Fonctions Monogènes

William T. Ross

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COLLECTION DE MONOGRAPHIES SUR LA THÉORIE DES FONCTIONS
PUBLIÉE SOUS LA DIRECTION DE M. ÉMILE BOREL.

LEÇONS

SUR LES

FONCTIONS MONOGÈNES

UNIFORMES

D'UNE VARIABLE COMPLEXE,

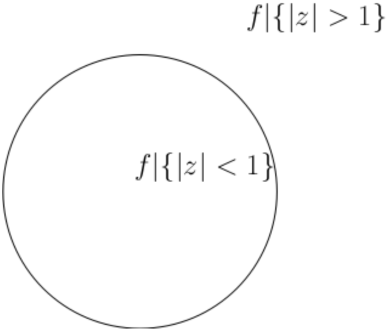
PAR

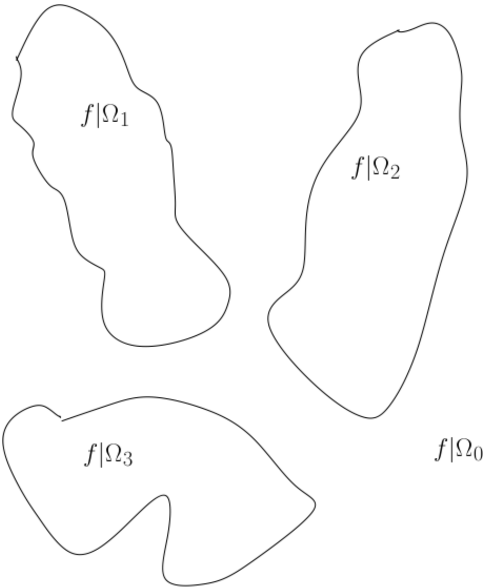
ÉMILE BOREL,

PROFESSEUR DE THÉORIE DES FONCTIONS A L'UNIVERSITÉ DE PARIS,

RÉDIGÉES PAR GASTON JULIA.

PARIS,
GAUTHIER-VILLARS ET Cie, ÉDITEURS,
LIBRAIRES DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,


$f|\{|z| > 1\}$
$f|\{|z| < 1\}$


$f|\Omega_1$
$f|\Omega_2$
$f|\Omega_3$
$f|\Omega_0$

How do we assign a number to

$$1 - 1 + 1 - 1 + 1 - \cdots ?$$

Answer # 1:

$$1 - 1 + 1 - 1 + 1 - 1 \cdots$$

Let s_n be the n-th partial sum

$$s_n = \begin{cases} 0, & n \text{ even}; \\ 1, & n \text{ odd}. \end{cases}$$

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Answer # 2: Cesàro

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\begin{aligned}
1-1+1\cdots &= \lim_{n\to\infty}\frac{s_0+s_1+\cdots+s_n}{n+1}\\
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$$S_C \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

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If S exists, then $S = S_A = S_C$.

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Uses of divergent series

Theorem (du Bois Reymond - 1873)

If f is continuous on $[0, 2\pi]$ with $f(0) = f(2\pi)$ with FS

$$(*) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta),$$

then the FS of f need not *converge pointwise.*

Theorem (Poisson)

If $f \in C[0, 2\pi]$ *with FS (*), then*

$$f(\theta) = \lim_{r \to 1^-} \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n a_n \cos(n\theta) + r^n b_n \sin(n\theta) \right\}$$

Theorem (Fejér - 1904)

If $f \in C[0, 2\pi]$, then $\sigma_n(f) \to f$ uniformly.

$$\sigma_n(f, \theta) = \frac{s_0(f, \theta) + \cdots + s_n(f, \theta)}{n+1}$$

Theorem (Fatou - 1906)

If $f \in L^1[0, 2\pi]$ with FS (*), then

$$f(\theta) = \lim_{r \to 1^-} \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n a_n \cos(n\theta) + r^n b_n \sin(n\theta) \right\} \quad a.e.$$

Theorem (Lebesgue - 1920)

If $f \in L^1[0, 2\pi]$, *then* $\sigma_n(f) \to f$ *a.e.*

Generalized Analytic Continuation

We want to explore ways in which

$$f|\Omega_j, \quad j = 1, 2, \cdots$$

can be related beyond analytic continuation.

- Ω is a disconnected open set in $\widehat{\mathbb{C}}$
- $\{\Omega_j : j = 1, 2, \cdots\}$ are the connected components of Ω.
- $f \in \mathrm{Mer}(\Omega)$.

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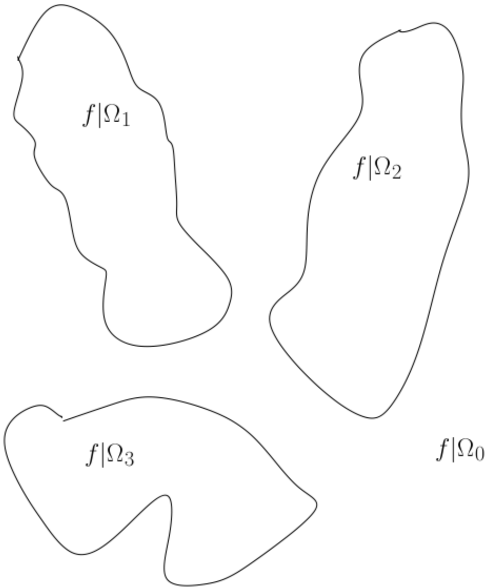
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Overview of GAC

- Define six types of continuations
  - analytic continuation (the 'gold standard')
  - pseudocontinuation
  - Gonçar continuation
  - continuation via formal mult. of series
  - Bochner-Bohnenblust continuation
  - continuation via overconvergence
- how these compare to analytic continuation
- how they prove themselves useful
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Theorem (Poincaré - 1883)

Let $\{e^{i\theta_n}\}$ *be a dense sequence in* $\mathbb{T} = \{|z| = 1\}$ *and* $\{c_n\}$ *be an abs. summable sequence in* $\mathbb{C} \setminus \{0\}$. *Define*

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{1 - e^{-i\theta_n} z} \in Hol(\widehat{\mathbb{C}} \setminus \mathbb{T}).$$

Then f *does not have an AC across any* $e^{i\theta}$.

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{1 - e^{-i\theta_n} z}$$

Still...

$$f(re^{i\theta}) - f(\frac{1}{r}e^{i\theta}) = \int \frac{1-r^2}{|re^{i\theta} - e^{it}|^2} d\mu(e^{it}), \quad d\mu = \sum_{n \geqslant 1} c_n \delta_{e^{i\theta_n}}.$$

Fatou's theorem (1906):

$$\lim_{r \to 1^-} \int \frac{1-r^2}{|re^{i\theta} - e^{it}|^2} d\mu(e^{it}) = \frac{d\mu}{d\theta}(e^{i\theta}) \text{ a.e.}$$

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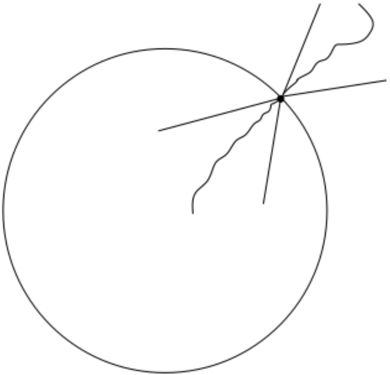
Actually

$$\measuredangle \lim_{\substack{z\to\zeta \\ |z|<1}} f(z) = \measuredangle \lim_{\substack{z\to\zeta \\ |z|>1}} f(z) \ \text{ a.e. } \ \zeta \in \mathbb{T}$$



Pseudocontinuation

Let $f \in \mathrm{Mer}(\mathbb{D}), F \in \mathrm{Mer}(\mathbb{D}_e)$. If

$$\measuredangle \lim_{\substack{z\to\zeta \\ |z|<1}} f(z) = \measuredangle \lim_{\substack{z\to\zeta \\ |z|>1}} F(z) \text{ a.e. } \zeta \in \mathbb{T}$$

then f and F are pseudocontinuations of each other.

Theorem (Lusin-Privalov)

If $f \in Mer(\mathbb{D})$ and

$$\measuredangle \lim_{z \to \zeta} f(z) = 0 \;\; \forall \, \zeta \in E \subset \mathbb{T}$$

with $|E| > 0$, then $f \equiv 0$.

Corollary

If $f \in Mer(\mathbb{D})$ has a PC $F \in Mer(\mathbb{D}_e)$ and f has an AC across $e^{i\theta}$, then this AC must be F.

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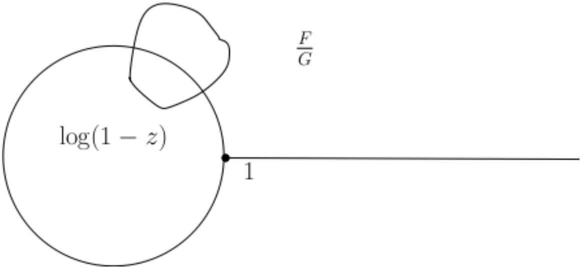
Example

$\log(1-z)$ does not have a pseudocontinuation $F/G \in \text{Mer}(\mathbb{D}_e)$.



Applications of pseudocontinuations

- cyclic vectors for the backward shift operator

$$Bf = \frac{f - f(0)}{z}$$

  on the Hardy space (Douglas-Shapiro-Shields - 1970)
- rational approximation with pre-assigned poles (Walsh 1935, Tumarkin 1966)
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does not have an AC across any $e^{i\theta}$

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Theorem (Aleksandrov - 1997)

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Theorem (Borel - 1917)

Let $\{z_n\}$ *be a seq. of points in* $\mathbb{C}$ *with* $|z_n| \to 1$ *and*

$$|c_n| \leq \exp(\exp(-n^2)),$$

and define

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - z_n}$$

If $f|\mathbb{D}$ *has an analytic continuation* $\widetilde{f}$ *to a neighborhood* U_w *with* $|w| > 1$*, then*

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Gonçar continuation

- Let A = compact set which separates the plane
- $\{\Omega_j : j = 1, 2, \cdots\}$ = connect. comp. of A^c
- $\{R_n\}$ = rat. funct., poles in A, $\deg(R_n) \leqslant n$

Definition

If $f \in \mathrm{Hol}(A^c)$, we say $f|\Omega_j$ is a Gonçar continuation of $f|\Omega_k$ if there are rational functions (as above) with

$$\sup_{K \subset \Omega_k \cup \Omega_j} \lim_{n \to \infty} \sup_{z \in K} |f(z) - R_n(z)|^{1/n} < 1$$

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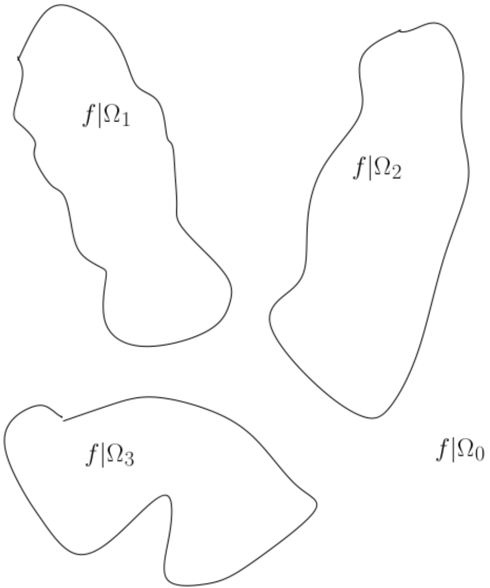
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Example

If

$$\limsup_n \sqrt[n]{|c_n|} < 1$$

and

$$f = \sum_n \frac{c_n}{z - e^{i\theta_n}},$$

then $f|\mathbb{D}_e$ is a G-continuation of $f|\mathbb{D}$.

An'Abelian' theorem

Theorem (Gonçar - 1967)

If $f|\Omega_j$ is a G-continuation of $f|\Omega_k$ and suppose $g = f|\Omega_j$ has an AC $\widetilde{g}$ along some path to a nbh U_w of $w \in \Omega_k$. Then $\widetilde{g}|U_w = f|U_w$.

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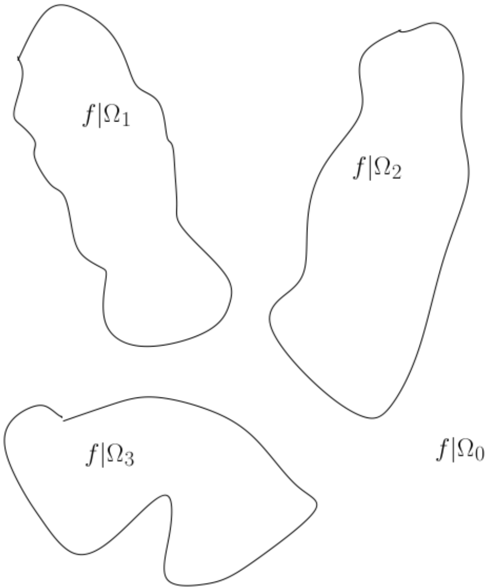
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Applications of G-continuation

- Cyclic vectors for linear operators with a spanning set of eigenvectors, corresponding to distinct eigenvalues (Sibilev)
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Continuation via formal mult. of series

$$f = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathrm{Hol}(\mathbb{D})$$

$$\frac{F}{G} \in \mathrm{Mer}(\mathbb{D}_e)$$

$$F = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots \quad G = B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \cdots$$

Definition

F/G is a formal series continuation of f if

$$\left\{ B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \cdots \right\}\left\{ a_0 + a_1 z + a_2 z^2 + \cdots \right\}$$

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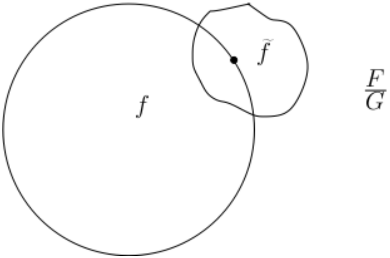
An ‘Abelian’ theorem for FS continuation

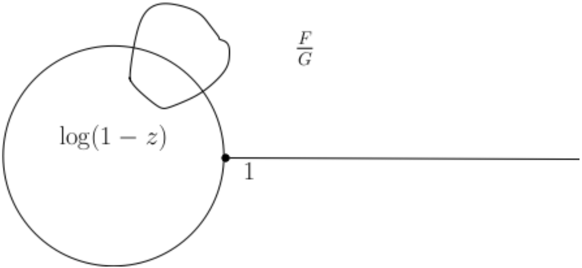
Theorem (R-Shapiro - 2000)

FS cont. are compatible with AC: If f has a FS continuation F/G and f has an AC $\widetilde{f}$ to a nbh U of $e^{i\theta}$, then $\widetilde{f} = F/G$ on U.



Example

$\log(1 - z)$ does not have a FS continuation.



Example

The Borel series

$$\sum_n \frac{c_n}{z - z_n}, \quad z_n \in \mathbb{D}_e$$

can have a FS continuation but not a PC.

Example

The gap series

$$\sum_n 2^{-n} z^{2^n}$$

does not have a FS continuation.

Note: It does not have an AC (Hadamard) nor a PC (Shapiro).

Applications of FS continuation

X is a B-space of analytic functions on $\mathbb{D}$.

$$B : X \to X, \quad Bf = \frac{f - f(0)}{z}.$$

$$[f] := \text{span}\{B^n f : n = 0, 1, 2, \cdots\} \neq X.$$

$$L \in [f]^{\perp} \setminus \{0\}.$$

$$f_L(\lambda) := L(\frac{f}{\cdot - \lambda}) / L(\frac{1}{\cdot - \lambda}), \quad \lambda \in \mathbb{D}_e \setminus \text{poles}$$

Then f_L is a formal series continuation of f.

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Bochner and Bohnenblust continuation

Let $\{A(n)\}_{n\in\mathbb{Z}}$ be a two-sided almost periodic sequence. For example

$$A(n) = \sum_{m=1}^{\infty} c_m e^{in\theta_m}, \quad n \in \mathbb{Z}, \quad \sum_{m=1}^{\infty} |c_m| < \infty.$$

$$f_1(z) := \sum_{n=0}^{\infty} A(n) z^n, \ |z| < 1,$$

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Theorem (B-B - 1934)

If f_1 has an AC across $e^{i\theta}$, then it must continue to f_2.

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Let ϕ be an almost periodic function on $\mathbb{R}$. For example,

$$\phi(x) = \sum_{n=1}^{\infty} c_n e^{i\lambda_n x}, \quad \lambda_n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} |c_n| < \infty.$$

$$f(z) = \int_0^{\infty} \phi(t) e^{-tz} dt, \quad \Re z > 0,$$

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Then

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - i\lambda_n}, \quad \Re z > 0,$$

$$F(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - i\lambda_n}, \quad \Re z < 0.$$

Then

$$\lim_{x \to 0^+} f(x+iy) = \lim_{x \to 0^-} F(x+iy), \quad \text{a.e. } y$$

Continuation via overconvergence

X is a B-space of analytic functions on $\mathbb{D}$ with

$$X = \text{span}\{\frac{1}{z-\lambda} : |\lambda| > 1\} \quad E = \text{span}\{\frac{1}{z-\lambda} : \lambda \in A\} \neq X.$$

If $f \in E$, then there are $f_n \in \text{Rat}(A)$ with $f_n \to f$ in norm.

Theorem (Shapiro - 1968)

$f_n \to S_f$ ucs of $\mathbb{D}_e \setminus A$.

So S_f is a 'continuation' of f.

Theorem (R-Shapiro - 2000)

If f has an AC continuation across $e^{i\theta}$, it must continue to S_f.

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