Hardy spaces of slit domains

Alexandru Aleman, Nathan S. Feldman, William T. Ross

Lund, Washington and Lee, Richmond

Set up:

- Ω is a bounded domain in $\mathbb{C}$
- $H^2(\Omega)$ is the Hardy space on Ω
- $S : H^2(\Omega) \to H^2(\Omega), (Sf)(z) = zf(z)$

Problem:

- Describe the S-invariant subspaces of $H^2(\Omega)$.

$\Omega = \mathbb{D}$:

Theorem (Beurling)

If M is a subspace (closed linear manifold) of $H^2(\mathbb{D})$ with $SM \subset M$, then $M = \Theta H^2(\mathbb{D})$, where Θ is a $\mathbb{D}$-inner function.

- Θ is $\mathbb{D}$-*inner* means $\Theta \in H^\infty(\mathbb{D})$ and $|\Theta^*| = 1$ a.e.
- Similar result when $\Omega = \text{ins}(\gamma)$, where γ is a closed smooth (enough) curve

$\Omega = A = \{r < |z| < R\}$

$\mathrm{Lat}(S, H^2(A))$ described by Sarason, Royden, Hitt

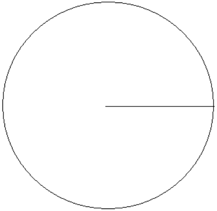
$\Omega = \mathrm{ins}(\Gamma) \setminus \bigcup_{j=1}^{n} \mathrm{ins}(\gamma_j)^-$

$\mathrm{Lat}(S, H^2(\Omega))$ described by Yakubovich, Aleman-Richter

Ω is a crescent domain

$\mathrm{Lat}(S, H^2(\Omega))$ described by Aleman and Olin

Aleman-Feldman-R consider **the** slit disk $G := \mathbb{D} \setminus [0, 1]$.



For $f \in H^2(G)$,

$$f^+(x) := \lim_{y \to 0^+} f(x+iy), \quad \text{a.e. } x \in [0,1]$$

$$f^-(x) := \lim_{y \to 0^-} f(x+iy), \quad \text{a.e. } x \in [0,1]$$

$$f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta), \quad \text{a.e. } \zeta \in \partial\mathbb{D}$$

If $\psi : G \to \mathbb{D}$, then

$$\|f\|^2_{H^2(G)} = \int_0^1 \left(|f^+|^2 + |f^-|^2\right) |\psi'| \frac{dx}{2\pi} + \int_0^{2\pi} |f^*|^2 |\psi'| \frac{d\theta}{2\pi}$$

$$|\psi'(\xi)| \asymp |\xi|^{-1/2} |\xi - 1|, \quad \xi \in \partial G$$

Lemma

Suppose M is a $H^\infty(G)$-invariant subspace of $H^2(G)$. Then $M = \Theta H^2(G)$, where Θ is a G-inner function.

- Θ is G-inner means $\Theta \circ \phi$ is $\mathbb{D}$-inner, where $\phi : \mathbb{D} \to G$.

There are **many other** S-invariant subspaces of $H^2(G)$.

Ex:

- $\rho : [0,1] \to \mathbb{C}$ be measurable
- E a measurable subset of $[0,1]$.

$$M(\rho, E) := \{f \in H^2(G) : f^+ = \rho f^- \text{a.e. on } E\}$$

- $M(\rho, E)$ is closed (since norm is an integral on ∂G) and is S-invariant.
- $M(\rho, E)$ is **not** always $\Theta H^2(G)$

Perhaps every S-invariant subspace of $H^2(G)$ is

$$M = \Theta M(\rho, E)$$

- $\rho : [0, 1] \to \mathbb{C}$,
- $E \subset [0, 1]$,
- Θ G-inner
- $M(\rho, E) := \{f \in H^2(G) : f^+ = \rho f^- \text{a.e. on } E\}$

Lemma

For $f \in H^2(G)$ TFAE:

- $f \in M(1, [0, 1])$
- *f has an AC across $[0, 1]$*

Lemma

$ball(M(1, [0, 1]))$ is a normal family of analytic functions on $\mathbb{D}$.

Ex: Let

$$M_0 := \{f \in M(1, [0,1]) : f(0) = 0\}.$$

- M_0 is closed and S-invariant
- $M_0 \neq \Theta M(\rho, E)$
- Θ can not account for the zero at $z = 0$ since $0 \in \partial G$

$$M_0 = \{f \in H^2(G) : \frac{f}{z} \in H^2(G), f^+ = 1f^- \text{a.e. on } [0,1]\}$$

$F(z) = z$ is G-outer

Fix $\epsilon \in (0,1)$ and let $G_\epsilon := \mathbb{D} \setminus [-\epsilon, 1]$.

Theorem

Let M be a non-trivial invariant subspace of $H^2(G)$ with greatest common G-inner divisor Θ_M. Then there exists a

- *a measurable set $E \subset [0,1]$,*
- *a measurable function $\rho : [0,1] \to \mathbb{C}$*
- *a G_ϵ-outer function F_ϵ,*

such that

$$M = \Theta_M \cdot \left\{ f \in H^2(G) : \frac{f}{F_\epsilon} \in H^2(G_\epsilon), f^+ = \rho f^- \text{ a.e. on } E \right\}$$

$$M = \Theta_M \cdot \left\{ f \in H^2(G) : \frac{f}{F_\epsilon} \in H^2(G_\epsilon), f^+ = \rho f^- \text{ a.e. on } E \right\}$$

- RHS is not necc closed for randomly chosen F_ϵ
- F_ϵ is not unique
- Θ_M, ρ, E are (essentially) unique

$M \in \mathrm{Lat}(S, H^2(G))$ is **cyclic** if there is an f so that

$$[f] = \bigvee \{z^n f : z \in \mathbb{N}_0\} = M.$$

Is every M cyclic?

Theorem

If

$$M = \Theta_M \cdot \left\{ f \in H^2(G) : \frac{f}{F_\epsilon} \in H^2(G_\epsilon), f^+ = \rho f^- \text{ a.e. on } E \right\},$$

and

$$m_1([0,1] \setminus E) > 0$$

then M is not cyclic.

Converse is not true.

What is $[f]$?

Theorem

If f and $1/f \in H^2(G)$, then

$$[f] = \{f \in H^2(G) : f^+ = \rho f^- \text{a.e. on } [0,1]\},$$

where $\rho = f^+/f^-$.

Is every M 2-cyclic?

Theorem

If $M \in Lat(S, H^2(G))$, then there $f, g \in H^2(G)$ such that

$$M = [f, g] := \bigvee \{z^n f, z^m g : m, n \in \mathbb{N}_0\}.$$

f, g can be chosen to be solutions to two extremal problems (just like in Beurling's theorem).

When is $[f, g] = H^2(G)$?

Theorem

If $f, g \in H^2(G) \setminus \{0\}$, then $[f, g] = H^2(G)$ if and only if f and g have no non-trivial common G-inner factor and the set

$$\left\{x \in [0,1) : \frac{f^+(x)}{f^-(x)} = \frac{g^+(x)}{g^-(x)}\right\}$$

has Lebesgue measure zero.

VOTCAM

Virginia **O**perator **T**heory and **C**omplex **A**nalysis **M**eeting

Saturday - November 10, 2007 - University of Richmond

- Vern Paulsen (Houston)
- Warren Wogen (Chapel Hill)
- David Sherman (Virginia)
- Leiba Rodman (William and Mary)