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RESEARCH ARTICLE

The Jordan curve theorem is non-trivial

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This article give some examples of Jordan curves where one questions the notions of interior and exterior.

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1. Introduction

The classical Jordan curve theorem: *Every Jordan curve separates the plane into exactly two components*, is often mentioned just in passing in undergraduate and graduate complex analysis courses and, unfortunately, is often given short shrift. One could argue there are good reasons for this. First, this theorem plays no role in the rest of the course. A professor has bigger fish to fry. There are the theorems of Cauchy, Hadamard, Morera, and the like which certainly comprise the nuts and bolts of complex analysis. The Jordan curve theorem appears to be a mere curiosity belonging in a topology course. Second, the Jordan curve theorem is impossible to prove, or even outline a proof, in any reasonable amount of time and in terms which uninitiated students can understand and, moreover, appreciate. Third, the result is clearly "obvious" and a professor does not want to put themselves or their students through a complicated proof of a theorem which seems to need no proof at all.

Perhaps the reader who teaches complex analysis from time to time and feels the need to, at the very least, mention the Jordan curve theorem might scribble a circle or an ellipse on the board and point out the interior and exterior regions – knowing full well that the students are unimpressed. The slightly more ambitious teacher might make some weak attempt (without practicing this before class) at drawing something more complicated and invite the students identify the interior and exterior regions. The students always do so very quickly and are still not impressed and don't see the difficulties. The Jordan curve theorem lesson usually ends with a few mumbled words like "Well, trust me on this one. Things can get pretty complicated out there and to make this all exact takes a lot of work. So, let's move on to...".

We certainly understand the issues mentioned above in teaching the Jordan curve theorem and the need to "move on" to more important and relevant things. We write this very brief survey, which a teacher can assign to their students, in order for them to appreciate some of the history of this result as well as the fact that it is non-trivial – both artistically and mathematically. We

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will not really get into the mathematics here but talk about this theorem through some recent art work of the first author which will give the reader (and student who is, perhaps, reading this paper as an assignment by their teacher!) an appreciation of the complexity and the real need to prove the Jordan curve theorem. In our opinion, the Jordan curve theorem is a wonderful result because it exposes us to amazing, pathological, counterintuitive, examples such as nowhere differentiable curves or curves with positive area. So, in a way, not appreciating the Jordan curve theorem is driven by a lack of imagination.

It is nearly impossible to accurately depict such pathological curves as ones which are nowhere differentiable or with positive area. Indeed there are the physical limitations of ink and paper. Instead, we would like to challenge the reader's preconceived notions of interior and exterior in the Jordan curve theorem. Along the way, we would also like to challenge the reader's notion of a curve as a cold, abstract, boring, object and give the reader a chance to explore the idea of a curve as leading the viewer through a story. We hope to convince the reader that indeed, whether through challenging the notions of interior and exterior or viewing a curve as a story, that the Jordan Curve Theorem is non-trivial.

2. Some history

The Jordan curve theorem:

> If C is the continuous image of the unit circle then $\mathbb{R}^2 \setminus C$ has two components.

We traditionally call the bounded component the "interior" and the unbounded component the "exterior". Bernhard Bolzano [3, 6] saw that the problem was non-trivial and officially posed it as a theorem needing a proof. Here is Bolzano's "prophetic version of the celebrated Jordan curve theorem" [3, p. 285]:

> If a closed line lies in a plane and if by means of a connected line one joins a point of the plane which is enclosed within the closed line with a point which is not enclosed within it, then the connected line must cut the closed line.

Bolzano also realized that the current notions of curve at the time were in desperate need of proper definitions! In fact, Felix Klein once said:

> Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.

The first proof, hence name Jordan curve theorem, was given in 1887 by Camille Jordan in his book *Cours d'analyse de l'École Polytechnique* [4] but was regarded by many to be incorrect. As an interesting side note, Thomas Hales [2] believes the charges of incorrectness against Jordan's original proof are trumped up and the errors are merely aesthetic. With a few relatively minor changes, he claims to make Jordan's proof rigorous.

If the Jordan curve is a polygon, one can prove the Jordan curve theorem quite easily: Start from a point not on the curve and draw a straight line from that point to the outside of the whole drawing. If the line meets the polygon an odd number of times, you are on the interior. If the line meets an even number of times, you are on the exterior. Try this for a few simple polygons. The above proof, which needs to be cleaned up a bit to make a completely rigorous, is the standard way to prove the Jordan curve theorem for polygons.

When the curve is not a polygon, the above argument needs to be changed since, as mentioned earlier, Jordan curves can be quite complicated. For the student, maybe you are thinking of defining the interior of the curve by using the rule you learned from calculus: When you travel around the curve, the interior is on your left. When you are doing this, notice how you are tacitly assuming there is a continuously turning normal to the curve (your left hand). When the curve is nowhere differentiable (which it can be!) this method breaks down. What is even more amazing here is that Jordan curves can have positive area [5]. It was Otto Veblen [7] who finally gave what many regard as the first correct proof of the Jordan curve theorem and others followed with different proofs, including ones using formal logic [1].

3. The work

The drawings in this survey are ink and graphite on Japanese paper, which depict humans and animals in a variety of land or seascapes rendered from a single curve. These types of drawings have been a four year project of the first author and did not start out as Jordan curve drawings. They started out as single line drawings which bind the figures, landscapes, and other elements of the drawing together in an unbroken thread-like line. Most of the works are unicursal labyrinths, whereby the viewer can visually trace a path through the entire work by following a single thread of inked line, whereby the path that leads into the labyrinth also will lead the viewer out. As in the most famous of mythological labyrinths, that of Theseus and the Minotaur, the viewer is led through a series of encounters with obstacles, fantastical creatures, and landscapes, emerging from their adventure with a sense of their very selves as the most complex of labyrinths.

These first set of single line drawings (*Blue Introverted Unicursal Labyrinth* - Figures 1, 2 and *Slippery Slope* - Figures 3, 4) are almost Jordan curves. Here the first author made a point of not uniting the ends of the curve that formed the labyrinth on purpose - just as there was no separation of the figures or landscape, there was no separation of the inside and outside of the labyrinth on the paper. Also, when using liquid inks on paper, she often intentionally used the tendency of ink to wick or spread into the paper surface in some areas of the drawing as a contrast to other more precisely drawn areas.

Intrigued by conversations with the second author, a mathematician (and in particular a complex analyst) and by Felix Klein's statement quoted earlier, the first author combined the idea of a curve as leading the viewer through a story, like Theseus and the Minotaur, with the mathematical notion of the complexities encountered when trying to prove the Jordan curve theorem. She began a new series of drawings (*When we could be diving for pearls* - Figure 5 and *A thread in the labyrinth* - Figure 6) rendered from a single (non-intersecting) curve without beginning or end, i.e., a Jordan curve. In these new drawings, the unicursal labyrinth is no longer open, with the curve leading into the drawing also being the means to exit it, but closed. The interior and exterior are very difficult to identify. Making it difficult to identify inside and outside spaces gives the "closed" drawings in Figures 5 and 6 a different meaning then the "open" drawings in Figures 1 - 4. The Jordan curve drawings have a more secretive and exclusive emotional content where they seem to hold their breath as the inked fingers of the external space unsuccessfully reach in and touch the interior, whereas the open drawings invite entry into the most remote areas of the work. The Jordan curve drawings created for this article (Figures 5 and 6) were drawn with micron ink markers or graphite on Denril drafting film. She chose this ink and drafting film because of the tendency of other inks to spread onto the surface of the Japanese paper she usually uses, resulting in lines that touch.

We invite the reader to start at a point on one of the drawings in Figures 5 or 6 and apply the "proof" of the Jordan curve theorem mentioned earlier (counting the number of crossings needed to exit the drawing) to determine whether you are in the interior or the exterior. Convince yourself, by means of the complexity of the drawing, that absent some sort of orderly system (counting crossings) it is quite difficult to determine where you are. Merely trying to "find your way out", stretches the ability of your attention span. While traveling along your line, also appreciate the figures you encounter as you exit the drawing. To realize the greater complexity involved in proving the theorem, imagine making the drawing even more complicated by adding a fractal, perhaps non-differential, element to the curve, and see how even more difficult this becomes. Can you think of an example of a Jordan curve where counting crossings no longer works? Do you appreciate the Jordan curve theorem more?

4. Drawings

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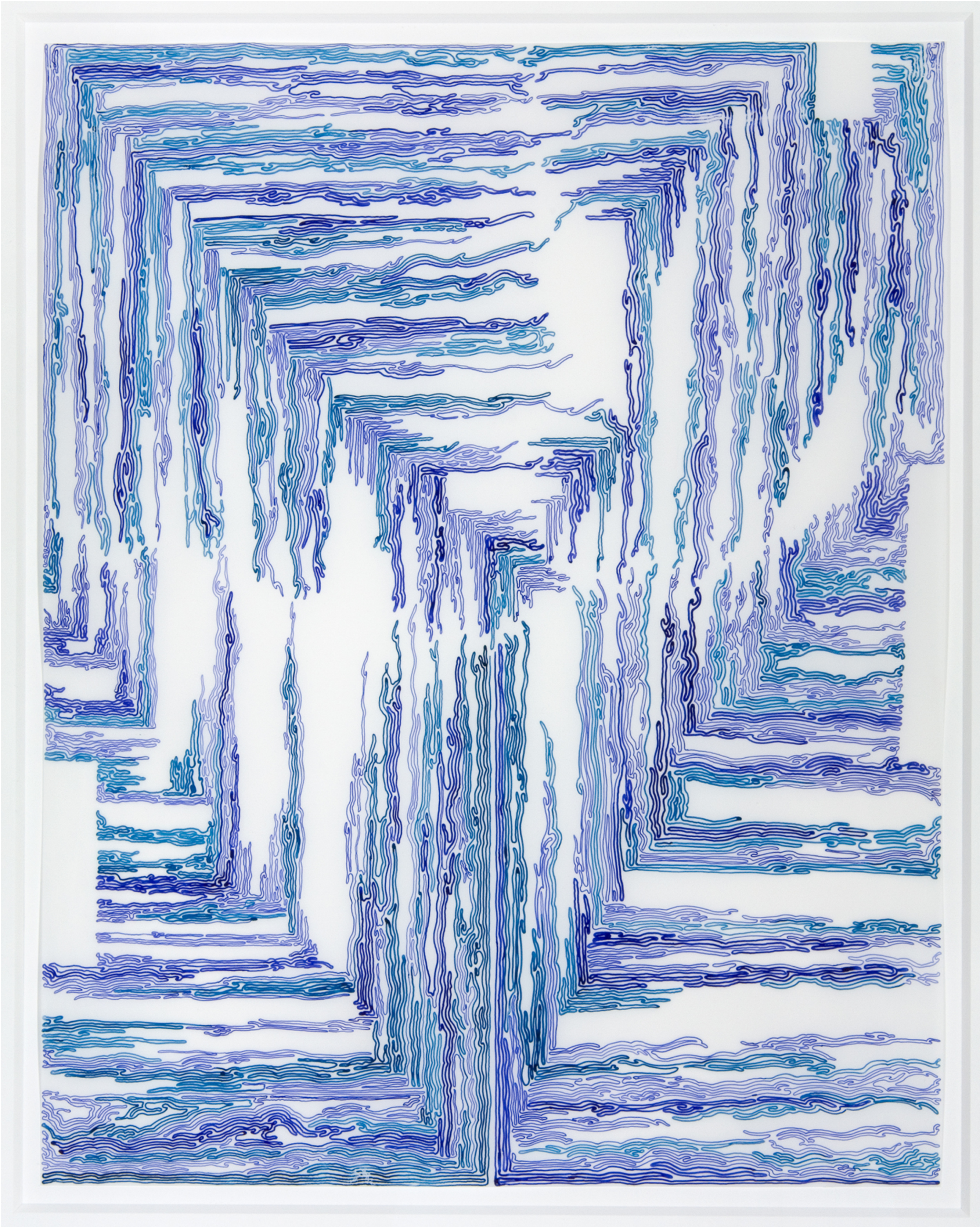

Figure 1. *Blue Introverted Unicursal Labyrinth*, 14" x 8", 2010, Ink on Yupo paper.

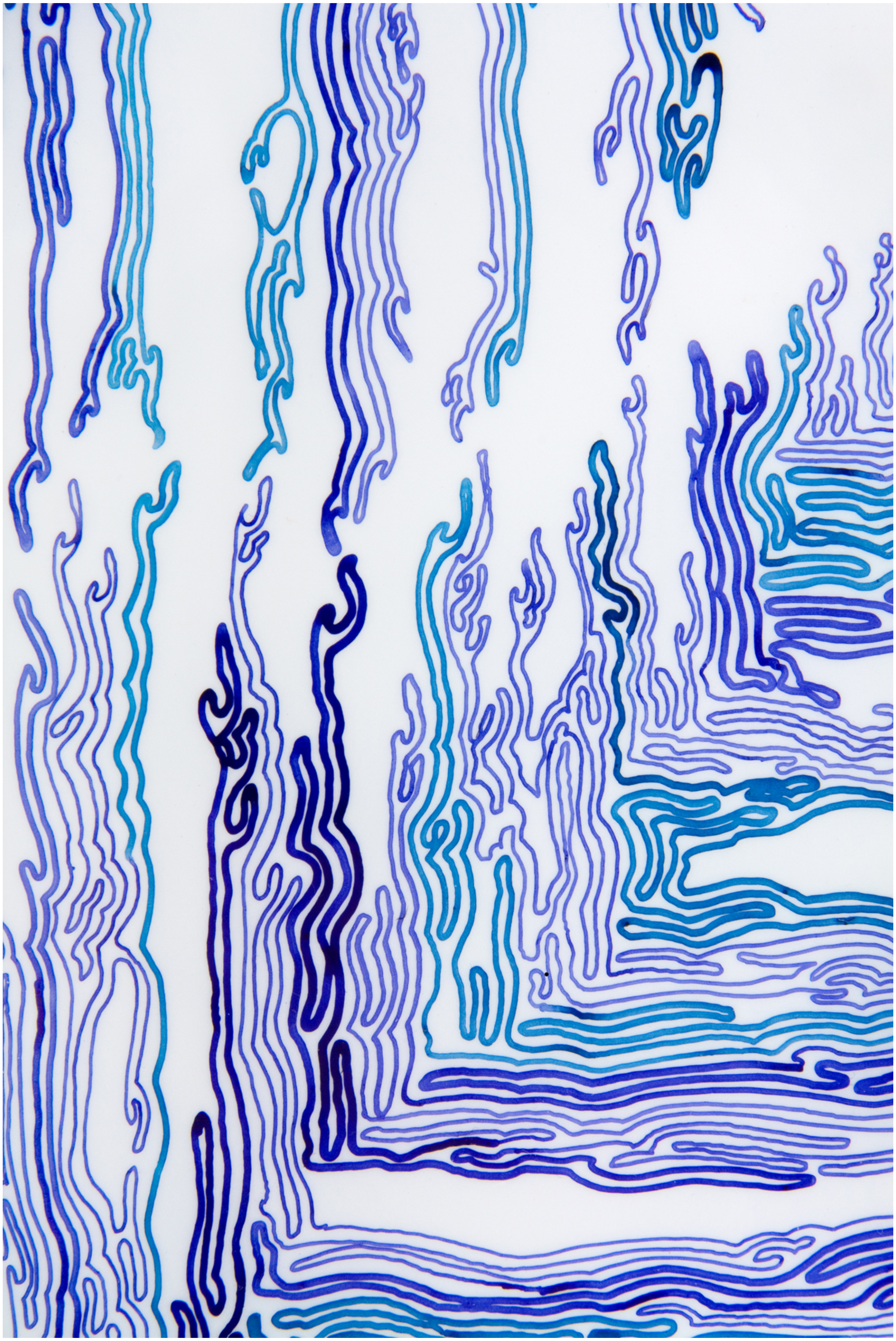

Figure 2. *Blue Introverted Unicursal Labyrinth*, detail

Figure 3. *Slippery Slope*, 14" x 8", 2010, Sumi ink on Yupo paper.

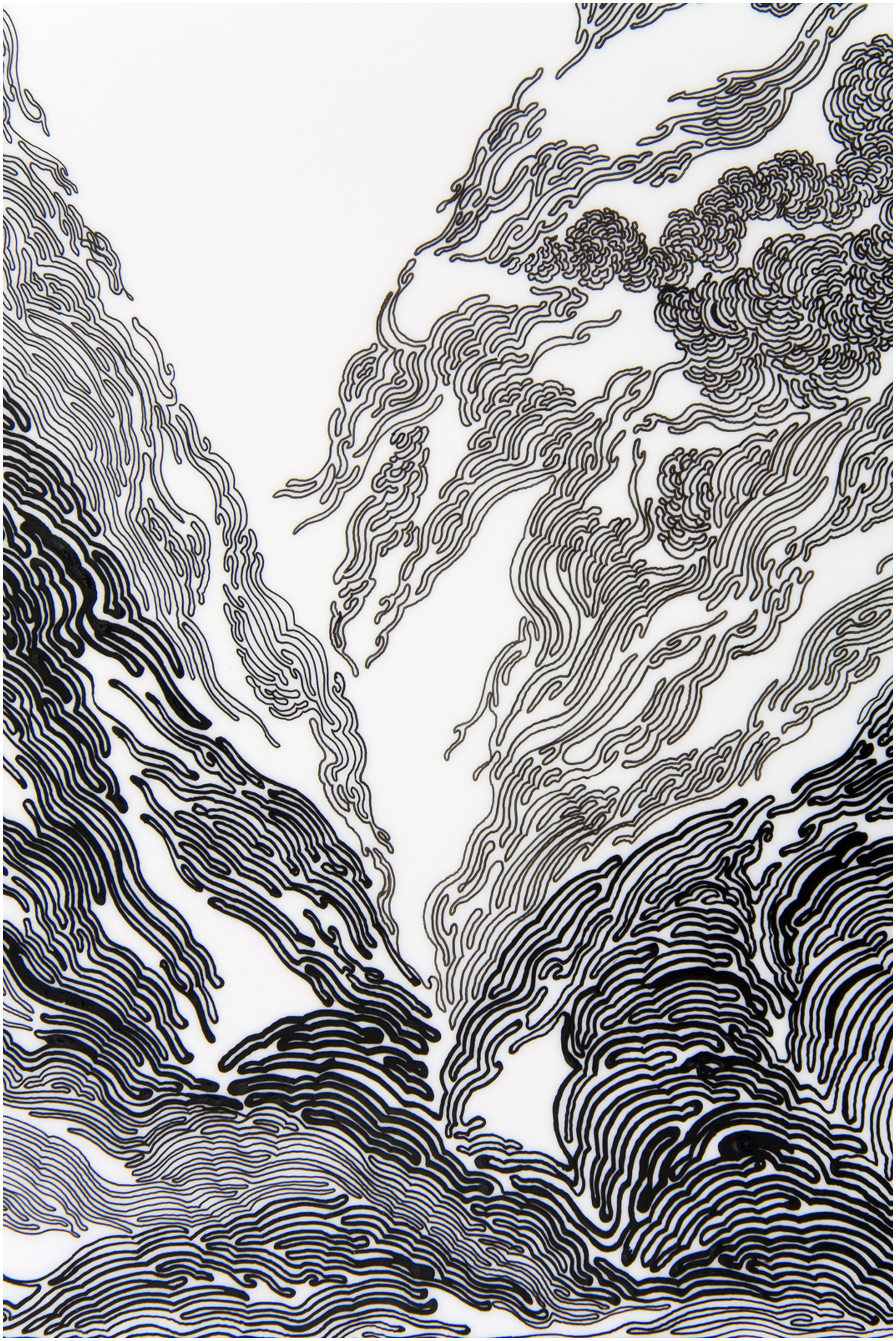

Figure 4. *Slippery Slope*, detail

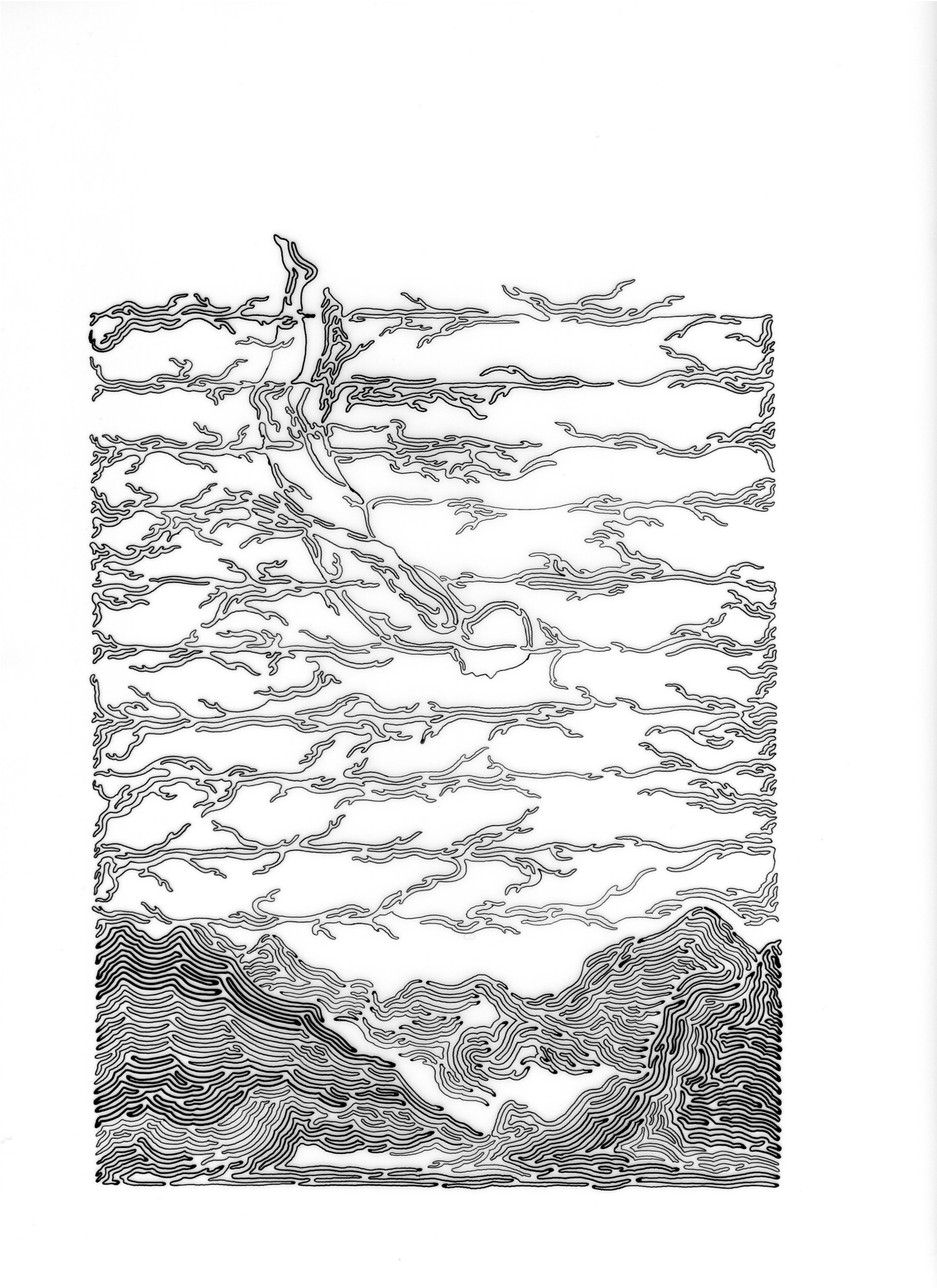

Figure 5. *When we could be diving for pearls*, 9 3/4" x 6", 2011, Micron ink on Denril paper.

Figure 6. *A thread in the labyrinth*, 6" x 6", 2011, Micron ink on Denril paper.