The backward shift on H^p

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1. Introduction

In this semi-expository paper, we examine the backward shift operator

$$Bf := \frac{f - f(0)}{z}$$

on the classical Hardy space H^p. Though there are many aspects of this operator worthy of study [20], we will focus on the description of its invariant subspaces by which we mean the closed linear manifolds $\mathcal{E} \subset H^p$ for which $B\mathcal{E} \subset \mathcal{E}$. When $1 < p < \infty$, a seminal paper of Douglas, Shapiro, and Shields [8] describes these invariant subspaces by using the important concept of a pseudocontinuation developed earlier by Shapiro [26]. When $p = 1$, the description is the same [1] except that in the proof, one must be mindful of some technical considerations involving the functions of bounded mean oscillation.

The $p \geqslant 1$ case involves heavy use of duality and especially the Hahn-Banach separation theorem where one gets at $\mathcal{E}$ by first looking at $\mathcal{E}^{\perp}$, the annihilator of $\mathcal{E}$, and then returning to $\mathcal{E}$ by ${}^{\perp}(\mathcal{E}^{\perp})$. On the other hand, when $0 < p < 1$, H^p is no longer locally convex and the Hahn-Banach separation theorem fails [12]. In fact, as we shall see in § 4, there are invariant subspaces $\mathcal{E} \neq H^p$, $0 < p < 1$, for which ${}^{\perp}(\mathcal{E}^{\perp}) = H^p$. Despite these difficulties, an ingenious *tour de force* approach of Aleksandrov [1] (see also [6]), using such tools as distribution theory and the atomic decomposition theorem, characterizes these invariant subspaces.

The first several sections of this paper are a leisurely, non-technical, treatment of the Douglas-Shapiro-Shields and Aleksandrov results. In § 5, we focus on some new results, based on techniques in [4], which give an alternative description of certain invariant subspaces of H^p. As a consequence, we eventually wind up characterizing the weakly closed invariant subspaces of H^p. In § 6, we make some remarks about the invariant subspaces of the standard Bergman spaces L^p_a when $0 < p < 1$.

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2. Preliminaries

We begin with some basic definitions and well-known results about the Hardy spaces H^p. A detailed treatment can be found in [11]. For $0 < p < \infty$, let H^p denote the space of analytic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ whose L^p integral means

$$\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}$$

are uniformly bounded for $r \in (0,1)$. These means increase as $r \nearrow 1$ and we define

$$\|f\|_p := \lim_{r \to 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

For almost every (with respect to Lebesgue measure on the unit circle $\mathbb{T} := \partial\mathbb{D}$) $e^{i\theta}$, the radial limit

$$\lim_{r \to 1^-} f(re^{i\theta})$$

exists and we denote its value by $f(e^{i\theta})$, or perhaps $f^*(e^{i\theta})$ when we want to emphasize this almost everywhere defined boundary function. Moreover,

$$\|f\|_p = \left(\int_0^{2\pi} |f^*(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

One can show that $f \in H^p$ satisfies the pointwise estimate

$$|f(z)| \leqslant 2^{1/p} \|f\|_p (1 - |z|)^{-1/p}, \quad z \in \mathbb{D}.$$

As a result, for $1 \leqslant p < \infty$, the quantity $\|f\|_p$ defines a norm that makes H^p a Banach space while for $0 < p < 1$, $\|f - g\|_p^p$ defines a translation invariant metric that makes H^p a complete metric space. In either case, $f(re^{i\theta}) \to f^*(e^{i\theta})$ almost everywhere and in the norm (metric) of L^p. When $p = \infty$, H^∞ will denote the bounded analytic functions on $\mathbb{D}$ with the sup-norm $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$.

Since $f \to f^*$ is an isometry of H^p to L^p, one can regard H^p as a closed subspace of L^p. In fact, at least when $1 \leqslant p < \infty$, we can think of H^p in the following way

$$H^p = \{f \in L^p : \widehat{f}(n) = 0 \text{ for all } n < 0\},$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient of f. This follows from the F. and M. Riesz theorem [11, p. 41].

Every function $f \in H^p$ can be factored as $f = \phi\Theta$, where $\phi \in H^\infty$ with $|\phi^*(e^{i\theta})| = 1$ almost everywhere (such functions are called 'inner functions') and $\Theta \in H^p$ has no zeros on $\mathbb{D}$ and satisfies

$$\log|\Theta(0)| = \int_0^{2\pi} \log|\Theta^*(e^{i\theta})| \frac{d\theta}{2\pi}$$

(such functions are called 'outer functions'). Moreover, except for a unimodular constant, this factorization is unique.

Identifying the dual, $(H^p)^*$, of H^p with a space of analytic functions on $\mathbb{D}$ is often, but not always, the key to understanding the structure of its invariant subspaces. The dual pairing between H^p and $(H^p)^*$, as a space of analytic functions, is the following 'Cauchy pairing'. For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

on the disk $\mathbb{D}$, define

$$\langle f, g \rangle := \lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n \overline{b_n} r^n, \tag{2.1}$$

whenever this limit exists. A simple computation with power series shows that if $\langle f, g \rangle$ exists, then

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_0^{2\pi} f(re^{i\theta}) \overline{g}(re^{i\theta}) \frac{d\theta}{2\pi}.$$

For $1 < p < \infty$, the dual of H^p can be identified with H^q, where q is the conjugate index to p, and this comes somewhat easily. Notice that for $f \in H^p$ and $g \in H^q$, we have $f_r \to f$ in H^p and $g_r \to g$ in H^q, where $f_r(z) := f(rz)$. Thus, by Hölder's inequality,

$$\langle f, g \rangle = \int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi}.$$

Certainly, the linear functional $f \to \langle f, g \rangle$ is continuous on H^p for fixed $g \in H^q$. On the other hand, if $\ell \in (H^p)^*$, the Hahn-Banach extension theorem says that

$$\ell(f) = \int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi} \tag{2.2}$$

for some $g \in L^q$. Using the continuity of the Riesz projection operator $P : L^q \to H^q$

$$P\left(\sum_{n=-\infty}^{\infty} \widehat{g}(n) e^{in\theta} \right) = \sum_{n=0}^{\infty} \widehat{g}(n) e^{in\theta}$$

and the identities

$$\int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi} = \int_0^{2\pi} f \overline{Pg} \frac{d\theta}{2\pi} = \langle f, Pg \rangle,$$

one can replace, in (2.2) and hence (2.1), the above $g \in L^q$ with a unique function in H^q. A little technical detail shows that norm of the linear functional $f \to \langle f, g \rangle$ is equivalent to the H^q norm of g. Thus $(H^p)^*$ can be identified with H^q via the dual pairing in (2.1).

When $p = 1$, the above analysis breaks down. Certainly if $\ell \in (H^1)^*$, then

$$\ell(f) = \int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi}$$

for some $g \in L^\infty$. However, when one tries to imitate the above analysis and replace g with Pg in the above integral, there are problems. For one, $P(L^\infty) =$

$BMOA \supsetneq H^\infty$, where $BMOA$ are the analytic functions of bounded mean oscillation[1]. Secondly, there are $f \in H^1$ and $g \in BMOA$, for which $f\overline{g} \notin L^1$. These technical problems are not insurmountable since, for $f \in H^1$ and $g \in BMOA$, the quantity $\langle f, g\rangle$ (as in (2.1)) does indeed exist and $f \to \langle f, g\rangle$ defines a continuous linear functional on H^1. In fact, these are all the linear functionals on H^1. Another technical detail says that the norm of $f \to \langle f, g\rangle$ is equivalent to the $BMOA$ norm of g. In summary, we can identify the dual of H^1 with $BMOA$ via the dual pairing in (2.1). See [13, Ch. 6] for more details on all this.

When $0 < p < 1$, surprisingly, there are non-trivial bounded linear functionals on H^p. Surprisingly since when $0 < p < 1$, $(L^p)^* = (0)$ [7]. The theorem here is one of Duren, Romberg, and Shields [12] and says that if ℓ is a bounded linear functional on H^p, then there is a unique g belonging O_p, a subspace of the disk algebra, so that $\ell(f) = \langle f, g\rangle$. Conversely, for $g \in O_p$, $f \to \langle f, g\rangle$ defines an element of $(H^p)^*$. The space O_β, for $\beta > 0$, is the set of analytic functions g on the disk for which

$$\|g\|_\beta := \sup_{|z|<1} |g^{[1/\beta]}(z)|(1 - |z|) < \infty,$$

where, if $g(z) = \sum a_n z^n$,

$$g^{[\alpha]}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} a_n z^n$$

is the fractional derivative of g of order α. The classes O_β can be equivalently characterized as Lipschitz or Zygmund spaces. For example, if $1/2 < p < 1$, then O_p is the space of analytic functions on $\mathbb{D}$ which have continuous extensions to $\mathbb{D}^-$ and such that

$$\sup_{\theta \neq t} \frac{|g(e^{i\theta}) - g(e^{it})|}{|\theta - t|^{1/p-1}} < \infty.$$

For other p's, one requires the derivatives (depending on p) of g to have certain smoothness on $\mathbb{T}$. In general, the smaller the p, the more derivatives of g that need to satisfy a Lipschitz or Zygmund condition on $\mathbb{T}$ in order for g to belong to O_p. One can show that the norm of the functional $f \to \langle f, g\rangle$ is equivalent to the O_p norm of g [2]. Thus we identify $(H^p)^*$ with O_p when $0 < p < 1$ via (2.1). Again, consult [12] for the details.

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[1] $g \in L^1$ is of bounded mean oscillation BMO if $\|g\| = \|g\|_{L^1} + \sup_I \frac{1}{|I|} \int_I |g - g_I| d\theta < \infty$, where $g_I = |I|^{-1} \int_I g d\theta$ and $|I|$ is the length of an arc $I \subset \mathbb{T}$. $BMOA := BMO \cap H^1$.

[2] Technically $\|g\|_\beta$ is only a semi-norm on O_β. One can make this a true norm by adding in $|g(0)| + |g'(0)| + \cdots + |g^{(\lfloor 1/\beta \rfloor)}|$, where $\lfloor x \rfloor$ is the greatest integer less than x.

3. The backward shift on H^p for $1 \leqslant p < \infty$

If $1 < p < \infty$, notice that the backward shift B on H^p is the Banach space adjoint of the forward shift operator $Sf = zf$ on H^q, that is to say

$$\langle Bf, g\rangle = \langle f, Sg\rangle, \quad f \in H^p, \quad g \in H^q.$$

Thus if $\mathcal{E} \subsetneq H^p$ is an invariant subspace for B, then

$$\mathcal{E}^{\perp} := \{g \in H^q : \langle f, g\rangle = 0 \ \ \forall f \in \mathcal{E}\},$$

the 'annihilator' of $\mathcal{E}$, is an S-invariant subspace of H^q. A celebrated theorem of Beurling [11, p. 114] says that $\mathcal{E}^{\perp} = \phi H^q$ for some non-constant inner function ϕ. By the Hahn-Banach separation theorem,

$$\mathcal{E} = {}^{\perp}(\mathcal{E}^{\perp}) = {}^{\perp}(\phi H^q),$$

where for $A \subset H^q$, ${}^{\perp}A := \{f \in H^p : \langle f, g\rangle = 0, \forall g \in A\}$ is the 'pre-annihilator' of A. So the problem of describing $\mathcal{E}$ is reduced to characterizing, in some function-theoretic way, this pre-annihilator ${}^{\perp}(\phi H^q)$.

The function theoretic tool, the concept of a pseudocontinuation, used here was developed by Shapiro in some earlier work [26] and we now take a few moments to point out some basic facts about pseudocontinuations. Suppose that h is a meromorphic function on $\mathbb{D}$ and H is a meromorphic on $\mathbb{D}_e$. There is no *a priori* reason why the non-tangential limits of h (from $\mathbb{D}$) and H (from $\mathbb{D}_e$) need to exist. But *if* they do, *and* they are equal almost everywhere, we say that H is a 'pseudocontinuation' of h. Two representative examples of functions with a pseudocontinuation are the following.

Example 3.1.

1. If h is an inner function, then

$$H(z) = \frac{1}{\overline{h}(1/\overline{z})}$$

is a pseudocontinuation of h. This follows from that fact that $h^* \overline{h^*} = 1$ almost everywhere. Also notice, for example, that if h is a Blaschke product whose zeros accumulate on all of the circle, then h, although a pseudocontinuable function, will not have an analytic continuation across any point of the unit circle.

2. Another example of a pseudocontinuation is when h is a Cauchy integral

$$h(z) := \int \frac{1}{1 - e^{-i\theta} z} d\mu(e^{i\theta}),$$

where μ is a finite Borel measure on $\mathbb{T}$ that is singular with respect to Lebesgue measure. If H is the above Cauchy integral but with $z \in \mathbb{D}_e$, one can show that h and H are H^p functions (for $0 < p < 1$) on their respective domains [11, p. 39][3] and so have finite non-tangential limits

[3] The Hardy space of the extended exterior disk $\mathbb{D}_e := \{z \in \widehat{\mathbb{C}} : 1 < |z| \leqslant \infty\}$ is defined by $H^p(\mathbb{D}_e) := \{f(1/z) : f \in H^p\}$. Note that if $f \in H^p$, then $\overline{f}(e^{i\theta})$ is the boundary function for a function belonging to $H^p(\mathbb{D}_e)$, the function being $\overline{f}(1/\overline{z})$.

almost everywhere. Notice that

$$h(z) - h(1/\overline{z}) = \int_0^{2\pi} P_z(e^{i\theta}) d\mu(e^{i\theta}),$$

where $P_z(e^{i\theta})$ is the Poisson kernel. Using a classical theorem of Fatou [11, p. 39], which says that

$$\int_0^{2\pi} P_z(e^{i\theta}) d\mu(e^{i\theta}) \to 2\pi\mu^{'}(e^{i\theta})$$

for almost every $e^{i\theta}$ as $z \to e^{i\theta}$, and the fact that μ is singular (and so $\mu' = 0$ almost everywhere), one can show that the non-tangential limits of h and H are equal almost everywhere.

Let us make a few general comments about pseudocontinuations. The first is that they are unique. Indeed, if H_1 and H_2 are two pseudocontinuations of h, then $H_1 - H_2$ is a meromorphic function on $\mathbb{D}_e$ that has zero non-tangential limits almost everywhere. A classical theorem of Privalov [16, p. 62] says that any meromorphic function that has zero non-tangential limits on a subset of $\mathbb{T}$ with positive Lebesgue measure must be identically zero. Hence h can have only one pseudocontinuation. Here is why we use *non-tangential limits* rather than *radial limits* in the definition of a pseudocontinuation. If radial limits were used, then pseudocontinuations would not be unique. Indeed, there are non-trivial analytic functions on $\mathbb{D}$ which have radial limits equal to zero almost everywhere [5] - thus the zero function would be a pseudocontinuation without the original function being the zero function. Certainly, when we are talking about H^p functions this cannot happen since the non-tangential limits exist almost everywhere anyway. But in general, we need to make this important distinction.

Another consequence of Privalov's uniqueness theorem is that if h has an analytic continuation to a neighborhood U of $e^{i\theta}$ and a pseudocontinuation H, then $h = H$ on $U \cap \mathbb{D}_e$, that is to say, pseudocontinuation is compatible with the classical notion of analytic continuation.

The point at infinity is important. The function $h(z) = e^z$ certainly has an analytic continuation across $\mathbb{T}$. However, it does not have a pseudocontinuation as we have defined it above since $H(z) = e^z$ has an essential singularity at infinity. The interested reader is invited to consult [23] for a more detailed discussion of pseudocontinuations.

The function theoretic description of ${}^{\perp}(\phi H^q)$ is the following well-known theorem.

Proposition 3.2 (Douglas-Shapiro-Shields). *Let ϕ be an inner function and $1 < p < \infty$. For $f \in H^p$, the following are equivalent:*

1. $f \in {}^{\perp}(\phi H^q)$
2. $f^* \in H^p \cap \phi\overline{H_0^p}$, *where* $H_0^p = \{f \in H^p : f(0) = 0\}$.

3. *The meromorphic function f/ϕ on $\mathbb{D}$ has a pseudocontinuation to a function $\widetilde{f_\phi} \in H^p(\mathbb{D}_e)$ with $\widetilde{f_\phi}(\infty) = 0$.*

It is important to note that the space

$$H^p \cap \phi\overline{H_0^p} = \{f \in H^p : f^* = \phi^*\overline{h^*}, h \in H_0^p\}^4 \tag{3.3}$$

must be understood as a space of functions on the circle and not on the disk. For fixed $1 < p < \infty$ and inner function ϕ, we let $\mathcal{E}^p(\phi)$ be the collection of H^p functions that satisfy one of the equivalent conditions in Proposition 3.2. Since $\mathcal{E}^p(\phi)$ is an annihilating subspace, it is closed in H^p. It also follows from the above argument that $\mathcal{E}^p(\phi)$ is invariant. Combining this with what was said above, we have the following summary theorem.

Theorem 3.4 (Douglas-Shapiro-Shields)**.** *For $1 < p < \infty$, a subspace $\mathcal{E} \subsetneq H^p$, is invariant if and only if $\mathcal{E} = \mathcal{E}^p(\phi)$ for some inner function ϕ.*

Before proceeding to the $p = 1$ case, we mention a few other items of interest. Using a Morera type argument, one can show that every $f \in \mathcal{E}^p(\phi)$ has an analytic continuation to the set

$$\widehat{\mathbb{C}} \setminus \{1/\overline{z} : z \in \sigma(\phi)\},$$

where

$$\sigma(\phi) := \left\{ z \in \mathbb{D}^- : \liminf_{\lambda \to z} |\phi(\lambda)| = 0 \right\}^5.$$

Note that ϕ has an analytic continuation to $\widehat{\mathbb{C}} \setminus \{1/\overline{z} : z \in \sigma(\phi)\}$ [13, p. 75-76]. Furthermore, by the compatibility of pseudocontinuation with analytic continuation, the analytic continuation of f/ϕ to $\mathbb{D}_e \setminus \{1/\overline{z} : z \in \sigma(\phi)\}$ must be equal to $\widetilde{f_\phi}$, the pseudocontinuation of f/ϕ.

Another interesting item is that if $f = B\phi$, then

$$[f]_{H^p} := \bigvee\{B^n f : n = 0, 1, 2, \cdots\} = \mathcal{E}^p(\phi),$$

where $\bigvee$ is the closed linear span in the H^p norm. This says that $\mathcal{E}^p(\phi)$ is a 'cyclic invariant subspace' generated by $f = B\phi$. While we are mentioning cyclic vectors, there is a celebrated result that determines exactly when a particular $f \in H^p$ is 'cyclic', that is to say $[f]_{H^p} = H^p$.

Theorem 3.5 (Douglas-Shapiro-Shields)**.** *For $1 \leqslant p < \infty$, a vector $f \in H^p$ is noncyclic for the backward shift if and only if f has a pseudocontinuation of bounded type, i.e., there is a meromorphic function $\widetilde{f}$ on $\mathbb{D}_e$, that can be written as a quotient of two bounded analytic functions on $\mathbb{D}_e$, such that $\widetilde{f}$ is a pseudocontinuation of f.*

[4] Recall from the preliminaries that $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ almost everywhere.

[5] A basic fact about inner functions is that if $\phi = bs_\mu$, where b is a Blaschke product and s_μ is a singular inner function with associated positive singular measure μ on $\mathbb{T}$ (all inner functions can be factored this way), then $\sigma(\phi)$ is the closure of the zeros of b together with the support of μ.

Though this theorem is both necessary and sufficient, the hypothesis (having a pseudocontinuation of bounded type) is not something easily tested. There are some obvious examples of cyclic vectors like

$$e^{1/(z-2)} \quad \text{and} \quad e^z$$

which are not meromorphic on $\mathbb{D}_e$, and

$$f = \sum_{n=1}^{\infty} \frac{2^{-n}}{z - (1 + 1/n)}$$

which has a pseudocontinuation, but not of bounded type (too many poles). Notice that we are using the uniqueness of pseudocontinuations here and the fact that if a function has an analytic continuation across a point of the circle, then the analytic continuation must agree with its pseudocontinuation. Along these lines, the vector $\sqrt{1-z}$ is a cyclic vector since its pseudocontinuation, which must be $\sqrt{1-z}$, can not have a branch cut. Less obvious examples of cyclic vectors are H^p functions given by Hadamard gap series [27] such as

$$f(z) = \sum_{n=0}^{\infty} 2^{-n} z^{2^n}$$

or Fabry gap series [2]

$$f(z) = \sum_{n=0}^{\infty} 2^{-n} z^{n^2}.$$

Actually, both of these gap series have the following stronger pathological property: There exists no $1 < R < \infty$ and no meromorphic function $\widetilde{f}$ on $\{z : 1 < |z| < R\}$ such that the nontangential limits of $\widetilde{f}$ and f agree almost everywhere. See also [23] for further details and other pathological examples of this type.

The $p = 1$ case is a bit pesky and poses some technical challenges that were overcome by Aleksandrov (see [1] or [6, p. 101]). If $\mathcal{E} \subset H^1$ is invariant, then $\mathcal{E}^{\perp}$ is an S-invariant subspace of $BMOA$, closed in the weak-* topology $BMOA$ inherits by being the dual of H^1. However, the description of these S-invariant subspaces is not as simple as $\phi BMOA$ (ϕ inner), as in Beurling's theorem for H^p. In fact, $\phi BMOA$ may not even be a subset of $BMOA$. That is to say, ϕ may not be a 'multiplier' of $BMOA$ [30]. The second technical challenge is that the dual pairing between H^1 and $BMOA$ is

$$\lim_{r \to 1^-} \int_0^{2\pi} f \overline{g_r} \frac{d\theta}{2\pi} \tag{3.6}$$

and not simply

$$\int_0^{2\pi} f \overline{g} \frac{d\theta}{2\pi}. \tag{3.7}$$

This may not seem like a major difference but the proof of the Douglas-Shapiro-Shields theorem makes use of the F. and M. Riesz theorem [11, p. 40-41]

$$\int_{[0,2\pi]} e^{in\theta} d\mu(\theta) = 0 \quad n = 0, 1, 2, \cdots \quad \Leftrightarrow \quad d\mu(\theta) = f(e^{i\theta}) \frac{d\theta}{2\pi}, \quad f \in H^1, \tag{3.8}$$

for which we need to write the dual pairing $\langle f, g \rangle$ as an integral, as in (3.7), and not as a limit of integrals, as in (3.6). Nevertheless, one can show that $\mathcal{E} \cap H^2$ is not equal to H^2, is closed in the norm of H^2, and is invariant and hence takes the form $\mathcal{E}^2(\phi)$ (Theorem 3.4). Using the $(H^1, BMOA)$ duality, one can show that $\mathcal{E}^2(\phi)$ is dense in $\mathcal{E}^1(\phi)$. Here $\mathcal{E}^1(\phi) := H^1 \cap \phi \overline{H^1_0}$, or equivalently, the space of H^1 functions f such that f/ϕ has a pseudocontinuation to a function $\widetilde{f_\phi} \in H^1(\mathbb{D}_e)$ that vanishes at infinity. Thus $\mathcal{E}^1(\phi) \subset \mathcal{E}$. The other inclusion is also a bit tricky but nevertheless true. The summary theorem here is the following.

Theorem 3.9 (Aleksandrov).

1. *A subspace $\mathcal{E} \subsetneq H^1$ is invariant if and only if $\mathcal{E} = \mathcal{E}^1(\phi)$ for some inner function ϕ.*
2. *If $f = B\phi$, then $[f]_{H^1} = \mathcal{E}^1(\phi)$, that is to say $\mathcal{E}^1(\phi)$ is cyclic.*
3. *A vector $f \in H^1$ is non-cyclic, i.e., $[f]_{H^1} \neq H^1$, if and only if f has a pseudocontinuation of bounded type.*

4. The backward shift on H^p for $0 < p < 1$

Characterizing the invariant subspaces of H^p when $0 < p < 1$ poses special challenges. For example, H^p $(0 < p < 1)$, with its metric topology, is no longer locally convex and the Hahn-Banach separation theorem, a key tool in understanding the $p \geqslant 1$ case, fails[6].

Example 4.1. For each $\theta \in [0, 2\pi]$, the function $(1 - e^{-i\theta} z)^{-1}$ belongs to H^p for all $0 < p < 1$ and so we can consider the following subspace of H^p:

$$\mathcal{E} := \bigvee \left\{ \frac{1}{1 - e^{-i\theta} z} : 0 \leqslant \theta < 2\pi \right\}.$$

When z is on the unit circle, we have

$$\frac{1}{1 - e^{-i\theta} z} = \frac{\overline{z}}{\overline{z} - e^{-i\theta}} \in \overline{H^p_0}$$

and so $\mathcal{E} \subset H^p \cap \overline{H^p_0} \neq H^p$. As an aside, one can show that indeed $\mathcal{E} = H^p \cap \overline{H^p_0}$ (see [1] or [6, p. 116]). Again we remind the reader that $H^p \cap \overline{H^p_0}$ is a space of functions on the unit circle (see (3.3)). We claim that $\mathcal{E}^\perp = (0)$. Indeed, if $g = \sum_n b_n z^n \in O_p = (H^p)^*$ belongs to $\mathcal{E}^\perp$, then for all θ,

$$0 = \left\langle \frac{1}{1 - e^{-i\theta} z}, g \right\rangle = \lim_{r \to 1} \sum_{n=0}^{\infty} e^{-in\theta} \overline{b_n} r^n = \lim_{r \to 1^-} \overline{g}(re^{i\theta}) = \overline{g}(e^{i\theta}),$$

[6] The Hahn-Banach extension theorem also fails in H^p $(0 < p < 1)$. Indeed, there is a closed subspace A of H^p and a continuous linear functional on A which cannot be extended continuously to all of H^p [12].

making g the zero function.

Example 4.1 shows that describing an invariant subspace $\mathcal{E}$ of H^p $(0 < p < 1)$ by first examining $\mathcal{E}^{\perp}$ and then returning to $\mathcal{E}$ via the Hahn-Banach separation theorem $\mathcal{E} = {}^{\perp}(\mathcal{E}^{\perp})$ is of no use here. In the above example, $\mathcal{E} \neq H^p$, but ${}^{\perp}(\mathcal{E}^{\perp}) = {}^{\perp}(0) = H^p$. As it turns out though, the invariant subspaces of H^p $(0 < p < 1)$ can be characterized but the description is not the same as before (namely $\mathcal{E}^p(\phi)$ spaces) and the proof is much more difficult, involving many advanced tools in analysis. This complicated but beautiful characterization was accomplished by Aleksandrov [1] and we spend a few moments stating his result.

With the use of duality out, one must discover what functions belong to a given invariant subspace almost by hand. Given $0 < p < 1$ and an invariant subspace $\mathcal{E} \subset H^p$, we notice that $\mathcal{E} \cap H^2$ is a closed (in the H^2 norm) invariant subspace of H^2 which, by the Douglas-Shapiro-Shields theorem (Theorem 3.4), equals $\mathcal{E}^2(\phi)$ for some inner function ϕ. If $\mathcal{E} \cap H^2 = (0)$, which can indeed be the case by Example 4.1, we take the ϕ to be the constant function $\phi = 1$. This makes sense since $\mathcal{E}^2(1) = H^2 \cap \overline{H^2_0} = (0)$ (F. and M. Riesz theorem - (3.8)).

Let $F \subset \mathbb{T}$ be the following set

$$F := \left\{ e^{i\theta} \in \mathbb{T} : \frac{1}{1 - e^{-i\theta} z} \in \mathcal{E} \right\}.$$

One can show that F is a closed subset of $\mathbb{T}$ and that $\sigma(\phi) \cap \mathbb{T} \subset F$. Also consider a function

$$k : F \to \mathbb{N} \cap [1, n_p],$$

where

$$n_p := \max\{n \in \mathbb{N} \cap [1, 1/p)\}, \tag{4.2}$$

defined by

$$k(e^{i\theta}) := \max \left\{ j \in \mathbb{N} \cap [1, n_p] : \frac{1}{(1 - e^{-i\theta} z)^j} \in \mathcal{E} \right\}.$$

Note that a simple integral calculation shows that $(1 - e^{-i\theta} z)^{-j} \in H^p$ for all $j \in \mathbb{N} \cap [1, n_p]$. One can show that if F_0 is the set of isolated points of F, then $k(e^{i\theta}) = n_p$ whenever $e^{i\theta} \in (F \setminus F_0) \cup (\sigma(\phi) \cap \mathbb{T})$.

With these three parameters ϕ, F, k, form the space $\mathcal{E}^p(\phi, F, k)$ of functions $f \in H^p$ such that

1. $f^* \in H^p \cap \phi \overline{H^p_0}$, or equivalently f/ϕ has a pseudocontinuation to a function $\widetilde{f_\phi} \in H^p(\mathbb{D}_e)$ that vanishes at infinity;
2. f has an analytic continuation to a neighborhood of $\mathbb{T} \setminus F$;
3. At each $e^{i\theta} \in F_0 \setminus \sigma(\phi)$, f has a pole of order at most $k(e^{i\theta})$.

Before moving on, let us give a non-trivial example of a function belonging to $\mathcal{E}^p(\phi, F, k)$. This example will become important later on.

Example 4.3. Suppose $F \subset \mathbb{T}$ is a closed set of Lebesgue measure zero. We assume, as usual, that $\sigma(\phi) \cap \mathbb{T} \subset F$. Consider the function

$$(K\mu)(z) := \int \frac{d\mu(e^{i\theta})}{1 - e^{-i\theta} z},$$

where μ is a finite Borel measure on $\mathbb{T}$ whose support is exactly F. As mentioned earlier in Example 3.1, $K\mu|\mathbb{D} \in H^p$ and $K\mu|\mathbb{D}_e \in H^p(\mathbb{D}_e)$ and moreover, since μ is singular with respect to Lebesgue measure, these two functions are pseudocontinuations of each other. Furthermore, $K\mu$ has an analytic continuation across $\mathbb{T} \setminus F$ and at each isolated point of F, $K\mu$ has a pole of order one. Finally, note from Example 3.1, that the inner function ϕ has a pseudocontinuation

$$\widetilde{\phi}(z) = \frac{1}{\overline{\phi}(1/\overline{z})}, \ z \in \mathbb{D}_e$$

and so $K\mu/\phi$ has a pseudocontinuation $K\mu/\widetilde{\phi}$ which belongs to $H^p(\mathbb{D}_e)$ and vanishes at infinity. Thus $K\mu \in \mathcal{E}^p(\phi, F, k)$, at least when F has Lebesgue measure zero.

Though somewhat involved to prove, one can show that $\mathcal{E}^p(\phi, F, k)$ is a non-trivial closed invariant subspace of H^p (invariance is clear, closed is what is difficult to prove). Furthermore,

$$\mathcal{E} \subset \mathcal{E}^p(\phi, F, k).$$

To obtain the reverse inclusion, Aleksandrov defines the space

$$e^p(\phi, F, k) := \mathcal{E}^2(\phi) \bigvee \left\{ \frac{1}{(1 - e^{-i\theta} z)^j} : e^{i\theta} \in F; j = 1, 2, \cdots, k(e^{i\theta}) \right\}. \tag{4.4}$$

From the very definition of the parameters ϕ, F and k, it follows that

$$e^p(\phi, F, k) \subset \mathcal{E}.$$

What is very difficult to prove here is that

$$e^p(\phi, F, k) = \mathcal{E}^p(\phi, F, k). \tag{4.5}$$

Aleksandrov's proof of this fact is quite involved and uses, among other tricks, distribution theory and the Coifman atomic decomposition theorem for H^p. To summarize, we have the following.

Theorem 4.6 (Aleksandrov)**.** *For fixed $0 < p < 1$ and parameters $\phi, F,$ and k above, the space $\mathcal{E}^p(\phi, F, k)$ is an invariant subspace of H^p. Moreover, every proper invariant subspace of H^p is of the form $\mathcal{E}^p(\phi, F, k)$.*

We close this section with a few remarks. The characterization of the cyclic vectors remains the same: f is non-cyclic if and only if f has a pseudocontinuation of bounded type. One can also show, as in the H^p case when $p \geqslant 1$ but with a more complicated vector, that $\mathcal{E}^p(\phi, F, k)$ is a cyclic subspace (i.e., generated by one vector). Later on (Theorem 5.6) we will give an alternative characterization of $\mathcal{E}^p(\phi, F, k)$. The curious reader might be wondering why the parameters F and

k are not needed in the $1 \leqslant p < \infty$ case. Notice that $(1 - e^{-i\theta} z)^{-j} \notin H^1$ for any $\theta \in [0, 2\pi)$ and $j \in \mathbb{N}$.

5. A closer look at Aleksandrov's theorem

Aleksandrov's theorem says that when $0 < p < 1$, a non-trivial invariant subspace of H^p takes the form $\mathcal{E}^p(\phi, F, k)$ (as described in §4). In this section, we show that under certain natural conditions, there is an alternative description of $\mathcal{E}^p(\phi, F, k)$. To do this, we will characterize the weakly closed invariant subspaces of H^p.

Let us say a few words about the weak topology on H^p $(0 < p < 1)$. The reader can refer to [12] for further details and examples. Recall from § 2 that $(H^p)^*$ can be identified (with equivalent norm) with a Lipschitz or Zygmund space O_p by means of the pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n \overline{b_n} r^n.$$

A set $U \subset H^p$ is 'weakly open' if given any $f_0 \in U$, there is an $\varepsilon > 0$ and $g_1, \cdots, g_n \in O_p$ so that

$$\bigcap_{j=1}^{n} \{f \in H^p : |\langle f - f_0, g_j \rangle| < \varepsilon\} \subset U.$$

Since the family of semi-norms

$$\{\rho_g(f) := |\langle f, g \rangle| : g \in O_p\}$$

on H^p separates points, standard functional analysis says that (H^p, wk) (H^p endowed with the weak topology) is a locally convex topological vector space [24, p. 64]. Furthermore, $(H^p, wk)^* = O_p$. As a consequence, a linear manifold $E \subset H^p$ is weakly closed if and only if it satisfies the Hahn-Banach separation property: If $f \notin E$, there is a $g \in O_p$ so that $g \perp E$ but $\langle f, g \rangle = 1$, i.e., each point not in E can be separated from E by a bounded linear functional [24, p. 60]. Viewing this another way,

$$\text{clos}_{(H^p, wk)} E = {}^{\perp}(E^{\perp}), \tag{5.1}$$

where, for $C \subset O_p$, ${}^{\perp}C := \{f \in H^p : \langle f, c \rangle = 0 \ \ \forall c \in C\}$ is the pre-annihilator of C. Finally notice that if E is weakly closed then E is closed in the metric topology.

There is a containing Banach space B^p of H^p namely, the weighted Bergman space of analytic functions f on $\mathbb{D}$ for which the quantity

$$\|f\|_{B^p} := \int_0^{2\pi} \int_0^1 |f(re^{i\theta})| (1 - r)^{1/p - 2} dr \, \frac{d\theta}{2\pi}$$

is finite. Certain standard facts about B^p are that H^p is a dense subset of B^p and

$$\|f\|_{B^p} \leqslant A_p \|f\|_{H^p}, \ \ f \in H^p,$$

that is to say, the containment $H^p \subset B^p$ is continuous. Moreover, B^p is a Banach space and $(B^p)^*$ can be identified (with equivalent norm) with the space O_p via the dual pairing in (2.1), i.e.,

$$\langle f, g\rangle = \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n \overline{b_n} r^n.$$

Thus B^p and H^p have the same continuous linear functionals. Using this fact along with the Hahn-Banach separation theorem, applied to the Banach space B^p, one can show that if E is a linear manifold in H^p, then

$$\operatorname{clos}_{(H^p,wk)} E = \big(\operatorname{clos}_{B^p} E\big) \cap H^p.$$

See [12, Lemma 8] for details.

For the rest of this section we will assume that $1/2 < p < 1$. For other values of p, most of the results are still true but the notation becomes cumbersome since the description of O_p changes very much with p. That being said, we fix $1/2 < p < 1$, an inner function ϕ, and a closed set $F \subset \mathbb{T}$. Without loss of generality, we assume that $\sigma(\phi) \cap \mathbb{T} \subset F$. Define

$$\mathcal{I}(\phi, F) := \{g \in O_p : g \in \phi H^\infty, g|F = 0\}^7.$$

One can easily observe that $\mathcal{I}(\phi, F)$ is an ideal of O_p. What is more difficult to prove is that when O_p is endowed with the weak-* topology it naturally inherits by being the dual of B^p, then $\mathcal{I}(\phi, F)$ is weak-* closed. In fact, every non-zero weak-* closed ideal of O_p is of the form $\mathcal{I}(\phi, F)$. There is a direct proof of this result (with an equivalent weak-* topology on O_p) in [21]. Another, perhaps more indirect, proof is found in [4, Thm. 3.2][8] Also, $\mathcal{I}(\phi, F) \neq (0)$ if and only if

$$\int_0^{2\pi} \log \operatorname{dist}(e^{i\theta}, \sigma(\phi) \cup F) \frac{d\theta}{2\pi} > -\infty \tag{5.2}$$

(see [32]). In fact, if (5.2) holds, then there is a $g \in A^\infty$ ($g^{(k)}$ has a continuous extension to $\mathbb{D}^-$ for all k) such that $g \in \mathcal{I}(\phi, F) \setminus (0)$ and g generates $\mathcal{I}(\phi, F)$ in the sense that the smallest weak-* closed ideal containing g is $\mathcal{I}(\phi, F)$. In this case, ϕ_g, the inner part of g, must be ϕ and $g^{-1}(\{0\}) \cap \mathbb{T}$ must be F [15].

It is worth repeating here that we are assuming, to avoid technical details, that $1/2 < p < 1$. In this case, $n_p = 1$ (see (4.2)) and so for ϕ, F, k as before,

$$\mathcal{E}^p(\phi, F, k) = \mathcal{E}^p(\phi, F, 1).$$

Proposition 5.3. $\mathcal{E}^p(\phi, F, 1)^\perp = \mathcal{I}(\phi, F)$.

---

[7] Recall that functions in O_p have a continuous extension to $\mathbb{D}^-$ and so the notation $g|F = 0$ makes sense.

[8] The characterization of the ideals of functions 'smooth up to the boundary' has been well worked over [15, 17, 18, 19, 25, 31].

Proof. Let $g \in \mathcal{I}(\phi, F)$. Then $g \in \phi H^\infty$ and so g annihilates $\mathcal{E}^2(\phi)$ (being the annihilator of ϕH^2 in H^2). Also, for $e^{i\theta} \in F$,

$$\left\langle \frac{1}{1 - e^{-i\theta} z}, g \right\rangle = \overline{g}(e^{i\theta}) = 0.$$

Recalling the definition of $e^p(\phi, F, 1)$ from (4.4), we see that g annihilates $e^p(\phi, F, 1)$ and hence, by Aleksandrov's approximation (4.5), $\mathcal{E}^p(\phi, F, 1)$. For the other direction, suppose $g \in O_p$ annihilates $\mathcal{E}^p(\phi, F, 1)$. Then g annihilates $\mathcal{E}^2(\phi)$ as well as $(1 - e^{-i\theta} z)^{-1}$ for all $e^{i\theta} \in F$. It follows now that $g \in \mathcal{I}(\phi, F)$. □

This proposition yields the following corollary.

Corollary 5.4. *The following are equivalent.*

1. $\mathcal{E}^p(\phi, F, 1)$ *is weakly closed.*
2. *Condition (5.2) is satisfied.*
3. $\mathcal{E}^p(\phi, F, 1)$ *is not weakly dense.*

Before getting into the proof, let us set some notation. For $f \in H^p$, let $[f]$ denote the linear span of $\{B^n f : n = 0, 1, \cdots\}$, $[f]_{H^p}$ the closure of $[f]$ in the H^p metric, and $[f]_w$ denote the weak-closure of $[f]$. From the definitions of the metric and weak topologies follow the inclusions

$$[f] \subset [f]_{H^p} \subset [f]_w. \tag{5.5}$$

Proof of Corollary 5.4. We will show that (1) ⇔ (2) and (2) ⇔ (3). If $\mathcal{E}^p(\phi, F, 1)$ is weakly closed, it is not weakly dense and so by Proposition 5.3,

$$(0) \neq \mathcal{E}^p(\phi, F, 1)^\perp = \mathcal{I}(\phi, F).$$

Since $\mathcal{I}(\phi, F) \neq (0)$, then (5.2) must be satisfied. So (1) ⇒ (2).

For the other direction, we assume (5.2) is satisfied. We will show that $\mathcal{E}^p(\phi, F, 1)$ is weakly closed by showing it has the Hahn-Banach separation property. Let $f_0 \in H^p \setminus (0)$ satisfy $\langle f_0, g\rangle = 0$ for all $g \in \mathcal{E}^p(\phi, F, 1)^\perp = \mathcal{I}(\phi, F)$. We will show that $f_0 \in \mathcal{E}^p(\phi, F, 1)$.

Since $\mathcal{I}(\phi, F)$ is an ideal, then $z^n \mathcal{I}(\phi, F) \subset \mathcal{I}(\phi, F)$ and, by using the identity

$$\langle B^n f_0, g\rangle = \langle f_0, z^n g\rangle = 0, \quad n = 0, 1, 2, \cdots, \quad g \in \mathcal{I}(\phi, F),$$

we see that

$$\langle f, g\rangle = 0 \text{ for all } g \in \mathcal{I}(\phi, F) \text{ and } f \in [f_0].$$

But since $[f_0]^\perp \neq (0)$ (since $\mathcal{I}(\phi, F) \neq (0)$), then $[f_0]_w \neq H^p$ and hence, by (5.5), $[f_0]_{H^p} \neq H^p$.

It follows now, by Aleksandrov's theorem (Theorem 4.6), that

$$[f_0]_{H^p} = \mathcal{E}^p(\psi, H, 1) = e^p(\psi, H, 1) = \mathcal{E}^2(\psi) \bigvee \left\{ \frac{1}{1 - e^{-i\theta} z} : e^{i\theta} \in H \right\},$$

where ψ is inner and H is a closed subset of $\mathbb{T}$. We assume, as always, that $\sigma(\psi) \cap \mathbb{T} \subset H$. Let $g_1 \in \mathcal{I}(\phi, F)$ so that ϕ_{g_1} (the inner part of g_1) is equal to ϕ

and $g_1^{-1}(\{0\}) \cap \mathbb{T} = F$. This is possible since we are assuming (5.2) and so we can invoke a result in [15] (the ideals are singly generated).

Since $g_1 \perp [f_0]_{H^p}$, then $g_1 \perp \mathcal{E}^2(\psi)$ and so $g_1 \in \psi H^\infty$. It follows now, since $\phi_{g_1} = \phi$, that ψ divides ϕ and so $\mathcal{E}^2(\psi) \subset \mathcal{E}^2(\phi)$. Notice again that $g_1 \perp [f_0]_{H^p}$ and so

$$g_1 \perp \bigvee \left\{ \frac{1}{1 - e^{i\theta} z} : e^{i\theta} \in H \right\}.$$

This means that

$$\left\langle \frac{1}{1 - e^{-i\theta} z}, g_1 \right\rangle = \overline{g_1}(e^{i\theta}) = 0, \quad e^{i\theta} \in H.$$

Since $g^{-1}(\{0\}) \cap \mathbb{T} = F$, then $H \subset F$ and so, again using Aleksandrov's approximation theorem $\mathcal{E}^p(\phi, F, 1) = e^p(\phi, F, 1)$ and $\mathcal{E}^p(\psi, H, 1) = e^p(\psi, H, 1)$,

$$f_0 \in [f_0]_{H^p} = \mathcal{E}^p(\psi, H, 1) \subset \mathcal{E}^p(\phi, F, 1).$$

Thus $\mathcal{E}^p(\phi, F, 1)$ satisfies the Hahn-Banach separation property and hence is weakly closed. Hence (2) $\Rightarrow$ (1).

Finally, from (5.1) and Proposition 5.3, notice that for any ϕ and F, the weak closure of $\mathcal{E}^p(\phi, F, 1)$ is

$$^\perp(\mathcal{E}^p(\phi, F, 1)^\perp) = {}^\perp\mathcal{I}(\phi, F).$$

Thus $\mathcal{E}^p(\phi, F, 1)$ is not weakly dense if and only if (5.2) is satisfied. Hence (2) $\Leftrightarrow$ (3). □

The following is our alternative description of $\mathcal{E}^p(\phi, F, 1)$. The theorem and proof is very similar to a result for weighted Bergman spaces in [4] but, for the sake of completeness, and since there are enough differences, we include it anyway.

Theorem 5.6. *If (5.2) is satisfied, then $\mathcal{E}^p(\phi, F, 1)$ is the space of functions $f \in H^p$ such that*

1. $fg \in H^1$
2. *f/ϕ has a pseudocontinuation to an $\widetilde{f_\phi} \in H^p(\mathbb{D}_e)$ with $\widetilde{f_\phi}(\infty) = 0$,*

where $g \in A^\infty$ with $\phi_g = \phi$ and $g^{-1}(\{0\}) \cap \mathbb{T} = F$.

Proof. Since $\mathcal{I}_g$ (the weak-* closed ideal generated by g) is equal to $\mathcal{I}(\phi, F)$, then, by the equality $\mathcal{E}^p(\phi, F, 1) = {}^\perp\mathcal{I}(\phi, F)$, we need to show that an $f \in H^p$ satisfies the two hypotheses of the theorem if and only if $f \in {}^\perp\mathcal{I}_g$.

Let $\phi\Theta = g$ be the inner-outer factorization of g. If $f \in H^p$ satisfies the two hypotheses of the theorem, then for almost every θ,

$$(f\overline{g})(e^{i\theta}) = \widetilde{f_\phi}(e^{i\theta})\overline{\Theta}(e^{i\theta}).$$

The right-hand side of the above equation is the boundary function for

$$\widetilde{f_\phi}(z)\overline{\Theta}(1/\overline{z})$$

which belongs to $H^p(\mathbb{D}_e)$. Moreover, by the assumption that $fg \in H^1$, this boundary function belongs to L^1 and so, by a classical theorem of Smirnov [11, p. 28],

$(f\overline{g})(e^{i\theta})$ is the boundary function for a function belonging to $H^1(\mathbb{D}_e)$. Hence, by the F. and M. Riesz theorem (3.8),

$$\int_0^{2\pi} (f\overline{g})(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = 0, \quad n = 0, 1, 2, \cdots$$

By our dual pairing between H^p and O_p, and the fact that $fg \in H^1$, we conclude that

$$\langle f, z^n g \rangle = \int_0^{2\pi} (f\overline{g})(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = 0, \quad n = 0, 1, 2, \cdots [9]$$

This shows that f annihilates the weak-* closed S-invariant subspace of O_p containing g. One can prove (see [4, Thm. 3.2]) that any weak-* closed S-invariant subspace of O_p is an ideal and so $f \in {}^{\perp}\mathcal{I}_g$ (the weak-* closed ideal generated by g).

Conversely, suppose $f \in {}^{\perp}\mathcal{I}_g$, or equivalently $f \in \mathcal{E}^p(\phi, F, 1)$. By the definition of $\mathcal{E}^p(\phi, F, 1)$, f satisfies the second (pseudocontinuation) condition of the theorem and so we just need to show that fg belongs to H^1. To this end note that for any integer $n \geqslant 1$,

$$\langle f, g \rangle = n! \int f \overline{(z^{n+1} g)^{(n+1)}} (1 - |z|^2)^n \frac{dA}{\pi},$$

where dA is area measure on the disk $\mathbb{D}$. For ease in notation, let

$$g_n := \overline{(z^{n+1} g)^{(n+1)}}.$$

We also assume that $n > 1/p$ so that $f g_n (1 - |z|^2)^n$ is bounded on $\mathbb{D}$. This is possible since g_n is bounded on $\mathbb{D}$, since we are assuming that $g \in A^{\infty}$, and all H^p functions f satisfy the growth estimate $|f(z)| \leqslant C_f (1 - |z|)^{-1/p}$ (recall this from § 2).

With this fixed n, let $\lambda \in \mathbb{D}$ and note, using the definition of $\mathcal{E}^p(\phi, F, 1)$, that

$$\frac{f - f(\lambda)}{z - \lambda} \in \mathcal{E}^p(\phi, F, 1)$$

and so, since g annihilates $\mathcal{E}^p(\phi, F, 1)$,

$$0 = \langle \frac{f - f(\lambda)}{z - \lambda}, g \rangle = n! \int \frac{f - f(\lambda)}{z - \lambda} g_n (1 - |z|^2)^n \frac{dA}{\pi}. \tag{5.7}$$

Let

$$G(\lambda) := n! \int \frac{g_n (1 - |z|^2)^n}{z - \lambda} \frac{dA}{\pi}, \qquad H(\lambda) := n! \int \frac{f g_n (1 - |z|^2)^n}{z - \lambda} \frac{dA}{\pi}.$$

Elementary facts about Cauchy transforms of bounded functions on the plane [33, p. 40] show that G and H are continuous functions on $\mathbb{C}$ and satisfy the Lipschitz-type condition

$$|G(\lambda_1) - G(\lambda_2)| \leqslant C_G |\lambda_1 - \lambda_2| \log \frac{1}{|\lambda_1 - \lambda_2|}. \tag{5.8}$$

---

[9] Note, by the dominated convergence theorem and the fact that $fg \in H^1$ and so $f_r g_r \to fg$ as $r \to 1$, that $f_r \overline{g_r} - f\overline{g} = (f_r g_r - fg)\overline{g_r}/g_r + fg(\overline{g_r}/g_r - \overline{g}/g)$ converges to zero as $r \to 1$.

Furthermore, by (5.7),

$$f(\lambda)G(\lambda) = H(\lambda), \quad \lambda \in \mathbb{D}. \tag{5.9}$$

A computation with power series shows that for $0 < r < 1$

$$G(e^{i\theta}/r) = re^{-i\theta}\overline{g}(re^{i\theta})$$

and so for $r > 1/2$,

$$|(fg)(re^{i\theta})| \leqslant C\left[|f(re^{i\theta})G(re^{i\theta})| + |f(re^{i\theta})|\left|G(re^{i\theta}) - G(e^{i\theta}/r)\right|\right]. \tag{5.10}$$

By (5.9), the first term on the right-hand side of the above is equal to $|H(re^{i\theta})|$ which is uniformly bounded in r and θ. For the second term, notice from (5.8) that

$$\left|G(re^{i\theta}) - G(e^{i\theta}/r)\right| \leqslant C_G(1-r)\log\frac{1}{1-r}$$

and so, by (5.10),

$$|(fg)(re^{i\theta})| \leqslant C_1 + C_2|f(re^{i\theta})|(1-r)\log\frac{1}{1-r}. \tag{5.11}$$

Since $H^p \subset B^p$ then, for any $0 < r < 1$,

$$\begin{aligned}
\|f\|_{B^p} &\geqslant \int_r^1 (1-s)^{1/p-2}\int_0^{2\pi}|f(se^{i\theta})|\frac{d\theta}{2\pi}ds \\
&\geqslant \int_r^1 (1-s)^{1/p-2}\int_0^{2\pi}|f(re^{i\theta})|\frac{d\theta}{2\pi}ds \\
&\geqslant (1/p-1)^{-1}(1-r)^{1/p-1}\int_0^{2\pi}|f(re^{i\theta})|\frac{d\theta}{2\pi}
\end{aligned}$$

and so

$$\int_0^{2\pi}|f(re^{i\theta})|\frac{d\theta}{2\pi} \leqslant C_p(1-r)^{1-1/p}.$$

Combining this with (5.11) along with the fact that $1/2 < p < 1$ (and so

$$(1-r)^{2-1/p}\log\frac{1}{1-r}$$

is bounded in r), we conclude that

$$\int_0^{2\pi}|(fg)(re^{i\theta})|\frac{d\theta}{2\pi}$$

is uniformly bounded in r. Hence $fg \in H^1$. □

For other p, not in $(1/2, 1)$, the above theorem is still true, though the proof is more technical since the resulting ideal $\mathcal{E}^p(\phi, F, k)^{\perp}$ will involve the zeros of the *derivatives* of g on the circle. The proof presented here needs to be changed slightly and for this we refer the reader to [4] where there is a similar result for the invariant subspaces of B^p. Notice that since every weakly closed invariant subspace is also closed in the metric of H^p, we have shown the following corollary.

Corollary 5.12. *For $0 < p < 1$ and ϕ, F, k satisfying the condition*

$$\int_0^{2\pi} \log dist(e^{i\theta}, \sigma(\phi) \cup F) \frac{d\theta}{2\pi} > -\infty, \tag{5.13}$$

$\mathcal{E}^p(\phi, F, k)$ is a non-trivial weakly closed invariant subspace of H^p. Conversely, every non-trivial weakly closed invariant subspace of H^p takes the form $\mathcal{E}^p(\phi, F, k)$ for some ϕ, F, k satisfying (5.13).

6. The Bergman spaces L^p_a, $0 < p < 1$

We end with some remarks about the invariant subspaces of the Bergman spaces[10] L^p_a $(0 < p < \infty)$ of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_p := \left(\int_{\mathbb{D}} |f|^p dA \right)^{1/p} < \infty.$$

The quantity $\|f - g\|_p$ defines a norm when $1 \leqslant p < \infty$ while $\|f - g\|_p^p$ defines a translation invariant metric when $0 < p < 1$. In either case, one can use the pointwise estimate

$$|f(z)| \leqslant \pi^{-1/p} \|f\|_p (1 - |z|)^{-2/p}, \quad z \in \mathbb{D}$$

to show that L^p_a is an F-space [9, p. 51]. For $f \in L^p_a$, routine integral estimates show that $Bf \in L^p_a$. Using the above pointwise estimate, one proves that the graph of B is closed and so, by the closed graph theorem (which is valid in an F-space [9, p. 57]), B is continuous on L^p_a.

When $1 \leqslant p < \infty$, one can make heavy use of duality to show that if $\mathcal{E}$ is a non-trivial invariant subspace of L^p_a, then every $f \in \mathcal{E}$ has a pseudocontinuation of bounded type. Moreover, when $1 \leqslant p < 2$, there is a complete description of $\mathcal{E}$ [3, 4, 22]. We pause for a moment to remark that in order for $f \in \mathcal{E}$ to have a pseudocontinuation, it must first have non-tangential limits almost everywhere on $\mathbb{T}$. This is automatic for H^p but not for L^p_a. There are indeed examples of functions in L^p_a (or in any of the weighted Bergman spaces such as B^p) which do not even have radial limits almost everywhere [11, p. 86]. Such poorly behaved functions do not belong to non-trivial invariant subspaces of L^p_a.

When $0 < p < 1$, can we say anything about the invariant subspaces of L^p_a? In this case, L^p_a is not locally convex and so, as in H^p $(0 < p < 1)$, duality is of little use. J. Shapiro [28, 29] showed, assuming $0 < p < 1$ as we will do from now on, that $L^p_a \subset B^{p/2}$ and this containment is continuous. Moreover, L^p_a and $B^{p/2}$ have the same set of continuous linear functionals (via the same 'Cauchy duality' as in (2.1)), namely $O_{p/2}$. As in the H^p case, there is a corresponding weak topology on L^p_a induced by $O_{p/2}$ and if A is a linear manifold in L^p_a, then

$$\operatorname{clos}_{(L^p_a, wk)} A = \left(\operatorname{clos}_{B^{p/2}} A\right) \cap L^p_a. \tag{6.1}$$

[10]Two nice references about Bergman spaces are the following books [10, 14].

If $\mathcal{E}$ is a norm-closed invariant subspace of $B^{p/2}$, then $\mathcal{E}^\perp$ is an S-invariant subspace of $O_{p/2}$ and hence a weak-* closed ideal which, as before (see also [4, Thm. 3.2]), is of the form $\mathcal{I}_g$ (the weak-* closed ideal generated by g) for some $g \in A^\infty$. By the Hahn-Banach theorem, which is applicable here since $B^{p/2}$ is a Banach space, we have $\mathcal{E} = {}^\perp\mathcal{I}_g$. One can prove [4] that $f \in B^{p/2}$ belongs to ${}^\perp\mathcal{I}_g$ if and only if (i) $fg \in H^1$; (ii) f/ϕ_g (where ϕ_g is the inner part of g) has a pseudocontinuation $\widetilde{f_{\phi_g}} \in N^+(\mathbb{D}_e)$ [11] which vanishes at infinity. Combining this with (6.1) we have the following result.

Theorem 6.2. *Let $0 < p < 1$ and $\mathcal{E}$ be a non-trivial weakly closed invariant subspace of L^p_a. Then there is a $g \in A^\infty$ such that $\mathcal{E}$ is the set of $f \in L^p_a$ such that*

1. *$fg \in H^1$.*
2. *f/ϕ_g has a pseudocontinuation $\widetilde{f_{\phi_g}} \in N^+(\mathbb{D}_e)$ which vanishes at infinity.*

Certainly if $\mathcal{E}$ is a weakly closed invariant subspace of L^p_a, then $\mathcal{E}$ is closed in the metric of L^p_a. Is every closed invariant subspace weakly closed? In H^p, this is not the case (see Corollary 5.12). Though we do not have a proof, we conjecture that every closed invariant subspace of L^p_a is indeed weakly closed. In H^p, the space $\mathcal{E} = \bigvee\{(1 - e^{-i\theta}z)^{-1} : 0 \leqslant \theta < 2\pi\}$, where $\bigvee$ is the closed linear span in the metric topology of H^p, is a proper closed invariant subspace that is weakly dense. This same example, with the linear span in H^p replaced by the closed linear span in the metric topology of L^p_a, is certainly weakly dense. However, since $L^1_a \subset L^p_a$ with continuous inclusion, and since the linear span of $\{(1-e^{-i\theta}z)^{-1} : 0 \leqslant \theta < 2\pi\}$ is norm dense in L^1_a, we see that $\mathcal{E}$ is dense in the metric topology of L^p_a. We end with the following open question.

Question 6.3. For $0 < p < 1$, what are the closed (in the metric topology) invariant subspaces of L^p_a?

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[11] $N^+(\mathbb{D}_e)$ is the Smirnov class (see [11, p. 25] for a definition).

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