bounds and projections

Our next several results concern lower bounds on \( \|A_\varphi\| \) involving the orthogonal projection \( P_\Theta : L^2 \to K_\Theta \).

**Theorem 2.** If \( \Theta \) is an inner function and \( \varphi \in L^2 \), then

\[ \frac{\|P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)\|}{(1 - |\Theta(0)|^2)^{\frac{1}{2}}} \leq \|A_\varphi\|. \]

**Proof.** First observe that \( \|k_0\| = (1 - |\Theta(0)|^2)^{\frac{1}{2}} \). Next we see that if \( \varphi \in L^2 \) and \( g \in K_\Theta \) is any unit vector, then

\[ (1 - |\Theta(0)|^2)^{\frac{1}{2}} \|A_\varphi\| \geq |\langle A_\varphi k_0, g \rangle| = |\langle P_\Theta(\varphi k_0), g \rangle| = |\langle P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi), g \rangle|. \]

Setting

\[ g = \frac{P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)}{\|P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)\|} \]

yields the desired inequality. \( \square \)

In light of the fact that \( P_\Theta(\Theta\varphi) = 0 \) whenever \( \varphi \in H^2 \), Theorem 2 leads us immediately to the following corollary:

**Corollary 4.** If \( \Theta \) is inner and \( \varphi \in H^2 \), then

\[ \frac{\|P_\Theta(\varphi)\|}{(1 - |\Theta(0)|^2)^{\frac{1}{2}}} \leq \|A_\varphi\|. \quad (7) \]

It turns out that (7) has a rather interesting function-theoretic implication. Let us first note that for \( \varphi \in H^\infty \), we can expect no better inequality than

\[ \|\varphi\| \leq \|\varphi\|_\infty \]

(with equality holding if and only if \( \varphi \) is a scalar multiple of an inner function). However, if \( \varphi \) belongs to \( K_\Theta^\infty \), then a stronger inequality holds.

**Corollary 5.** If \( \Theta \) is an inner function, then

\[ \|\varphi\| \leq (1 - |\Theta(0)|^2)^{\frac{1}{2}}\|\varphi\|_\infty \quad (8) \]

holds for all \( \varphi \in K_\Theta^\infty \). If \( \Theta \) is a finite Blaschke product, then equality holds if and only if \( \varphi \) is a scalar multiple of an inner function from \( K_\Theta \).
Proof. First observe that the inequality

\[ \| \varphi \| \leq (1 - |\Theta(0)|^2)^{\frac{1}{2}} \| \varphi \|_\infty \]

follows from Corollary 4 and the fact that \( P_\varphi \varphi = \varphi \) whenever \( \varphi \in K_\Theta \). Now suppose that \( \Theta \) is a finite Blaschke product and assume that equality holds in the preceding for some \( \varphi \in K_\Theta^\infty \). In light of (7), it follows that \( \| A_\varphi \| = \| \varphi \|_\infty \). From [6, Thm. 2] we see that \( \varphi \) must be a scalar multiple of the inner part of a function from \( K_\Theta \). But since \( \varphi \in K_\Theta^\infty \), then \( \varphi \) must be a scalar multiple of an inner function from \( K_\Theta \). \( \square \)

When \( \Theta \) is a finite Blaschke product, then \( K_\Theta \) is a finite dimensional subspace of \( H^2 \) consisting of bounded functions [3, 5, 9]. By elementary functional analysis, there are \( c_1, c_2 > 0 \) so that

\[ c_1 \| \varphi \| \leq \| \varphi \|_\infty \leq c_2 \| \varphi \| \]

for all \( \varphi \in K_\Theta \). This prompts the following question:

**Question.** What are the optimal constants \( c_1, c_2 \) in the above inequality?

4. LOWER BOUNDS FROM THE DECOMPOSITION OF \( K_\Theta \)

A result of Sarason [9, Thm. 3.1] says, for \( \varphi \in L^2 \), that

\[ A_\varphi \equiv 0 \iff \varphi \in \Theta H^2 + \overline{\Theta H^2}. \]  \hfill (9)

It follows that the most general truncated Toeplitz operator on \( K_\Theta \) is of the form \( A_{\psi + \chi} \) where \( \psi, \chi \in K_\Theta \). We can refine this observation a bit further and provide another canonical decomposition for the symbol of a truncated Toeplitz operator.

**Lemma 1.** Each bounded truncated Toeplitz operator on \( K_\Theta \) is generated by a symbol of the form

\[ \varphi = \psi + \overline{\chi \Theta} \]

where \( \psi, \chi \in K_\Theta \).

Before getting to the proof, we should remind the reader of a technical detail. It follows from the identity \( K_\Theta = H^2 \cap \Theta \overline{zH^2} \) (see [2, p. 82]) that

\[ C : K_\Theta \rightarrow K_\Theta, \quad Cf := \overline{z}\Theta, \]

is an isometric, conjugate-linear, involution. In fact, when \( A_\varphi \) is a bounded operator we have the identity \( CA_\varphi C = A_\varphi^* \) [9, Lemma 2.1].

**Proof of Lemma 1.** If \( T \) is a bounded truncated Toeplitz operator on \( K_\Theta \), then there exists some \( \varphi \in L^2 \) such that \( T = A_\varphi \). We claim that this \( \varphi \) can be chosen to have the special form (10). First let us write \( \varphi = f + \overline{zg} \) where \( f, g \in H^2 \). Using the orthogonal decomposition \( H^2 = K_\Theta \oplus \Theta H^2 \), it follows that \( \varphi \) may be further decomposed as

\[ \varphi = (f_1 + \Theta f_2) + \overline{z(g_1 + \Theta g_2)} \]

where \( f_1, g_1 \in K_\Theta \) and \( f_2, g_2 \in H^2 \). By (9), the symbols \( \Theta f_2 \) and \( \overline{\Theta(g_2)} \) yield the zero truncated Toeplitz operator on \( K_\Theta \). Therefore we may assume that

\[ \varphi = f + \overline{zg} \]

for some \( f, g \in K_\Theta \). Since \( Cg = \overline{gz}\Theta \), we have \( \overline{zg} = (Cg)\overline{\Theta} \) and hence (10) holds with \( \psi = f \) and \( \chi = Cg \). \( \square \)
Corollary 6. Let $\Theta$ be an inner function. If $\psi_1, \psi_2 \in K_\Theta$ and $\varphi = \psi_1 + \psi_2\Theta$, then

$$\|\psi_1 - \Theta(0)\psi_2\| \leq \|A_\varphi\|.$$  

Proof. If $\varphi = \psi_1 + \psi_2\Theta$, then, since $\psi_1, \psi_2 \in K_\Theta$ and $\psi_2\Theta \in \mathcal{H}_{\mathbb{D}}$, we have

$$P_\Theta(\varphi) - \Theta(0)P_\Theta(\Theta\varphi) = \psi_1 - \Theta(0)\psi_2.$$  

The result now follows from Theorem 2. \qed

References


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