



Prolongations and Cyclic Vectors

William T. Ross and Harold S. Shapiro

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Abstract. For functions belonging to invariant subspaces of the backward shift operator $Bf := (f - f(0))/z$ on spaces of analytic functions on the unit disk \mathbb{D} , we explore, in a systematic way, the continuation properties of these functions.

Keywords. Backward shift operator, analytic continuation, pseudocontinuation.

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1. Introduction

In this paper, we continue, as well as refine, a discussion begun in [24] concerning the continuation properties of the non-cyclic vectors for the backward shift operator

$$Bf := \frac{f - f(0)}{z}$$

on a general Banach space X contained in $\text{Hol}(\mathbb{D})$, the analytic functions on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here we say a function $f \in X$ is *cyclic* if

$$[f] := \bigvee \{B^n f : n \in \mathbb{N}_0\} = X,$$

where \bigvee denotes the closed linear span and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. As observed in several papers including [1, 2, 9, 10, 16, 17, 21, 23, 30], non-cyclic vectors have ‘continuations’ of one sort or another to functions that are meromorphic on the extended exterior disk $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$. Here $\mathbb{D}^- := \{|z| \leq 1\}$ is the closure of the disk and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The continuations to be examined in this paper take the form

$$\tilde{f}_L(w) := L\left(\frac{f}{z-w}\right) / L\left(\frac{1}{z-w}\right),$$

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where L is a bounded linear functional on X for which $L[f] = 0$. We will call such a continuation \tilde{f}_L , which is a meromorphic function on \mathbb{D}_e , an L -prolongation of f . A more detailed discussion of this function follows in Section 2.

The idea to use L -prolongations as a tool to study cyclic vectors for the backward shift operator on various spaces can be traced back (apart from differences in the notations and formal apparatus) implicitly to ideas of Beurling and explicitly to the dissertation of his student Nyman. In Nyman's studies [21, §4], in different terminology, the problem of characterizing the non-cyclic vectors for the backward shift operator on a topological vector space of holomorphic functions in \mathbb{D} , namely the space ℓ_A^∞ of holomorphic functions with bounded Taylor coefficients (the topology being that induced by the weak* topology of the space $\ell^\infty(\mathbb{Z}_+)$ of bounded sequences on the non-negative integers). Actually, Nyman does not give details of this study, but observes that it proceeds analogously to that which he carried out in [21, §2] for the space of bounded measurable functions on the positive real axis assuming that one has an element of $L^\infty(\mathbb{R}_+)$ that is non-cyclic for the backward shift (which in this setting is translation), and hence a non-trivial element k of $L^1(\mathbb{R}_+)$ annihilating its backward shifts, Nyman proceeds to construct a meromorphic function in the right half-plane (in our language, an L -prolongation based on the annihilating functional represented by k) and shows it is an analytic continuation of the Laplace transform of the given non-cyclic vector across an open subset of the imaginary axis whose complement has linear measure zero. For the disk, Nyman's half-line result translates to: every non-cyclic $f \in \ell_A^\infty$ has an analytic continuation across a subset of the circle $\mathbb{T} := \partial\mathbb{D}$, of full measure, to a meromorphic function on \mathbb{D}_e . In fact, for any non-trivial annihilating functional L , the formula for this continuation is \tilde{f}_L .

What is remarkable in Nyman's Theorem is that the L -prolongation is a true analytic continuation across a subset of \mathbb{T} which is the complement of a rather 'thin' closed set, namely the zero-set of some non-trivial absolutely convergent Taylor series. Later authors have extended this result to weighted sequence spaces and other Banach spaces of sequences [9, 16, 17, 30] (which ultimately use an idea dating back to Carleman [5]). These investigations have two main things in common. For one, they use Banach algebra techniques and thus only work for backward shifts on topological spaces whose duals are Banach algebras with certain favorable properties. Secondly, they lead to a true analytic continuation of the non-cyclic vector (encoded as a Taylor series) across an open subset of \mathbb{T} having full measure (and so, *a fortiori*, to the L -prolongation being independent of the choice of the annihilating functional). The general theorem along these lines is that f and \tilde{f}_L are analytic continuations of each other across any arc I of the circle \mathbb{T} for which the function $w \mapsto L((z - w)^{-1})$, $w \in \mathbb{D}_e$, is bounded away from zero near I . This analytic continuation result was used to characterize the closed ideals of various Banach algebras of analytic functions on \mathbb{D} such as disk algebra type spaces, Lipschitz classes and certain Besov classes.

Our set-up is more general insofar as the non-cyclic vectors on some Banach spaces really may not possess any analytic continuation across any point of the circle. Moreover, the L -prolongations may not be unique; and even when they are unique this may be because non-cyclic vectors in the space in question necessarily are pseudocontinuable (see definition below), but need not have an analytic continuation across any point of the circle. An example along these lines arises when our Banach space X is the Hardy space

$$H^2 := \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\}.$$

A result of Douglas, Shapiro and Shields [10] states that to each non-cyclic vector $f \in H^2$, there is a unique meromorphic function \tilde{f} of bounded type (quotient of two bounded analytic functions) on \mathbb{D}_e such that the non-tangential limits of f (from \mathbb{D}) and \tilde{f} (from \mathbb{D}_e) agree almost everywhere with respect to Lebesgue measure on the unit circle $\mathbb{T} := \partial\mathbb{D}$. In language developed in [25], we say that \tilde{f} is a *pseudocontinuation* of f . In fact, the existence of such a pseudocontinuation \tilde{f} of bounded type completely characterizes the non-cyclic vectors of H^2 . Moreover, \tilde{f} turns out to be an L -prolongation \tilde{f}_L of f , where L is any non-zero continuous linear functional on H^2 for which $L[f] = 0$. That \tilde{f}_L is independent of L follows from the Lusin-Privalov Uniqueness Theorem [15, p. 62]: if g is meromorphic on \mathbb{D} and has non-tangential limits equal to zero on a subset of \mathbb{T} of positive measure, then $g \equiv 0$.

This concept of a pseudocontinuation is really more general than ordinary analytic continuation in that there are non-cyclic vectors, which have a pseudocontinuation by the above, but which do not have an analytic continuation across any point of \mathbb{T} . For example, if ϕ is an inner function whose zeros accumulate at every point of \mathbb{T} , then ϕ certainly does not have an analytic continuation across any point of \mathbb{T} . However, ϕ does have a pseudocontinuation given by

$$(1.1) \quad \tilde{\varphi}(z) := \frac{1}{\varphi(1/\bar{z})}.$$

The same pseudocontinuation results hold for Hardy spaces

$$H^p := \left\{ f \in \text{Hol}(\mathbb{D}) : \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty \right\}, \quad p > 1.$$

For the Bergman space

$$L_a^2 := \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty \right\},$$

where dA is planar area measure, a similar result is true: each non-cyclic $f \in L_a^2$ is of bounded type on \mathbb{D} (Bergman space functions are, in general, not of bounded type [11, p. 86]) and moreover, there is a unique \tilde{f} of bounded type on \mathbb{D}_e which

is a pseudocontinuation of f [23]. However, unlike the Hardy space case, this is not a complete characterization of the non-cyclic vectors. For example, there are inner functions φ , that are cyclic for B on L_a^2 , and have pseudocontinuations given by (1.1) (see [6, p. 108]). As is the case for the Hardy space, the pseudocontinuation \tilde{f} of f is an L -prolongation \tilde{f}_L (and is independent of the annihilating L). The same result is true for the more general weighted Bergman spaces

$$L_a^p(w) := \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^p w(|z|) dA < \infty \right\}, \quad p > 1,$$

for suitable weights w , for example $w(r) = (1-r)^s$, $s > -1$, [1, 2].

When $X = \mathcal{D}$, the classical Dirichlet space

$$\mathcal{D} := \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} (n+1) |a_n|^2 = \int_{\mathbb{D}} |(zf)'|^2 \frac{dA}{\pi} < \infty \right\},$$

or the closely related spaces

$$\mathcal{D}_p := \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |(zf)'|^p dA < \infty \right\}, \quad p > 1,$$

$$D_\alpha := \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty \right\}, \quad \alpha > 0,$$

the non-cyclic vectors f need not have pseudocontinuations across any arc of \mathbb{T} [1, Thm. 6.2]. Nevertheless, for non-cyclic f , the L -prolongation \tilde{f}_L can be thought of, in some sense, as a 'continuation' of f . For example, in \mathcal{D}_p ,

$$L\left(\frac{1}{z-w}\right) [\tilde{f}_L(w) - f(e^{i\theta})] \rightarrow 0$$

for almost every $e^{i\theta}$ as $w \rightarrow e^{i\theta}$ non-tangentially [1, Lemma 7.4]. So, for example, if the function $w \mapsto L((z-w)^{-1})$ has non-tangential limits almost everywhere on \mathbb{T} , then \tilde{f}_L is a pseudocontinuation of f .

In this paper, we demonstrate both the ubiquity and the utility of these L -prolongations. In the above discussion, we have already seen the appearance of L -prolongations as 'continuations' of non-cyclic vectors. L -prolongations also appear in the analysis of $\sigma(B|\mathcal{M})$, the spectrum of B restricted to one of its invariant subspaces \mathcal{M} . This has been observed in [1] and we review these results in Section 2 of this paper. In Section 3 we will show, for a wide class of Banach spaces X including H^p , $L_a^p(w)$, D_α and \mathcal{D}_p , that the L -prolongation of f is compatible with analytic continuation in the sense that if f has an analytic continuation (also denoted by f) to an open neighborhood U_ζ of $\zeta \in \mathbb{T}$, then f is equal to \tilde{f}_L on $U_\zeta \cap \mathbb{D}_e$. For these spaces X , the proof presented here will be simpler than the one presented in [24, p. 101]. From the other direction, we shall

examine the question. If \tilde{f}_L has an analytic continuation (also denoted by \tilde{f}_L) to an open neighborhood U_ζ of $\zeta \in \mathbb{T}$, then is \tilde{f}_L equal to f on $U_\zeta \cap \mathbb{D}$? These results yield interesting corollaries about the nature of non-cyclic vectors. For example, if $f \in X$ has an isolated winding point on the circle, then f is cyclic. This is a well-known fact in the H^2 setting [10]. These results also make connections to certain convolution equations [4, 12] and to determining the orbit of an $f \in X$ under both the forward and backward shift operators, that is

$$\bigvee \{z^n f, B^n f : n \in \mathbb{N}_0\}.$$

Finally, in Section 4 and Section 5, we will show how L -prolongations relate to several questions, originally asked in [26], about overconvergence of rational functions. We will also show the relationship between L -prolongations and approximate spectral synthesis.

The paper includes some known results to enhance its readability and coherence, but is by no means purely expository. The main novelties are as follows: in Section 3 (Theorem 3.1) a new proof is given for one of the main results of [24, p. 101]. Of course, it is less general than its forerunner but suffices for most of the interesting applications, and its proof (based on the idea of L -prolongations) is much simpler. Also, Section 5 contributes to the study of approximation of analytic functions by rational ones in various norms, in the context of ‘tableaux’ prescribing the possible poles of the n -th approximant, as in earlier works by Walsh, Tumarkin, V. Katsnelson, N. K. Nikol’skii, and others. The new idea, based on the notion of a ‘uniformly deficient tableau’ is embodied in Theorem 5.16.

2. Preliminaries

Let X be a Banach space of analytic functions on \mathbb{D} which satisfies the following properties:

$$(2.1) \quad X \hookrightarrow \text{Hol}(\mathbb{D}),$$

$$(2.2) \quad M_z X \subset X, \quad M_z f := z f,$$

$$(2.3) \quad 1 \in X,$$

$$(2.4) \quad \bigvee \{1, z, z^2, \dots\} = X,$$

$$(2.5) \quad \frac{f - f(\lambda)}{z - \lambda} \in X \quad \text{whenever } f \in X \text{ and } \lambda \in \mathbb{D},$$

$$(2.6) \quad \sigma(M_z) = \mathbb{D}^-,$$

$$(2.7) \quad \|M_{(z-\lambda)^{-1}}\| \rightarrow 0 \quad \text{as } |\lambda| \rightarrow \infty.$$

Remark 2.8. 1. In (2.1), the inclusion map from X (with the norm topology) to $\text{Hol}(\mathbb{D})$ (with the topology of uniform convergence on compact sets) is both injective and continuous. In particular, for each compact $K \subset \mathbb{D}$,

$$(2.9) \quad \sup\{|f(z)| : z \in K\} \leq C_K \|f\| \quad \text{for all } f \in X.$$

2. Note that

$$\bigvee \{1, z, z^2, \dots\} = \bigvee \left\{ \frac{1}{z - \lambda} : \lambda \in \mathbb{D}_e \right\} = X$$

[1, Prop. 2.2].

3. For an operator T on a Banach space V , $\sigma(T)$, the spectrum of T , denotes the set of complex numbers λ such that $(\lambda I - T)$ is not invertible.

4. For $|\lambda| > 1$, the operator $M_{(z-\lambda)^{-1}} f := f/(z - \lambda)$ is well-defined by (2.6).

Examples of such X include the spaces H^p , $L^p_a(w)$ for suitable w , \mathcal{D}_p , D_α , $-\infty < \alpha < \infty$, as well as spaces like

$$\ell^p_A(w) := \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^p w_n < \infty \right\}, \quad p \geq 1,$$

for suitable weight sequences $w = (w_n)_{n \geq 0}$, [27].

If X' is the dual space of X and $L \in X'$, the function

$$\lambda \mapsto L\left(\frac{1}{z - \lambda}\right)$$

is analytic on \mathbb{D}_e and vanishes at infinity (see (2.7)). It follows from the Hahn-Banach Theorem and basic complex function theory that

$$(2.10) \quad X = \bigvee \left\{ \frac{1}{z - \lambda} : \lambda \in E \right\}$$

whenever $E \subset \mathbb{D}_e$ has a cluster point in \mathbb{D}_e . Furthermore, for fixed $|\lambda| > 1$,

$$(2.11) \quad X = \bigvee \left\{ \frac{1}{(z - \lambda)^n} : n \in \mathbb{N} \right\}.$$

We will say the dual pair (X, X') is an ℓ^2 dual pair if X' can be identified with a Banach space of analytic functions on \mathbb{D} such that the dual pairing is given by

$$(2.12) \quad (f, g) := \sum_{n=0}^{\infty} a_n \bar{b}_n,$$

where the above series converges absolutely, $(a_n)_{n \geq 0}$ are the Taylor series coefficients of $f \in X$ and $(b_n)_{n \geq 0}$ are those of $g \in X'$. In this case, X' also satisfies the conditions (2.1) through (2.7) [1, Prop. 5.2]. Spaces like $\ell^p_A(w)$, for $p > 1$ and $w = (w_n)_{n \geq 0}$, satisfy this condition since $(\ell^p_A)'$ can be identified with $\ell^q_A(w')$,

where $q = p/(p - 1)$ and $w' = (1/w_n)_{n \geq 0}$. For example, (L_a^2, \mathbb{D}) is an ℓ^2 dual pair. Note that for $g \in X', \lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$,

$$(2.13) \quad \left(\frac{n!z^n}{(1 - \bar{\lambda}z)^{n+1}}, g \right) = \overline{g^{(n)}(\lambda)}.$$

We will not always require (X, X') to be an ℓ^2 dual pair, but will require it from time to time.

From the hypothesis on X , it follows that the operators $M_z f = zf$ and

$$Bf = \frac{f - f(0)}{z}$$

are continuous on X . In fact, if (X, X') is an ℓ^2 dual pair, then B on X is the adjoint of M_z on X' (and vice versa). We will denote the collection of (closed) B -invariant subspaces by $\text{Lat}(B, X)$. In this general setting, one can prove (see [1, §2]) that

$$\sigma(B) = \mathbb{D}^-,$$

and that for $w \in \mathbb{D}$,

$$(2.14) \quad (I - wB)^{-1}f = \frac{zf - wf(w)}{z - w},$$

$$(2.15) \quad (I - wB)^{-1}Bf = \frac{f - f(w)}{z - w}.$$

From the spectral radius formula, it follows that

$$(2.16) \quad \sigma(B|\mathcal{M}) \subset \mathbb{D}^- \quad \text{for all } \mathcal{M} \in \text{Lat}(B, X).$$

Since,

$$B \frac{1}{1 - az} = a \frac{1}{1 - az},$$

each $a \in \mathbb{D}$ is an eigenvalue of B (with geometric multiplicity one) with corresponding eigenfunction $(1 - az)^{-1}$. Moreover,

$$(2.17) \quad \sigma_{ap}(B|\mathcal{M}) \cap \mathbb{D} = \sigma_p(B|\mathcal{M}) \cap \mathbb{D} = \left\{ a \in \mathbb{D} : \frac{1}{1 - az} \in \mathcal{M} \right\}.$$

Here σ_{ap} is the approximate point spectrum and σ_p is the point spectrum (the set of eigenvalues). Furthermore, by (2.10), the set in (2.17) is either a countable subset of \mathbb{D} with no cluster points in \mathbb{D} , or all of \mathbb{D} . Thus $\sigma_p(B|\mathcal{M}) \cap \mathbb{D} = \mathbb{D}$ if and only if $\mathcal{M} = X$. Under a mild regularity condition on X ,

$$(2.18) \quad \sigma_{ap}(B|\mathcal{M}) \cap \mathbb{T} = \sigma(B|\mathcal{M}) \cap \mathbb{T}$$

and this set is the complement (in the unit circle) of the set of points $\zeta \in \mathbb{T}$ such that every $f \in \mathcal{M}$ extends to be analytic in a neighborhood of $1/\zeta$. Furthermore, since $\partial\sigma(B|\mathcal{M}) \subset \sigma_{ap}(B|\mathcal{M})$, one can prove the following dichotomy: either

$$\sigma(B|\mathcal{M}) \cap \mathbb{D} = \sigma_{ap}(B|\mathcal{M}) \cap \mathbb{D}$$

and is a countable subset of \mathbb{D} with no cluster points in \mathbb{D} , or $\sigma(B|\mathcal{M}) = \mathbb{D}^-$.

For a set $A \subset X$, let A^\perp , the annihilator of A , denote the set

$$A^\perp := \{L \in X' : L|_A = 0\}.$$

For a non-cyclic vector $f \in X$ and $L \in [f]^\perp \setminus \{0\}$, define the L -prolongation of f to be the meromorphic function on \mathbb{D}_e defined by

$$(2.19) \quad \tilde{f}_L(w) := L\left(\frac{f}{z-w}\right) / L\left(\frac{1}{z-w}\right).$$

Observe from (2.10) that the denominator of the above expression is not identically zero. One can show, when $T = B|[f]$ and $\sigma(T) \cap \mathbb{D}$ is discrete, that for each non-trivial annihilating L and $|w| > 1$ with $1/w \notin \sigma(T)$,

$$(I - wT)^{-1}f = \frac{zf - \tilde{f}_L(w)}{z-w}.$$

Compare this formula to the one in (2.14). Furthermore, in this case, the meromorphic function \tilde{f}_L is independent of L . In fact, if \mathcal{M} is any B -invariant subspace, $T := B|\mathcal{M}$ and $|w| > 1$ with $1/w \notin \sigma_{ap}(T)$, then $(I - wT)^{-1}$ exists if and only if for each $f \in \mathcal{M}$, $\tilde{f}_L(w)$ is independent of the $L \in \mathcal{M}^\perp$ with $L((z-w)^{-1}) \neq 0$ [1, Prop. 2.6].

When X is either H^p or $L_a^p(w)$, \tilde{f}_L is a function of bounded type and is a pseudocontinuation of f . Thus, by the Lusin-Privalov Uniqueness Theorem, \tilde{f}_L is independent of $L \in [f]^\perp \setminus \{0\}$. For other spaces X such as the Dirichlet-type spaces \mathcal{D}_p or D_α , $\alpha > 0$, f_L need not be of bounded type on \mathbb{D}_e (quotient of two bounded analytic functions) nor be a pseudocontinuation of f . Indeed, it might depend on L [24, p. 122]. For instance, there is a non-cyclic $f \in \mathcal{D}$ and $L_1, L_2 \in [f]^\perp \setminus \{0\}$ such that f_{L_1} is a pseudocontinuation of f while f_{L_2} is not.

The L -prolongation \tilde{f}_L can also be thought of as a ‘continuation’ via formal multiplication of series as in [24, Chap. 8]. In cases where (X, X') is an ℓ^2 dual pair, then

$$L\left(\frac{1}{z-w}\right) f(w) = L\left(\frac{f}{z-w}\right),$$

where we understand the above equation (which is not technically defined since $L((f/(z-w)))$ and $L(1/(z-w))$ are defined on \mathbb{D}_e while f is defined on \mathbb{D}) as a formal multiplication of Laurent series. That is to say, if

$$\begin{aligned} f &= \sum_{n=0}^{\infty} a_n w^n, & |w| < 1, \\ L\left(\frac{f}{z-w}\right) &= \sum_{n=0}^{\infty} \frac{A_n}{w^n}, & |w| > 1, \\ L\left(\frac{1}{z-w}\right) &= \sum_{n=0}^{\infty} \frac{B_n}{w^n}, & |w| > 1, \end{aligned}$$

then a formal computation with Laurent series — gathering up the terms that are associated with each w^n and $1/w^n$ — along with the fact that

$$0 = L(B^n f) = \sum_{k=0}^{\infty} a_{N+k} B_k, \quad N \in \mathbb{N}_0,$$

yields

$$\left(B_0 + \frac{B_1}{w} + \cdots \right) (a_0 + a_1 w + \cdots) = A_0 + \frac{A_1}{w} + \cdots.$$

3. Compatibility with analytic continuation

As mentioned in the introduction, in certain cases (namely when the dual space of X can be identified with a Banach algebra of analytic functions on \mathbb{D} which are continuous on \mathbb{D}^-), the functions f and \tilde{f}_L are analytic continuations of each other across any arc of I of the circle where the function $w \mapsto L((z-w)^{-1})$, $w \in \mathbb{D}_e$, is bounded away from zero near I . In this section, we will prove two results about the compatibility of L -prolongations with ordinary analytic continuation. The first is the following.

Theorem 3.1. *Let X satisfy (2.1) through (2.7) along with the following additional condition: if $f \in X$ extends analytically to an open neighborhood of a point $\zeta \in \mathbb{T}$, then*

$$(3.2) \quad \frac{f - f(w)}{z - w} \rightarrow \frac{f - f(\zeta)}{z - \zeta} \quad \text{in the norm of } X \text{ as } w \rightarrow \zeta.$$

Suppose that $f \in X$ extends analytically, to a function also denoted by f , on an open neighborhood U_ζ of a boundary point $\zeta \in \mathbb{T}$. Then any L -prolongation \tilde{f}_L agrees with f on $U_\zeta \cap \mathbb{D}_e$.

Proof. Let $T = B|[f]$. Since, by (2.16), $\sigma(T) \subset \mathbb{D}^-$, then $(I - wT)^{-1}$ exists for all $w \in \mathbb{D}$. Moreover, by (2.15),

$$(I - wT)^{-1} T f = \frac{f - f(w)}{z - w}.$$

So for any $L \in [f]^\perp \setminus \{0\}$,

$$L \left(\frac{f - f(w)}{z - w} \right) = 0 \quad \text{for all } w \in \mathbb{D}.$$

By (2.6),

$$\frac{f}{z - w}, \frac{1}{z - w} \in X \quad \text{for all } w \in U_\zeta \cap \mathbb{D}_e$$

and moreover, since f is analytic on U_ζ , we can use our hypothesis in (3.2) — as well as Morera's Theorem — to observe that

$$w \mapsto \frac{f - f(w)}{z - w}$$

is an analytic X -valued function on U_ζ . Thus,

$$H(w) := L\left(\frac{f - f(w)}{z - w}\right)$$

is analytic on U_ζ . But since H is zero on $U_\zeta \cap \mathbb{D}$, it is zero on all of U_ζ . It follows now that

$$L\left(\frac{f}{z - w}\right) = f(w)L\left(\frac{1}{z - w}\right) \quad \text{for all } w \in U_\zeta \cap \mathbb{D}_e$$

and so $\tilde{f}_L = f$ on $U_\zeta \cap \mathbb{D}_e$. ■

The hypothesis in (3.2) seems a bit mysterious. Nevertheless, spaces X with a norm satisfying

$$(3.3) \quad \|f\|^p \asymp \sum_{k=0}^K \int_{\mathbb{D}} |f^{(k)}|^p w(|z|) dA$$

satisfy this hypothesis. Indeed, if $f \in X$ continues analytically to $B(\zeta, 2r) := \{w : |w - \zeta| < 2r\}$ and $C = \{w : |w - \zeta| = 3r/2\}$, then for $a, b, z \in B(\zeta, r)$,

$$f(z; a) := \frac{f(z) - f(a)}{z - a} = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t - z)(t - a)} dt$$

and each $k = 0, 1, \dots, K$,

$$\frac{d^k}{dz^k} [f(z; a) - f(z; b)] = \frac{k!}{2\pi i} (a - b) \int_C \frac{f(t)}{(t - z)^{k+1}(t - a)(t - b)} dt.$$

Thus

$$\sup_{z \in B(\zeta, r)} \left| \frac{d^k}{dz^k} [f(z; a) - f(z; b)] \right| \leq C(k, r)$$

and is independent of the points a and b . Now estimate $\|f(z; a) - f(z; b)\|$ by computing the integrals in (3.3) separately over $\mathbb{D} \cap B(\zeta, r)$ and $\mathbb{D} \setminus B(\zeta, r)$ and use the Dominated Convergence Theorem.

Certainly all the Bergman-type spaces $L_a^p(w)$ or even \mathcal{D}_p satisfy the extra condition of (3.2). The Dirichlet-type spaces D_α , $-\infty < \alpha < \infty$, also satisfy the extra hypothesis of Theorem 3.1 since

$$(3.4) \quad \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 \asymp \int_{\mathbb{D}} |f|^2 (1 - |z|^2)^{-1-\alpha} dA, \quad \alpha < 0,$$

[32, Lemma 2] and for any $\alpha \in \mathbb{R}$, $f \in \mathcal{D}_\alpha$ if and only if $f' \in \mathcal{D}_{\alpha-2}$. For other spaces such as $\ell_A^p(w)$, the condition in (3.2) seems difficult to verify, although for a wide class of weight sequences $w = (w_n)_{n \geq 0}$, there is an alternate but more involved proof of Theorem 3.1 [24, p. 101].

Since L -prolongations must be single-valued, the following useful corollary follows.

Corollary 3.5. *Any $f \in X$ which has an isolated winding point on the circle must be cyclic.*

For example, the function $\log(1-z)$, whenever this belongs to X , is cyclic. Other functions such as

$$\exp\left(\frac{1}{z-2}\right) \quad \text{and} \quad \exp(z)$$

are also cyclic since they have analytic continuations across \mathbb{T} but are not meromorphic on \mathbb{D}_e . Note that ∞ lies in the interior of \mathbb{D}_e (the *extended* exterior disk) and so e^z is not meromorphic on \mathbb{D}_e .

Now we come from the other direction and start with the L -prolongation \tilde{f}_L . If \tilde{f}_L has an analytic continuation across a boundary point $\zeta \in \mathbb{T}$, must this continuation be equal to f ? For a general Banach space X , this question seems difficult so let us first focus on the important special case when \tilde{f}_L is identically zero. We restrict ourselves to spaces which satisfy our usual properties, (2.1) through (2.7), along with the following two additional conditions: we assume that (X, X') is an ℓ^2 dual pair and that if $f = \sum_n a_n z^n \in X \setminus \{0\}$ and $g = \sum_n b_n z^n \in X'$, then

$$(3.6) \quad \sum_{l=0}^{\infty} a_{l-k} \bar{b}_l = 0 \quad \text{for all } k \in \mathbb{Z} \quad \Rightarrow \quad (b_n)_{n \geq 0} \equiv (0),$$

(where $a_s := 0$ when $s < 0$). The sequence defined by the left-hand side of the above is the *convolution* $(a_{-n}) * (\bar{b}_n)$ of the sequences $(a_{-n})_{n \in \mathbb{Z}}$ and $(\bar{b}_n)_{n \in \mathbb{Z}}$ (where $b_n := 0$ for $n < 0$). In a moment, we will discuss examples of these spaces.

Theorem 3.7. *Let X satisfy the conditions (2.1) through (2.7) along with the additional assumptions that (X, X') is an ℓ^2 dual pair and the condition in (3.6) holds. If $f \in X$ is a non-cyclic vector and $\tilde{f}_L \equiv 0$ for some $L \in [f]^\perp \setminus \{0\}$, then $f \equiv 0$.*

Assuming the condition in (3.6) holds, here is the proof of the theorem. If \tilde{f}_L is the zero function, that is to say, in our ℓ^2 pairing notation (equating $L \in X'$ with $g = \sum_n b_n z^n$),

$$\left(\frac{f}{z-\lambda}, g\right) / \left(\frac{1}{z-\lambda}, g\right) = 0, \quad |\lambda| > 1,$$

then

$$\left(\frac{f}{z-\lambda}, g\right) = 0, \quad |\lambda| > 1$$

and a computation with power series shows that

$$0 = \left(\frac{f}{z-\lambda}, g\right) = - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} \sum_{l=k}^{\infty} a_{l-k} \bar{b}_l$$

and so

$$(3.8) \quad (M_z^k f, g) = \sum_{l=k}^{\infty} a_{l-k} \bar{b}_l = 0, \quad k \in \mathbb{N}_0.$$

But g annihilates the B -invariant subspace generated by f and so

$$(3.9) \quad 0 = (B^k f, g) = \sum_{l=0}^{\infty} a_{l+k} \bar{b}_l, \quad k \in \mathbb{N}_0.$$

Combining (3.8) and (3.9) we see that the convolution $(a_{-n}) * (\bar{b}_n)$ is the zero sequence (where we understand that $a_n = b_n = 0$ when $n < 0$). But if $(a_n)_{n \geq 0}$ is not the zero sequence, then, by our assumption on the space X , $(b_n)_{n \geq 0}$ must be the zero sequence, which we are assuming is not the case. Thus $f \equiv 0$.

Remark 3.10. Theorem 3.7 was stated in a slightly different way in [24, p. 134] for the spaces D_α , $\alpha \in \mathbb{R}$, but the proof was deficient. This proof corrects this.

As it turns out, the spaces D_α with $\alpha > 0$, and many others as well, satisfy (3.6) (see remark below). For Hardy and weighted Bergman spaces, a stronger result than Theorem 3.7 is true (see Proposition 3.13 below) and so we focus our efforts here on the Dirichlet-type spaces, or more generally, spaces of analytic functions on \mathbb{D} which are ‘smooth up to the boundary’. For the sake of completeness, and to give the reader a flavor of what goes on here, we will show that spaces like $\ell_A^1(w)$ with, for example, $w_n = (1+n)^s$, $s > 0$, satisfy (3.6). The following argument was kindly communicated to us by Yngve Domar.

To do this, let $w = (w_n)_{n \in \mathbb{Z}}$ be a (two-sided) weight sequence such that

$$(3.11) \quad \begin{aligned} w_n &= w_{-n}, \\ w_0 &= 1, \quad w_n \geq 1, \\ w_{n+m} &\leq w_m w_n, \\ \sum_{n \in \mathbb{Z}} \frac{\log w_n}{1+n^2} &< \infty. \end{aligned}$$

Weights like $w_n = (1+|n|)^s$, with $s > 0$, satisfy these conditions. Let $\ell^1(w)$ be the space of sequences $x = (x(n))_{n \in \mathbb{Z}}$ with

$$\sum_{n \in \mathbb{Z}} |x(n)| w_n < \infty.$$

The dual space of $\ell^1(w)$ can be identified with $\ell^\infty(w)$, the space of sequences $y = (y(n))_{n \in \mathbb{Z}}$ with

$$\sup_{n \in \mathbb{Z}} \frac{|y(n)|}{w_n} < \infty$$

via the pairing

$$\langle x, y \rangle := \sum_{n \in \mathbb{Z}} x(n)y(-n).$$

One can check that $\ell^1(w)$ is a Banach algebra with respect to the operations of pointwise addition and convolution: for $x_1, x_2 \in \ell^1(w)$,

$$\begin{aligned} (x_1 + x_2)(n) &:= x_1(n) + x_2(n), \\ (x_1 * x_2)(n) &:= \sum_{m \in \mathbb{Z}} x_1(n - m)x_2(m). \end{aligned}$$

The Banach algebra $\ell^1(w)$ is also *regular*: given any open arc $I \subset \mathbb{T}$, there is an $x \in \ell^1(w)$ whose Gel'fand transform

$$\widehat{x}(\zeta) := \sum_{n \in \mathbb{Z}} x(n)\zeta^n, \quad \zeta \in \mathbb{T},$$

(which is continuous on \mathbb{T} since its coefficients are absolutely summable) has support in I . In fact, given $N \in \mathbb{Z}$, we can even choose x to satisfy the extra condition that $x(N) \neq 0$. For weight sequences like $w_n = (1 + |n|)^s$ with $s > 0$, sequences like $(x(n))_{n \in \mathbb{Z}}$, where $x(n)$ is the n -th Fourier coefficient of a C^∞ function with support in I , are contained in $\ell^1(w)$. For general weights w_n satisfying the ‘non-quasianalyticity condition’ in (3.11), the regularity is a bit more delicate [7].

The result here is the following: suppose $x \in \ell^1(w) \setminus \{0\}$, $y \in \ell^\infty(w)$ with $y(n) = 0$ for all $n > 0$, and $\langle x(\cdot - m), y \rangle = 0$, that is,

$$\sum_{n \in \mathbb{Z}} x(n - m)y(-n) = 0$$

for all $m \in \mathbb{Z}$. Then $y = 0$. In other words, $\ell^1_A(w)$ satisfies the condition in (3.6).

To see this, note that since x is non-trivial and \widehat{x} is continuous on \mathbb{T} , there is an arc $I \subset \mathbb{T}$ so that $\widehat{x}(\zeta) \neq 0$ for all $\zeta \in I$. Fix $N \geq 0$ and choose $v \in \ell^1(w)$ with \widehat{v} supported in I so that $v(N) \neq 0$ (regularity of $\ell^1(w)$). For real t , define $u_t \in \ell^1(w)$ by

$$u_t(n) := v(n)e^{int}$$

and observe that for small positive c , \widehat{u}_t is again supported in I provided $|t| < c$. The quotient $\widehat{u}_t/\widehat{x}$ is the Gel'fand transform of an $r_t \in \ell^1(w)$ with $u_t = r_t * x$. Thus since $\langle x(\cdot - m), y \rangle = 0$ for all $m \in \mathbb{Z}$, then $\langle u_t(\cdot - m), y \rangle = 0$ for all $m \in \mathbb{Z}$, $|t| < c$, or equivalently,

$$\sum_{n \in \mathbb{Z}} y(-n)v(n - m)e^{int}e^{-imt} = 0 \quad m \in \mathbb{Z}, |t| < c.$$

Letting $m = 0$ and noting that $y(-n) = 0$ for all $n < 0$, we conclude

$$\sum_{n=0}^{\infty} y(-n)v(n)e^{int} = 0, \quad |t| < c.$$

But since the sequence $(y(-n)v(n))_{n \geq 0}$ is absolutely summable,

$$h(z) := \sum_{n=0}^{\infty} y(-n)v(n)z^n$$

is analytic on \mathbb{D} , continuous on \mathbb{D}^- , and is zero on an arc of \mathbb{T} . Thus, by a classical theorem of Riesz [11, p. 17], $h \equiv 0$ and so $y(-n)v(n) = 0$ for all $n \geq 0$. But we are assuming $v(N) \neq 0$ and so $y(-N) = 0$. Since N was arbitrary, $y(n) = 0$ for all $n \leq 0$. But we are already assuming $y(n) = 0$ for all $n > 0$ and so $y = 0$.

Remark 3.12. 1. There are some spaces that are not of the form $\ell_A^1(w)$ which satisfy the condition in (3.6), for example, D_α , $\alpha > 0$, [4].

2. Recently, Jean Esterle [12, Thm. 4.10] showed there are $\ell_A^2(w)$ spaces with weights w_n increasing to infinity as slowly as desired, containing non-trivial functions whose orbit under all backward and forward shifts fails to span $\ell_A^2(w)$, or in our terminology, using (3.8) and (3.9): a non-cyclic vector f for the backward shift for which some L -prolongation is identically zero (both f and L of course being non-trivial). The crucial feature of this example is that the weights, while monotone, lack finer regularity properties like convexity which are needed in the theorems of [4].

For our three spaces H^p , $L_a^p(w)$ with a suitable weight, and \mathcal{D}_p , the following stronger compatibility result is true. Note that the Dirichlet-type spaces D_α , $\alpha < 0$, can be included in this list since $L_a^2((1 - |z|)^{-1-\alpha}) \cong D_\alpha$ for $\alpha < 0$ (recall (3.4)).

Proposition 3.13. *Suppose that X is one of the three spaces mentioned above. If $f \in X$ is a non-cyclic vector and \tilde{f}_L has an analytic continuation, also denoted by \tilde{f}_L , to an open neighborhood U_ζ of $\zeta \in \mathbb{T}$, then \tilde{f}_L agrees with f on $U_\zeta \cap \mathbb{D}$.*

Proof. For the Hardy and Bergman spaces, \tilde{f}_L is of bounded type on \mathbb{D}_e and is a pseudocontinuation of f . Thus if \tilde{f}_L has an analytic continuation F across an arc $I \subset \mathbb{T}$, then \tilde{f}_L will have two pseudocontinuations across I , f and F . By the Lusin-Privalov Uniqueness Theorem, f and F must be the same.

For the Dirichlet spaces \mathcal{D}_p , the proof is somewhat similar. In [1, Lemma 7.4], the authors show, using very special calculations with the local Dirichlet integral, that

$$L\left(\frac{1}{z-w}\right) [\tilde{f}_L(w) - f(e^{i\theta})] \rightarrow 0$$

for almost every $e^{i\theta}$ as $w \rightarrow e^{i\theta}$ non-tangentially. If \tilde{f}_L has finite non-tangential limits on an arc $I \subset \mathbb{T}$, then since

$$L\left(\frac{1}{z-w}\right)$$

cannot go to zero as $w \rightarrow e^{i\theta}$ (non-tangentially) on a subset of I of positive measure (Lusin-Privalov Uniqueness Theorem), it must be the case that

$$\tilde{f}_L(w) \rightarrow f(e^{i\theta})$$

as $w \rightarrow e^{i\theta}$ non-tangentially for almost every $e^{i\theta} \in I$. Thus \tilde{f}_L must be a pseudo-continuation of f across I . So if \tilde{f}_L has an analytic continuation across an arc $I \subset \mathbb{T}$, then, by combining the above along with the Lusin-Privalov Uniqueness Theorem, \tilde{f}_L must be an analytic continuation of f across I . ■

To prove the analogue of Proposition 3.13 for a wide class of Banach spaces X , say even something like $\ell_A^p(w)$, remains an open problem. As pointed out in Remark 3.12 (Esterle's example), Proposition 3.13 does not hold for certain pathological $\ell_A^2(w)$ spaces since there are non-cyclic $f \in \ell_A^2(w) \setminus \{0\}$ and $L \in [f]^\perp \setminus \{0\}$ for which $\tilde{f}_L \equiv 0$.

4. Overconvergence and spectral synthesis

Let T be a bounded linear operator on a Banach space V . Given $t \in \mathbb{C}$ and $r \in \mathbb{N}$, we say that t is an eigenvalue of algebraic multiplicity r , and the non-zero vector $v \in V$ is a *root vector* of order r corresponding to t if

$$(T - tI)^r v = 0 \quad \text{but} \quad (T - tI)^{r-1} v \neq 0.$$

The subspace

$$\bigvee \{(T - tI)^j v : j = 0, 1, \dots, r-1\}$$

is called a root space of V and one can show it has dimension r . An invariant subspace \mathcal{M} of T has the *spectral synthesis property* if \mathcal{M} is the closed linear span of the root spaces it contains, where the root spaces are of the type $\ker(T - tI)^r$ as above. From elementary linear algebra, every finite dimensional \mathcal{M} has the spectral synthesis property. When \mathcal{M} is infinite dimensional, this is no longer the case, though there is a reasonable substitute for spectral synthesis (see below).

For example, if T is the backward shift operator B on our Banach space X of analytic functions on \mathbb{D} , observe that for $a \in \mathbb{D}_e$ and $r(a) \in \mathbb{N}$,

$$(4.1) \quad \ker(aI - B)^{r(a)} = \bigvee \left\{ \frac{1}{(z - a)^s} : s = 1, \dots, r(a) \right\}.$$

Note, from (2.10) and (2.11), that if $\mathcal{M} \neq X$ has the spectral synthesis property, there is a sequence $(a_n)_{n \geq 1} \subset \mathbb{D}_e$ with no cluster points in \mathbb{D}_e and $r(a_n) \in \mathbb{N}$ so that

$$(4.2) \quad \mathcal{M} = \bigvee \left\{ \frac{1}{(z - a_n)^s} : s = 1, \dots, r(a_n) \right\}.$$

Moreover, if $\lambda \notin (a_n)_{n \geq 1}$, then $(z - \lambda)^{-1} \notin \mathcal{M}$.

Remark 4.3. To avoid needless technicalities, we will assume the eigenvalues of B do not lie on the unit circle. If they do, what was stated above, and what follows below can be suitably altered.

Since (H^2, H^2) is an ℓ^2 dual pair, we can use (2.13) to see that $\mathcal{M} = (\varphi H^2)^\perp$ does not have the spectral synthesis property whenever φ is an inner function with a non-constant singular inner factor. A thorough discussion of the spectral synthesis property can be found in [18]. An interesting fact worth pointing here is the following.

Proposition 4.4. *Let $A = (a_n)_{n \geq 1}$ be a sequence of distinct points of \mathbb{D} and let $(c_n)_{n \geq 1}$ be a sequence of non-zero complex numbers for which*

$$\overline{\lim}_{n \rightarrow \infty} |d_n|^{1/n} < 1,$$

where $d_n := c_n \|(1 - a_n z)^{-1}\|_X$. If

$$f := \sum_{n=1}^{\infty} \frac{c_n}{1 - a_n z},$$

then $f \in X$ and

$$(4.5) \quad [f] = \mathcal{M}(A) := \bigvee \left\{ \frac{1}{1 - a_n z} : n \in \mathbb{N} \right\}.$$

That is to say, the spectral synthesis invariant subspace $\mathcal{M}(A)$ is singly generated.

To prove this, we begin with a theorem of Beurling [3].

Lemma 4.6. *Suppose $(z_j)_{j \geq 1}$ is a sequence of distinct points of \mathbb{D} and V is a matrix of Vandermonde type, that is to say, the j -th column of V is $1, z_j, z_j^2, z_j^3, \dots$. If $w = (w_n)_{n \geq 1}$ is a column vector of complex numbers such that*

$$\overline{\lim}_{n \rightarrow \infty} |w_n|^{1/n} < 1$$

and $V \cdot w$ is the zero vector, then w is the zero vector.

Proof of Proposition 4.4. Suppose that $L \in [f]^\perp$, that is to say $L(B^n f) = 0$ for all $N \in \mathbb{N}_0$. Then, using the identity

$$B^n \frac{1}{1 - a_n z} = a_n^n \frac{1}{1 - a_n z},$$

we see that

$$0 = L(B^N f) = \sum_{n=1}^{\infty} c_n a_n^n L\left(\frac{1}{1 - a_n z}\right) \quad \text{for all } N \in \mathbb{N}_0.$$

By our hypothesis and Lemma 4.6,

$$L\left(\frac{1}{1 - a_n z}\right) = 0 \quad \text{for all } n.$$

But this means that $L \in \mathcal{M}(A)^\perp$ and so $[f] \supset \mathcal{M}(A)$. The other inclusion is obvious. ■

Remark 4.7. For a general bounded operator T on a Banach space V for which the eigenvectors (corresponding to distinct eigenvalues) of T span V , Nordgren and Rosenthal [20], in an unpublished paper brought to our attention by M. Putinar, prove the existence of a vector $v \in V$ such that

$$\bigvee \{T^n v : n \in \mathbb{N}_0\} = V.$$

In other words, T has a cyclic vector. We suspect that the same is true when V is spanned by root vectors (corresponding to distinct eigenvalues, which we assume all have one dimensional eigenspaces) and, under certain technical conditions, this is indeed the case. Another paper that considers Beurling’s result (Lemma 4.6) and cyclic vectors for operators is one of Sibilev [31].

Our next observation concerns the backward shift B . If $(f_n)_{n \geq 1} \subset \mathcal{M}$ is a sequence of finite linear combinations of root vectors of X (which in this case are rational functions whose poles lie in \mathbb{D}_e) which converge in norm to f , then certainly $f_n \rightarrow f$ uniformly on compact subsets of \mathbb{D} (see (2.9)). This next result says that this sequence of rational functions ‘overconverges’. This result is found in [26] but we include a proof anyway, both for completeness and since the version stated here is slightly more general. We will also be using some of the techniques of the proof in the next section.

Theorem 4.8. *Let $\mathcal{M} \in \text{Lat}(B, X)$, $\mathcal{M} \neq X$ have the spectral synthesis property and $(f_n)_{n \geq 1}$ be a sequence of finite linear combinations of root vectors in \mathcal{M} with $f_n \rightarrow f$ in norm. Then there is a meromorphic function S_f on \mathbb{D}_e such that $f_n \rightarrow S_f$ uniformly on compact subsets of $\mathbb{D}_e \setminus (a_n)_{n \geq 1}$. Moreover, for any $L \in \mathcal{M}^\perp \setminus \{0\}$, $S_f = \tilde{f}_L$.*

Proof. Let $\lambda \in \mathbb{D}_e \setminus (a_n)_{n \geq 1}$ and $L \in \mathcal{M}^\perp \setminus \{0\}$ with

$$L\left(\frac{1}{z - \lambda}\right) \neq 0.$$

This is possible since $(z - \lambda)^{-1} \notin \mathcal{M}$. Since $L((z - \lambda)^{-1})$ is analytic, there is an open neighborhood U_λ of λ so that

$$L\left(\frac{1}{z - w}\right) \neq 0 \quad \text{for all } w \in U_\lambda.$$

Fix $w \in U_\lambda$, $a = a_n$, and $k \leq r(a)$ (see (4.2)). Select constants $c_j = c_j(a, w)$ (independent of z) so that

$$\frac{1}{z - w} + \frac{c_1}{z - a} + \cdots + \frac{c_k}{(z - a)^k} = \frac{(w - a)^k}{(z - w)(z - a)^k}.$$

Thus, since

$$L\left(\frac{1}{(z-a)^j}\right) = 0, \quad j = 1, \dots, k,$$

we have

$$L\left(\frac{1}{z-w}\right) = (w-a)^k L\left(\frac{1}{(z-w)(z-a)^k}\right)$$

and so it follows that

$$L\left(\frac{1}{(z-w)(z-a)^k}\right) = \frac{1}{(w-a)^k} L\left(\frac{1}{z-w}\right).$$

Now, if f is a finite linear combination of root vectors in \mathcal{M} then it follows that for $w \in U_\lambda$,

$$(4.9) \quad L\left(\frac{f}{z-w}\right) = f(w)L\left(\frac{1}{z-w}\right).$$

Thus for any $w \in U_\lambda$,

$$|f(w)| \leq \frac{\|L\| \|M_{(z-w)^{-1}}\| \|f\|}{\left|L\left(\frac{1}{z-w}\right)\right|} \leq C_\lambda \|f\|.$$

The first part of the theorem now follows.

For the last part of the theorem, note that if $f_n \rightarrow f$ in norm as in the hypothesis, then for any $L \in \mathcal{M}^\perp \setminus \{0\}$ (note that $L \in [f_n]^\perp \setminus \{0\}$) (4.9) shows that

$$f_n = \tilde{f}_{nL}$$

on \mathbb{D}_e . Using the norm convergence of $f_n \rightarrow f$ and the definitions of \tilde{f}_{nL} and \tilde{f}_L (from (2.19)), one can show that

$$\tilde{f}_L = \lim_{n \rightarrow \infty} \tilde{f}_{nL}$$

pointwise on \mathbb{D}_e (minus appropriate poles). Thus the limit function S_f is \tilde{f}_L , the L -prolongation of f . \blacksquare

Remark 4.10. 1. Since $S_f = \tilde{f}_L$ for any $L \in \mathcal{M}^\perp \setminus \{0\}$ it follows that \tilde{f}_L is independent of $L \in \mathcal{M}^\perp \setminus \{0\}$ and so, using the results mentioned in Section 2 (see also [1, Prop. 2.6]),

$$\sigma(B|\mathcal{M}) \cap \mathbb{D} = \left(\frac{1}{a_n}\right)_{n \geq 1}.$$

2. As a small technical aside, notice that $f_n = \tilde{f}_{nL}$ for all $L \in \mathcal{M}^\perp \setminus \{0\}$. It is true that $f_n = \tilde{f}_{nL}$ for all $L \in [f_n]^\perp \setminus \{0\}$. To see this, one can use a variant of the proof of Proposition 4.4 to show that if $\{b_1, \dots, b_k\} \subset (a_n)_{n \geq 1}$, and

$$h = \sum_{j=1}^k \frac{c_j}{(z-b_j)^{s(j)}},$$

then

$$[h] = \bigvee \left\{ \frac{1}{(z - b_j)^s} : j = 1, \dots, k, s = 1, \dots, s(j) \right\}.$$

Thus, by replacing \mathcal{M} with $[f_n]$ in equation (4.9), we see that $f_n = \tilde{f}_{nL}$ for all $L \in [f_n]^\perp \setminus \{0\}$. When X satisfies the condition in (3.2), this follows directly from Theorem 3.1.

3. For $f \in \mathcal{M}$ we know that \tilde{f}_L is independent of $L \in \mathcal{M}^\perp \setminus \{0\}$ and that \tilde{f}_{nL} is independent of $L \in [f_n]^\perp \setminus \{0\}$. One might be tempted to say that \tilde{f}_L is independent of $L \in [f]^\perp \setminus \{0\}$. However, this is not the case. By a construction in [14], there are two sequences A_1, A_2 contained in \mathbb{D} so that

$$\mathcal{I}(A_j) := \{f \in L_a^2 : f|_{A_j} = 0\} \neq \{0\}$$

and the M_z -invariant subspace $\mathcal{I} := \mathcal{I}(A_1) \vee \mathcal{I}(A_2)$ has index equal to two, that is to say $\dim(\mathcal{I}/M_z\mathcal{I}) = 2$. It is well-known [22, Thm. 4.5] that the B -invariant subspace (of the Dirichlet space \mathcal{D}) \mathcal{I}^\perp satisfies $\sigma(B|\mathcal{I}^\perp) = \mathbb{D}^-$. Using (2.13) and the ℓ^2 dual pairing between L_a^2 and \mathcal{D} , we can show that

$$\mathcal{I}(A_1)^\perp = \bigvee \left\{ \frac{1}{1 - \bar{a}z} : a \in A_1 \right\}$$

and thus has the spectral synthesis property. By using [1, Prop. 2.6], there is an $f \in \mathcal{I}^\perp$ and annihilating L_1, L_2 of $\mathcal{I}^\perp \setminus \{0\}$ (and hence $L_1, L_2 \in [f]^\perp \setminus \{0\}$) so that $\tilde{f}_{L_1} \neq \tilde{f}_{L_2}$. Notice that $f \in \mathcal{I}^\perp \subset \mathcal{I}(A_1)^\perp$.

4. If X' has the property that every (non-zero) M_z -invariant subspace has index equal to one (for example H^2 , \mathcal{D} , and many others [1, Cor. 5.10]), then $\sigma(B|[f]) \cap \mathbb{D}$, where f is any non-cyclic vector, is a countable subset of \mathbb{D} with no accumulation points in \mathbb{D} and so [1, Prop. 2.6] \tilde{f}_L is independent of $L \in [f]^\perp \setminus \{0\}$.

5. For the above Borel series f , in (4.5), that generates $\mathcal{M}(A)$ (which we shall assume is not all of X), there are two functions defined on \mathbb{D}_e that, in some sense, can be associated with f . There is the function

$$\sum_{n=0}^{\infty} \frac{c_n}{1 - a_n w}$$

which is just the Borel series defined on $\mathbb{D}_e \setminus (1/a_n)_{n \geq 0}$ and then there is an L -prolongation of f for any $L \in [f]^\perp \setminus \{0\}$. By Theorem 4.8, these are the same.

6. In a way, the function S_f can be thought of as a ‘continuation’ of f across \mathbb{T} , though not necessarily an analytic continuation. At the end of the paper [26], the author posed the following compatibility questions about this continuation S_f :

- (i) If $S_f \equiv 0$, must $f \equiv 0$?
- (ii) If f has an analytic continuation across $e^{i\theta}$, must this analytic continuation be equal to S_f near $e^{i\theta}$?
- (iii) If S_f has an analytic continuation across $e^{i\theta}$, must S_f equal f near $e^{i\theta}$?

The identity $\tilde{f}_L = S_f$, along with Theorem 3.1, Theorem 3.7, and Proposition 3.13, give answers to these questions.

7. When the space X satisfies certain mild technical conditions, which spaces like H^2 , $L_a^p(w)$, D_α , \mathcal{D}_p do, then the sequence $(f_n)_{n \geq 1}$ converges uniformly on compact subsets of $\widehat{\mathbb{C}} \setminus (a_n)_{n \geq 1}^-$ (see [26] or Remark 5.18 below).

5. Overconvergence and approximate spectral synthesis

As we have seen earlier, there are invariant subspaces that do not have the spectral synthesis property. The following reasonable substitution was proposed by Nikolskiĭ [19] (see also [28] for a nice exposition): if T is a bounded operator on a Banach space V and $(\mathcal{M}_j)_{j \geq 1}$ is a sequence of T -invariant subspaces of V , define

$$(5.1) \quad \mathcal{M} = \varliminf_{j \rightarrow \infty} \mathcal{M}_j := \left\{ v \in V : \lim_{j \rightarrow \infty} \text{dist}(v, \mathcal{M}_j) = 0 \right\}.$$

The above ‘liminf’ space is closed and T -invariant. It is important to note here that \mathcal{M}_j need not lie in \mathcal{M} . We say a T -invariant subspace \mathcal{M} has the *approximate spectral synthesis* property if \mathcal{M} can be written as in (5.1) for a sequence $(\mathcal{M}_j)_{j \geq 1}$ of T -invariant subspaces with $\dim(\mathcal{M}_j) < \infty$. In this case, \mathcal{M}_j is a linear span of root spaces (and thus satisfies the approximate spectral synthesis property), and hence the name ‘approximate spectral synthesis’. Note that if $\mathcal{M}_j \subset \mathcal{M}_{j+1}$ for all $j \geq 1$, then \mathcal{M} has the spectral synthesis property.

When T is the backward shift operator B on X , we can ask the following important questions.

- (i) When is this ‘liminf space’ all of X ?
- (ii) Can every B -invariant subspace of X be written as a ‘liminf space’?

For the first question, there are theorems of [13, 33, 34] and others that give specific conditions for this to happen. For the second question, the answer is yes when $X = H^2$ (the Hardy space) [33] and when $X = \mathcal{D}$ (the Dirichlet space — or more generally certain weighted Dirichlet spaces) [28, 29]. For other spaces, question (ii) remains unanswered.

In the previous section, we showed that if $(f_n)_{n \geq 1}$ is a norm-convergent sequence of root vectors in some non-trivial B -invariant subspace satisfying the spectral synthesis property, then this sequence overconverges on \mathbb{D}_e (minus the appropriate poles). A little later we shall show an analogous overconvergence result for B -invariant subspaces satisfying the approximate spectral synthesis property.

Our set-up is as follows: for each $n \in \mathbb{N}$, choose a sequence

$$E_n := \{z_{n,1}, \dots, z_{n,N(n)}\}$$

of points of \mathbb{D} (multiplicities are allowed) to create the tableau \mathcal{S}

$$\begin{aligned} & z_{1,1}, z_{1,2}, \dots, z_{1,N(1)} \\ & z_{2,1}, z_{2,2}, \dots, z_{2,N(2)} \\ & \vdots \end{aligned}$$

For each n , create the finite dimensional B -invariant subspace (see (4.1))

$$(5.2) \quad R_n := \bigvee \left\{ \frac{1}{(1 - \overline{z_{n,j}}z)^s} : j = 1, \dots, N(n), s = 1, \dots, \text{mult}(z_{n,j}) \right\},$$

where $\text{mult}(z_{n,j})$ is the number of times $z_{n,j}$ appears in E_n , the n -th row of the tableau. If $z_{n,j} = 0$ with multiplicity k , then the functions $1, z, z^2, \dots, z^k$ are added to the spanning set for R_n . One can now form the ‘liminf space’, as in (5.1), associated with the tableau \mathcal{S} by

$$(5.3) \quad R(\mathcal{S}) := \underline{\lim} R_n.$$

When $X = H^2$, the following condition determines when $R(\mathcal{S}) \neq H^2$, [33, 34]: if

$$\gamma(E_n) := \sum_{j=1}^{N(n)} (1 - |z_{n,j}|),$$

then

$$(5.4) \quad R(\mathcal{S}) \neq H^2 \quad \Leftrightarrow \quad \underline{\lim}_{n \rightarrow \infty} \gamma(E_n) < \infty.$$

There are analogous results for other spaces [18] where the quantity $\gamma(E_n)$ is replaced by another ‘capacity’ suitable for the particular Banach space X . Recall also the results of [28, 33] which say, when our Banach space is either H^2 or the Dirichlet space \mathcal{D} , that every B -invariant subspace can be written as $R(\mathcal{S})$ for some tableau \mathcal{S} .

We say a tableau \mathcal{S} is *ample* if $R(\mathcal{S}) = X$, *deficient* if $R(\mathcal{S}) \neq X$ and *uniformly deficient* if

$$\underline{\lim}_{n \rightarrow \infty} \text{dist} \left(\frac{1}{z - \lambda}, R_n \right) > 0$$

for some $\lambda \in \mathbb{D}_e$. These uniformly deficient tableaux will be the focus of our attention. Before stating our overconvergence theorem, we make a few remarks.

Remark 5.5. 1. Assuming (X, X') is an ℓ^2 dual pair, recall from (2.13) that for $g \in X'$, $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$,

$$(5.6) \quad \left(\frac{n!z^n}{(1 - \overline{\lambda}z)^{n+1}}, g \right) = \overline{g^{(n)}(\lambda)}.$$

Thus g belongs to R_n^\perp if and only if g vanishes on E_n (up to appropriate orders).

2. With our ℓ^2 dual pairing, we equate a functional L with an analytic function g . With this set up, the L -prolongation \tilde{f}_L of a non-cyclic vector f looks like

$$(5.7) \quad \tilde{f}_L(w) = \left(\frac{f}{z-w}, g \right) / \frac{-1}{w} \overline{g\left(\frac{1}{w}\right)}.$$

3. If

$$(5.8) \quad d_n(\lambda) := \text{dist} \left(\frac{1}{1-\bar{\lambda}z}, R_n \right)$$

(note from (2.13) that $d_n(\lambda) = 0$ if and only if $\lambda \in E_n$) and

$$(5.9) \quad d(\lambda) := \varliminf_{n \rightarrow \infty} d_n(\lambda),$$

then \mathcal{S} is uniformly deficient if and only if $d(\lambda) > 0$ for some $\lambda \in \mathbb{D}$. It is well-known, and easily shown, that the distance $d(x, A)$ from a point x in a metric space to some fixed set A satisfies $|d(x, A) - d(y, A)| \leq d(x, y)$. So, for $z, w \in \mathbb{D}$,

$$|d_n(w) - d_n(z)| \leq |w - z|$$

and so

$$(5.10) \quad d(\lambda) > 0 \quad \Rightarrow \quad d(w) > 0 \quad \text{for all } w \in U_\lambda,$$

where U_λ is some open neighborhood containing λ . Therefore, the set

$$(5.11) \quad Q := \{\lambda \in \mathbb{D} : d(\lambda) > 0\}$$

is an open subset of \mathbb{D} which is non-empty whenever the tableau \mathcal{S} is uniformly deficient.

4. The liminf subspaces $R(\mathcal{S})$ (see (5.3)) arising from uniformly deficient tableaux need not satisfy the spectral synthesis property. For instance, if ϕ is a singular inner function, then the B -invariant subspace $(\phi H^2)^\perp$ does not satisfy the spectral synthesis property. Indeed, $\phi(\lambda) \neq 0$ for any $\lambda \in \mathbb{D}$ and so by (2.13), $(\phi H^2)^\perp$ contains no eigenvectors of B . However, it arises as a liminf space from a uniformly deficient tableau. One can see by using Frostman's Theorem [15, p. 85] to obtain a sequence $(B_n)_{n \geq 1}$ of finite Blaschke products such that $B_n \rightarrow \phi$ weakly in H^2 (equivalently $B_n \rightrightarrows \phi$ pointwise in \mathbb{D}). From here, it is routine to prove that

$$\varliminf_{n \rightarrow \infty} (B_n H^2)^\perp = (\phi H^2)^\perp$$

[18, p. 35]. Since, for fixed $\lambda \in \mathbb{D}$,

$$\frac{1}{1-\bar{\lambda}z} \notin (\phi H^2)^\perp,$$

we have

$$\lim_{k \rightarrow \infty} \text{dist} \left(\frac{1}{1-\bar{\lambda}z}, (B_{n_k} H^2)^\perp \right) > 0$$

for some subsequence $(B_{n_k})_{k \geq 1}$. Since $B_{n_k} \rightarrow \phi$ weakly in H^2 , then

$$\varinjlim_{k \rightarrow \infty} (B_{n_k} H^2)^\perp = (\phi H^2)^\perp.$$

Thus if E_k are the zeros of B_{n_k} , repeated according to multiplicity, then the tableau $\{E_k : k \in \mathbb{N}\}$ is uniformly deficient.

Our overconvergence will take place on the set $W^{-1} := \{1/w : w \in W\}$ (see Theorem 5.16 below), where

$$(5.12) \quad W := \mathbb{D} \setminus \bigcap_{n=1}^{\infty} S_n, \quad S_n := \left(\bigcup_{k=n}^{\infty} E_k \right)^-.$$

However, unlike Theorem 4.8, where the overconvergence took place on the set $\mathbb{D}_e \setminus (a_n)_{n \geq 1}$ and the removed sequence does not have any cluster points in \mathbb{D}_e , the overconvergence will take place on a much smaller set, that may not even be connected. In fact, without the hypothesis of uniformly deficient, W^{-1} might even be empty. For example, let $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$ be a Blaschke sequence, that is

$$\sum_{n=1}^{\infty} (1 - |\lambda_n|) < \infty,$$

and $(a_n)_{n \geq 1}$ be a countable dense subset of \mathbb{D} . Letting

$$(5.13) \quad E_{2n+1} := \{\lambda_1, \dots, \lambda_n\}, \quad E_{2n} := \{a_1, \dots, a_n\},$$

we see that

$$\gamma(E_{2n+1}) = \sum_{j=1}^n (1 - |\lambda_j|)$$

is uniformly bounded and so by (5.4), $R(\mathcal{S}) \neq H^2$. Note that $R(\mathcal{S}) \neq \{0\}$ since $(1 - \overline{\lambda_j} z)^{-1} \in R(\mathcal{S})$ for all j . Indeed choose a subsequence of $(a_n)_{n \geq 1}$ converging to λ_j and use that fact that the sets E_{2n+1} and E_{2n} are increasing. Finally, notice from (5.12) that $W = \emptyset$.

Before proceeding further, we also mention that overconvergence in the liminf setting will not come as smoothly as it did in the spectral synthesis case (Theorem 4.8). In that case, f was the norm limit of the rational functions f_n and, for any chosen $L \in \mathcal{M}^\perp \setminus \{0\}$, one saw that \tilde{f}_L was the pointwise limit of the functions \tilde{f}_{nL} . What made that work was the fact that $f_n \in \mathcal{M}$ and so whenever $L \in \mathcal{M}^\perp \setminus \{0\}$, then $L \in [f_n]^\perp$, making \tilde{f}_{nL} a bona fide L -prolongation of f_n . When $R(\mathcal{S}) = \varinjlim R_n$ and f is the norm limit of $f_n \in R_n$, then f_n may not belong to $R(\mathcal{S})$ and so $L \in R(\mathcal{S})^\perp \setminus \{0\}$ may not annihilate $[f_n]$, making the expression

$$L \left(\frac{f_n}{z - \lambda} \right) / L \left(\frac{1}{z - \lambda} \right)$$

a meromorphic function on \mathbb{D}_e but not an L -prolongation of f_n . The rub here is to judiciously choose $L_n \in [f_n]^\perp \setminus \{0\}$ converging weak* to $L \in R(\mathcal{S})^\perp \setminus \{0\}$.

Using the definitions of the sets Q , W , and the function $d(\lambda)$ (see definitions (5.11), (5.12), and (5.9) respectively), one can show that

$$Q \subset W.$$

Indeed, if $d(\lambda) > 0$, then $|d_k(\lambda)| \geq t > 0$ for all $k \geq n$. This certainly means that

$$\lambda \notin \bigcup_{k \geq n} E_k.$$

If $\lambda_{k_j} \in E_{k_j}$ converges to λ , then

$$\frac{1}{1 - \overline{\lambda_{k_j}} z} \rightarrow \frac{1}{1 - \overline{\lambda} z}$$

in X norm (since by (2.6), $M_{(1-az)^{-1}} = (I - aM_z)^{-1}$ is an operator-valued analytic function on \mathbb{D}). But this says that $d(\lambda) = \lim_{j \rightarrow \infty} d_{k_j}(\lambda) = 0$ which is a contradiction. Thus $\lambda \notin S_n$ for some n and so $\lambda \in W$.

Lemma 5.14. *If Q is non-empty, i.e. \mathcal{S} is uniformly deficient, and W is connected, then $Q = W$.*

Proof. Suppose $b \in Q$, that is $d(b) > 0$. We need to show that

$$d(c) = \underline{\lim}_{n \rightarrow \infty} d_n(c) > 0$$

for each $c \in W$. By the Hahn-Banach Theorem, there is, for each (large) n , some $g_n \in R_n^\perp$ (equivalently, by (2.13), g_n vanishing on E_n up to appropriate orders), $\|g_n\| = 1$, and $g_n(b) = d_n(b)$. The hypothesis of the theorem say that for large enough n , $|g_n(b)| \geq t' > 0$.

Assume the conclusion of the theorem is false, in that there is a subsequence n_1, n_2, n_3, \dots so that

$$d_{n_k}(c) \rightarrow 0.$$

Join the points b and c with a smooth arc in W (note that W is open and connected) and let V be an open neighborhood containing that arc and with $V^- \subset W$. Let V' be an open set with $V^- \subset V' \subset V'^- \subset W$. Since $\|g_{n_k}\| = 1$, g_{n_k} is uniformly bounded on compact subsets of \mathbb{D} (Since (X, X') is an ℓ^2 dual pair, then X' also satisfies (2.1) through (2.7) [1, Prop. 5.2]) and so there is a subsequence (still denoted by g_{n_k}) converging uniformly on compact subsets of V' to an analytic function g on V . Since $|g_{n_k}(b)| \geq t' > 0$, g is not identically zero on V' .

Observe that the number of zeros of g_{n_k} in V is uniformly bounded for all k . Indeed, since g is not identically zero, there is a Jordan curve $C \subset V'$ surrounding V on which g is non-vanishing. So, for large k , g_{n_k} is also non-vanishing on C and its change in argument as one goes around C is equal to that of g . Thus for all large k , g_{n_k} has the same number of zeros, say r , in V .

Let B_{n_k} denote the Blaschke product formed from these r zeros of g_{n_k} in V and let

$$G_{n_k} = \frac{g_{n_k}}{B_{n_k}}.$$

One can check that $G_{n_k} \in X'$ and is uniformly bounded in norm. To see this, note again that X' satisfies (2.1) through (2.7) and so, in particular, the spectrum of B on X' is \mathbb{D}^- . Thus from (2.15),

$$\frac{f}{z - \lambda} = (I - \lambda B)^{-1} B f$$

whenever $f \in X'$ with $f(\lambda) = 0$. Since the function $\lambda \rightarrow \|(I - \lambda B)^{-1} B\|$ is continuous on \mathbb{D} , one can argue, using the fact that for each k , B_{n_k} has only r zeros on V , that g_{n_k}/B_{n_k} is uniformly bounded in norm.

So, by (2.9), G_{n_k} is uniformly bounded in V' and thus has a subsequence (also denoted by G_{n_k}) converging uniformly on V to G which, since $|G_{n_k}(b)|$ is bounded away from zero, is not identically zero. By Hurwitz's Theorem, G vanishes nowhere in V since all the G_{n_k} are zero free. Hence $G(c) \neq 0$.

Since G_{n_k} vanishes on E_{n_k} (up to appropriate orders), G_{n_k} belongs to $R_{n_k}^\perp$ (see (5.6)). For any $f \in R_{n_k}$,

$$|G_{n_k}(c)| = \left| \left(\frac{1}{1 - \bar{c}z} + f, G_{n_k} \right) \right| \leq \left\| \frac{1}{1 - \bar{c}z} + f \right\|_X \|G_{n_k}\|_{X'}.$$

Now use that fact that G_{n_k} is norm-bounded and the definition of $d_{n_k}(c)$ to conclude

$$|G_{n_k}(c)| \leq C d_{n_k}(c) \rightarrow 0 \quad \text{as } n_k \rightarrow \infty.$$

Thus $G(c) = 0$ which is a contradiction. ■

Remark 5.15. If W is disconnected, the above argument can be adjusted to show that if Q meets any component of W , it contains that whole component.

Our overconvergence theorem is the following.

Theorem 5.16. *Let X be a Banach space satisfying the conditions (2.1) through (2.7) and such that (X, X') is an ℓ^2 dual pair. For a uniformly deficient tableau \mathcal{S} such that W is connected and $f \in R(\mathcal{S})$ with*

$$f = \lim_{n \rightarrow \infty} f_n, \quad f_n \in R_n,$$

the sequence $(f_n)_{n \geq 1}$ forms a normal family on W^{-1} .

Proof. Let $b \in W$ (which equals the set Q) and note that $d(b) > 0$. By (5.10), $d_n(\lambda) \geq t' > 0$ on an open neighborhood U_b of b with $U_b \subset W$ for all large enough n .

By the Hahn-Banach Theorem, there is, for each (large) n , some $g_n \in R_n^\perp$ (equivalently g_n vanishing on E_n up to appropriate orders), $\|g_n\| = 1$, and $g_n(b) = d_n(b)$. In fact,

$$\underline{\lim}_{n \rightarrow \infty} |g_n(\lambda)| \geq t'$$

for all $\lambda \in U_b$. Proceed as in the proof of Theorem 4.8 (using the ℓ^2 dual pairing and equating L with g , see (5.7)) to see that for all $\lambda \in U_b$,

$$\left| f_n \left(\frac{1}{\lambda} \right) \right| \leq \frac{\|g_n\| \|M_{(1-\bar{\lambda}z)^{-1}}\| \|f_n\|}{|g_n(\lambda)|} \leq C_b \|f_n\|.$$

The result now follows. ■

If

$$f = \lim_{n \rightarrow \infty} f_n, \quad f_n \in R_n,$$

then by the previous result, $(f_n)_{n \geq 1}$ is a normal family on W^{-1} and so, by passing to a subsequence if necessary, f_n converges uniformly on compact subsets of W^{-1} to an analytic function F .

Corollary 5.17. $F = \tilde{f}_L$ for some $L \in R(\mathcal{S})^\perp \setminus \{0\}$.

Proof. Since W^{-1} is connected, it suffices to show that $F = \tilde{f}_L$ for some $L \in R(\mathcal{S})^\perp \setminus \{0\}$ on some open set $U \subset W^{-1}$. For $b \in W$, let $(g_n)_{n \geq 1} \subset X'$ be as in the proof of Lemma 5.14. By Remark 4.10,

$$f_n = (\widetilde{f_n})_{L_n}.$$

Here we equate g_n with L_n . Since g_n is norm-bounded, $g_{n_k} \rightarrow g$ weak* for some subsequence and in fact, $g \in R(\mathcal{S})^\perp$. Indeed for any

$$h = \lim_{n \rightarrow \infty} h_n, \quad h_n \in R_n,$$

we get

$$|(h, g)| = \lim_{n_k \rightarrow \infty} |(h - h_{n_k}, g_{n_k})| \leq \overline{\lim}_{n_k \rightarrow \infty} \|h - h_{n_k}\|_X \|g_{n_k}\|_{X'} = 0.$$

Furthermore, also by weak* convergence, $g_{n_k} \rightarrow g$ pointwise on \mathbb{D} (see (2.13)) and since $|g_{n_k}(b)|$ is bounded away from zero, $g \neq 0$. Using (5.7), it is easy to show that

$$(\widetilde{f_{n_k}})_{L_{n_k}} \rightarrow \tilde{f}_L$$

pointwise on U_b^{-1} . Again, we are equating L with g . By assumption,

$$(\widetilde{f_{n_k}})_{L_{n_k}} = f_{n_k} \rightarrow F$$

pointwise on U_b^{-1} and so $\tilde{f}_L = F$ on U_b^{-1} . ■

Remark 5.18. 1. Is it the case that f_n converges uniformly on compact subsets of W^{-1} ? We do not know the answer to this. We only know the somewhat weaker result that $(f_n)_{n \geq 1}$ is a normal family on W^{-1} . With an extra hypothesis though, the result is true (see Corollary 5.21).

2. When W is disconnected, one can make the obvious changes to the above results to show that given a connected component W' of W , $(f_n)_{n \geq 1}$ forms a normal family on W'^{-1} .

3. For general deficient (but not necessarily uniformly deficient) tableaux, there is a weaker result [24, Thm. 8.7.6]. If $f = \lim_n f_n, f_n \in R_n$ and $w \in W$ (note that W might be the empty set), then there is an open neighborhood U_w of w and a subsequence $(f_{n_k})_{k \geq 1}$ so that $(f_{n_k})_{k \geq 1}$ forms a normal family on U_w^{-1} .

A theorem of Beurling (see [8]) says that if \mathcal{F} is a family of analytic functions on a domain H and there exists a function $\rho: H \rightarrow \mathbb{R}_+ \cup \{\infty\}$ such that $|f(z)| \leq \rho(z)$ for all $f \in \mathcal{F}$, where for some $r > 1$,

$$(5.19) \quad \int_H [\log^+ \log^+ \rho(z)]^r dA(z) < \infty,$$

then \mathcal{F} is a normal family on H . One can show that (under certain technical conditions - see comments below) if

$$U = \widehat{\mathbb{C}} \setminus \left(\bigcap_{n=1}^{\infty} S_n \right)^{-1},$$

and $f_n \rightarrow f$ in norm (as in either Theorem 4.8 or Theorem 5.16), then $(f_n)_{n \geq 1}$ forms a normal family on U . This means that if $\bigcap_n S_n$ does not contain \mathbb{T} , then functions from \mathcal{M} (in Theorem 4.8) or R (in Theorem 5.16) have analytic continuations across parts of the circle. What are these ‘certain technical conditions’? We just need to check, for all rational $f \in \mathcal{M}$ (or R), that

$$(5.20) \quad |f(z)| \leq \rho(z), \quad |z| < 1, \quad |\tilde{f}_L(z)| \leq \rho(z), \quad |z| > 1,$$

for a ρ defined as above on some open neighborhood of a point of the circle. For the standard spaces, $L_a^p(w), \ell_A^p(w)$ (with reasonable weights), \mathcal{D}_p, D_α , one can often take ρ to be something like

$$\rho(z) = \frac{1}{|1 - |z||^s}.$$

Such ρ easily satisfy the hypothesis (5.19) of Beurling’s Theorem. See [26] where this was done for \mathcal{M} as in Theorem 4.8. Putting all this together we have the following result.

Corollary 5.21. *Suppose the hypotheses of Theorem 5.16 hold as well as the condition that U contains an open arc $J \subset \mathbb{T}$. Furthermore, in a two-dimensional open neighborhood H of J , assume that (5.19) holds for some $\rho: H \rightarrow \mathbb{R}_+ \cup \{\infty\}$. Finally, assume that (5.20) holds for all rational $f \in R(\mathcal{S})$. Then $f_n \rightarrow F$*

uniformly on compact subsets of U and $F = \tilde{f}_L$ for some $L \in R(\mathcal{S})^\perp \setminus \{0\}$. In fact, F is an analytic continuation of f to U .

Proof. By the discussion following Theorem 5.16, it suffices to show that f_n converges ucs (uniformly on compact subsets) of U . Since $(f_n)_{n \geq 1}$ is a normal family on U , every subsequence of $(f_n)_{n \geq 1}$ has a further subsequence converging ucs of U to a analytic function F . *A priori*, this F depends on the subsequence. However, since $f_n \rightarrow f$ ucs of \mathbb{D} , then this F must be an analytic continuation of f and hence does *not* depend on the subsequence. Thus every subsequence of $(f_n)_{n \geq 1}$ has a further subsequence converging to F ucs of U . But since the topology of ucs of U is a metrizable topology on the space of analytic functions on U , using a standard fact from the theory of metric spaces, we conclude that $f_n \rightarrow F$ ucs of U . ■

Corollary 5.22. *With the assumptions in Corollary 5.21 and the hypotheses of Theorem 3.1 (which are valid for spaces like H^2 , D_α , $L_a^2(w)$ for example), $\sigma(B|R(\mathcal{S})) \cap \mathbb{D}$ is discrete.*

Proof. If $f \in R(\mathcal{S})$, then by Corollary 5.21 f has an analytic continuation F to U . By Theorem 3.1, any \tilde{f}_L , where $L \in R(\mathcal{S})^\perp \setminus \{0\}$, must be equal to F on $U \cap \mathbb{D}_e$. By [1, Prop. 2.6], $\sigma(B|R(\mathcal{S})) \cap \mathbb{D}$ must be discrete. ■

6. A final comment

The example of Esterle in Remark 3.12 of an $f \in \ell_A^2(w) \setminus \{0\}$ with $\tilde{f}_L \equiv 0$ for some $L \in [f]^\perp \setminus \{0\}$ inspires an intriguing question. Assuming that (X, X') is an ℓ^2 dual pair, we know from (3.8) and (3.9) that $\tilde{f}_L \equiv 0$ for some $L \in [f]^\perp \setminus \{0\}$ if and only

$$O_f := \bigvee \{S^n f, B^m f : m, n \geq 0\} \neq X,$$

where $S = M_z$. For a set $E \subset X$, let

$$[E]_B := \bigvee \{B^n h : n \geq 0, h \in E\},$$

$$[E]_S := \bigvee \{S^n h : n \geq 0, h \in E\}.$$

It is easy to show that for any non-trivial $f \in X$,

$$[[f]_B]_S = X.$$

Indeed, $[f]_B$ contains a g with $g(0) \neq 0$ (look at $B^n f$ for a suitable choice of n). Then $SBg = g - g(0) \in [[f]_B]_S$ and so $1 \in [[f]_B]_S$. Thus $S^n 1 = z^n \in [[f]_B]_S$ for all n , which, under the assumption that the polynomials are dense in X , implies $[[f]_B]_S = X$. Perhaps more interesting is that fact that

$$[[f]_S]_B = O_f.$$

To see this, first notice that the containment \supset is obvious (since clearly $B^n f$ and $S^n f$ belong to $[[f]_S]_B$). For the other direction, let $g \in [[f]_S]_B$ and $\varepsilon > 0$ be given. Then there is an $h \in [f]_S$ and a polynomial p such that $\|p(B)h - g\| < \varepsilon$. Moreover, there is another polynomial q such that $h = q(S)f + k$, where $\|p(B)k\| < \varepsilon$. Hence

$$\|p(B)(q(S)f + k) - g\| < \varepsilon,$$

which implies that

$$\|p(B)q(S)f - g\| < \varepsilon.$$

But $p(B)q(S)f \in O_f$, since every monomial $B^m S^n$ reduces to either B^{m-n} or to S^{n-m} (because $BS = I$).

So, $\tilde{f}_L \equiv 0$ for some $L \in [f]_B^\perp$ if and only if

$$[f]_B \subset [[f]_S]_B \neq X.$$

Of course, for many ‘standard’ spaces X this never happens. In certain pathological spaces X [12] though, one can find a non-trivial $f \in X$ for which

$$[f]_B \subsetneq [[f]_S]_B \neq X.$$

The question now is, can we find a space X and a non-trivial $f \in X$ for which

$$[f]_B = [[f]_S]_B \neq X?$$

If this were the case, then we would have produced a non-empty S -invariant subspace $[f]_S$ contained in a proper B -invariant subspace $[f]_B$ of X . Assuming (X, X') is an ℓ^2 dual pairing (and so S and B are adjoints of each other), then we would have also produced a non-empty S -invariant subspace $[f]_B^\perp$ of a proper B -invariant subspace $[f]_S^\perp$ of X' . We know of no specific examples of this pathology. We do know that such an example will not be found amongst the spaces $X = \ell_A^2(w)$ where the weight w is either log-convex and decreasing to zero or log-concave and increasing to infinity [4].

We also note that $[f]_B = [[f]_S]_B$ if and only if $\tilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$. Indeed, if $\tilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$, then by the above argument, $[f]_B^\perp \subset O_f^\perp$ and so

$$[f]_B = [[f]_S]_B.$$

Conversely, if $[f]_B = [[f]_S]_B$, then $[f]_S \subset [f]_B$ and so

$$\frac{f}{z - \lambda} \in [f]_B \quad \text{for all } |\lambda| > 1.$$

Hence $\tilde{f}_L \equiv 0$ for all $L \in [f]_B^\perp \setminus \{0\}$.

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William T. Ross

E-MAIL: wross@richmond.edu

ADDRESS: *Department of Mathematics and Computer Science, University of Richmond, Richmond, Virginia 23173, U.S.A.*

Harold S. Shapiro

E-MAIL: shapiro@math.kth.se

ADDRESS: *Mathematics Institute, Royal Institute of Technology, S 10044 Stockholm, Sweden.*