From here to H^∞

Definitions

The Hardy space H^1

$$H^1 := \left\{ f \in \mathcal{O}(\mathbb{D}) : \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)| \frac{|d\zeta|}{2\pi} < \infty \right\}$$

Theorem (Fatou – 1906, F. Riesz – 1928)

If $f \in H^1$, then

$$f(\zeta) := \lim_{r \to 1^-} f(r\zeta)$$

exists for almost every $\zeta \in \partial\mathbb{D}$. Moreover,

$$\sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)| \frac{|d\zeta|}{2\pi} = \int_{\partial\mathbb{D}} |f(\zeta)| \frac{|d\zeta|}{2\pi} := \|f\|_{H^1}.$$

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Cauchy integral formula

The Cauchy integral formula for H^1

Theorem (F. and M. Riesz – 1916)

If $f \in H^1$, then

$$f(z) = \oint_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}, \quad z \in \mathbb{D}.$$

Norm of the linear functional

For fixed $z \in \mathbb{D}$ we have

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\begin{aligned}
|f(z)| &= \left| \oint_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i} \right| \\
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Best constant?

Is $(1 - |z|)^{-1}$ the best constant in

$$|f(z)| \leqslant \frac{1}{1 - |z|} \|f\|_{H^1}?$$

In other words, for fixed $z \in \mathbb{D}$, what is

$$\sup_{f \in b(H^1)} \left| \oint_{\partial\mathbb{D}} f(\zeta) \frac{1}{\zeta - z} \frac{d\zeta}{2\pi i} \right|?$$

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In a similar way, for $f \in H^1$ and $n \in \mathbb{N}_0$,

$$f^{(n)}(z) = n! \oint_{\partial\mathbb{D}} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \frac{d\zeta}{2\pi i}, \quad z \in \mathbb{D},$$

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General classical problem

Classical extremal problem

Let ψ be a rational function with no poles on $\partial\mathbb{D}$ and define

$$\Lambda(\psi) := \sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, d\zeta \right|$$

Note that

$$\Lambda(\psi) \leqslant \|\psi\|_\infty := \max_{\zeta \in \partial\mathbb{D}} |\psi(\zeta)|$$

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Example

$$\psi(z) = \frac{n!}{(z-\lambda)^{n+1}}, \quad \Lambda(\psi) = \sup_{F \in b(H^1)} |F^{(n)}(\lambda)|$$

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$$\psi(z) = \sum_{j=1}^{n} \frac{c_j}{z-\lambda_j}, \quad \Lambda(\psi) = \sup_{F \in b(H^1)} \left| \sum_{j=1}^{n} c_j F(\lambda_j) \right|$$

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Extremal functions

$$\Lambda(\psi) := \sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, d\zeta \right|$$

- How do we compute $\Lambda(\psi)$?
- Can (how do) we compute $F_e \in b(H^1)$ so that

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Some history

Theorem (Fejér - 1927)

$$\Lambda\left(\frac{c_0}{z}+\cdots+\frac{c_n}{z^{n+1}}\right)=\left\|\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n & \\ c_2 & \cdots & c_n & & \\ \vdots & \iddots & & & \\ c_n & & & & \end{pmatrix}\right\|.$$

Theorem (Egerváry - 1928)

1.

$$\Lambda\left(\frac{1}{z}+\frac{1}{z^2}+\cdots+\frac{1}{z^{n+1}}\right)=\frac{1}{2}\sec\frac{(n+1)\pi}{2n+3}$$

2.

$$\Lambda=\frac{1}{2\pi i}\oint_{\partial\mathbb{D}}\left(\frac{1}{z}+\frac{1}{z^2}+\cdots+\frac{1}{z^{n+1}}\right)F_e\,dz,$$

$$F_e=\frac{4}{2n+3}\left[\sin\frac{(n+1)\pi}{2n+3}+z\sin\frac{n\pi}{2n+3}+\cdots+z^n\sin\frac{\pi}{2n+3}\right]^2.$$

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Theorem (Golusin - 1945)

For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$,

$$\sup_{F \in b(H^1)} |F^{(n)}(\lambda)| = \Lambda\left(\frac{n!}{(z-\lambda)^{n+1}}\right) = \frac{n!}{(1-|\lambda|^2)^{n+1}}\|T\|,$$

where T is the $(n+1)\times(n+1)$ lower triangular Toeplitz matrix

$$T_{ij} = \begin{cases} |\lambda|^{i-j}\binom{n+1}{i-j} & \text{if } i \geqslant j, \\ 0 & \text{if } i < j. \end{cases}$$

Example

n = 0

$$|F(\lambda)| \leqslant \frac{1}{(1-|\lambda|^2)} \|F\|_{H^1}, \quad F \in H^1.$$

* Shown independently by Macintyre and Rogosinski

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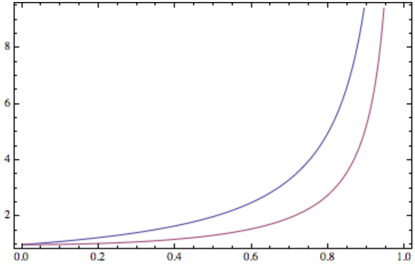

Figure : $\frac{1}{1-r}$ (blue) versus $\frac{1}{1-r^2}$ (purple)

Example

$\mathbf{n = 1}$

$$|F'(\lambda)| \leqslant \frac{|\lambda| + \sqrt{1 + |\lambda|^2}}{(1 - |\lambda|^2)^2} \|F\|_{H^1} \quad F \in H^1.$$

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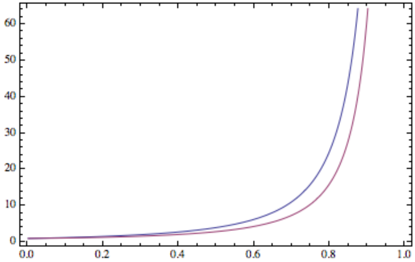

Figure : $\frac{1}{(1-r)^2}$ (blue) versus $\frac{r+\sqrt{1+r^2}}{(1-r^2)^2}$ (purple)

Example

n = 2

$$|F''(\lambda)| \leqslant \frac{2!}{(1-|\lambda|^2)^3} \left\| \begin{pmatrix} 1 & 0 & 0 \\ 3|\lambda| & 1 & 0 \\ 3|\lambda|^2 & 3|\lambda| & 1 \end{pmatrix} \right\| \|F\|_{H^1}, \quad F \in H^1.$$

Example

n = 3

$$|F'''(\lambda)| \leqslant \frac{3!}{(1-|\lambda|^2)^4} \left\| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\lambda| & 1 & 0 & 0 \\ 6|\lambda|^2 & 4|\lambda| & 1 & 0 \\ 4|\lambda|^3 & 6|\lambda|^2 & 4|\lambda| & 1 \end{pmatrix} \right\| \|F\|_{H^1} \quad F \in H^1.$$

Theorem (Golusin - 1945)

$$\Lambda\left(\frac{n!}{(z-r)^{n+1}}\right) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{n!}{(z-r)^{n+1}} F_e \, dz,$$

$$F_e = e^{i\theta}\frac{1-r^2}{(1-rz)^2}\left[\alpha_0 + \alpha_1 \frac{z-r}{1-rz} + \cdots + \alpha_n \left(\frac{z-r}{1-rz}\right)^n\right]^2.$$

Theorem (Macintyre, Rogosinski, Khavinson, Shapiro – 1945/50/53)

1. *For ψ rational (no poles on $\partial\mathbb{D}$),*

$$\Lambda(\psi) = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \psi F_e \, d\zeta,$$

   where $F_e \in b(H^1)$ and rational (not necessarily unique).

2. *F_e can be chosen to be outer.*

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More general problem

General L^∞ extremal problem

Let $\psi \in L^\infty(\partial\mathbb{D})$ and define

$$\Lambda(\psi) := \sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, d\zeta \right|$$

(Again) note that

$$\Lambda(\psi) \leqslant \|\psi\|_\infty := \operatorname{ess\,sup}_{\zeta \in \partial\mathbb{D}} |\psi(\zeta)|$$

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Linear versus quadratic

Theorem (Chalendar, Fricain, Timotin - 2009)

For $\psi \in L^\infty(\partial\mathbb{D})$,

$$\sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, d\zeta \right| = \sup_{f \in b(H^2)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi f^2 \, d\zeta \right|.$$

Computer calculations

Example

$$\Lambda\left(\frac{1}{z}+\frac{1}{z-1/2}\right)=\frac{1}{6}(7+\sqrt{37})$$

$$F_e(z)=\frac{z\sqrt{2-\frac{11}{\sqrt{37}}}}{z-2}-\frac{\sqrt{2+\frac{10}{\sqrt{37}}}}{z-2}$$

Functional analysis

The algebra H^∞

$$H^\infty := \left\{ f \in \mathcal{O}(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)| := \|f\|_\infty < \infty \right\}.$$

Theorem (Fatou – 1906)

If $f \in H^\infty$, then

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exists for almost every $\zeta \in \partial\mathbb{D}$. Moreover,

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This allows us to think

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From here to H^∞

Theorem (Khavinson (1949), Rogosinski-Shapiro (1953))

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$$\sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \psi F \, d\zeta \right| = dist(\psi, H^\infty).$$

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Operator theory

Back to Fejér for a moment:

$$\Lambda\left(\frac{c_0}{z} + \cdots + \frac{c_n}{z^{n+1}}\right) = \left\| \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_n & \\ c_2 & \cdots & c_n & & \\ \vdots & \iddots & & & \\ c_n & & & & \end{pmatrix} \right\|.$$

A Hankel matrix comes into the picture here! This is no accident.

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Hankel operators

For $\psi \in L^\infty(\partial\mathbb{D})$ define the *Hankel operator*

$$\mathfrak{H}_\psi : H^2 \to L^2 \ominus H^2,$$

$$\mathfrak{H}_\psi f = P_-(\psi f).$$

Note that

$$[\mathfrak{H}_\psi]^{\{\bar z, \bar z^2, \cdots\}}_{\{1, z, z^2, \cdots\}} = (\widehat{\psi}(-j-k-1))_{0 \leqslant j,k<\infty}.$$

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Norm and essential norm as distance problems

Theorem (Nehari – 1957)

If $\psi \in L^\infty(\partial\mathbb{D})$*, then*

$$\|\mathfrak{H}_\psi\| = dist(\psi, H^\infty).$$

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The Goldbach conjecture

Hardy-Littlewood

Hardy-Littlewood circle method wants bounds on

$$\left| \int_{\mathfrak{m}} f(x)^2 e(-nx)\, dx \right|,$$

$$e(x) = e^{2\pi i x},$$

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Attention: We do not have a solution to the Goldbach conjecture!

Moral justification

$$\left| \int_{\mathfrak{m}} f(x)^2 e(-nx)\, dx \right|$$

But we do have

$$\begin{aligned} \sup_{f \in b(H^2)} \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \psi f^2 \, d\zeta \right| &= \sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \psi F \, d\zeta \right| \\ &= \operatorname{dist}(\psi, H^\infty) \\ &= \|\mathfrak{H}_\psi\| \end{aligned}$$

Fourier extremal problem

Moral justification

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This makes the case for studying

$$\psi = \overline{\zeta}^n \chi_E, \quad n \in \mathbb{Z}, E \subset \partial\mathbb{D},$$

that is,

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Some trivial facts

1. $|E| = 0 \Rightarrow \mathrm{dist}(\overline{\zeta}^n \chi_E, H^\infty) = 0, n \in \mathbb{Z}$
2. $|E| = 2\pi \Rightarrow \mathrm{dist}(\overline{\zeta}^n \chi_E, H^\infty) = 1, n \in \mathbb{N}$
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Example

Suppose $E \subset \partial\mathbb{D}$ with $0 < |E| < 2\pi$. Then

$$\operatorname{dist}(\chi_E, H^\infty) \leqslant \frac{1}{2}.$$

Proof.

$$\operatorname{dist}(\chi_E, H^\infty) = \inf_{g \in H^\infty} \|\chi_E - g\|_\infty \leqslant \|\chi_E - \frac{1}{2}\|_\infty = \frac{1}{2}.$$

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Theorem (S. Ja. Khavinson - 1986)

If $E \subset \partial\mathbb{D}$, $0 < |E| < 2\pi$, is a finite union of arcs, then

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Theorem (Chalendar, Garcia, R, Timotin - 2013)

If $E \subset \partial\mathbb{D}$ *with* $0 < |E| < 2\pi$, *then*

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The difficult case: $n \geqslant 1$

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If $E \subset \partial\mathbb{D}$, $0 < |E| < 2\pi$, *then*

$$\max\left(\frac{1}{2}, \frac{|E|}{2\pi}\right) \leqslant dist(\overline{\zeta}^n \chi_E, H^\infty) < 1, \quad n \geqslant 1.$$

Proof.

$$\text{dist}(\overline{\zeta}^n \chi_E, H^\infty) \geqslant \text{dist}(\chi_E, H^\infty) = \frac{1}{2}.$$

$$\begin{aligned} \text{dist}(\overline{\zeta}^n \chi_E, H^\infty) &= \sup_{F \in b(H^1)} \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \overline{\zeta}^n \chi_E F \, d\zeta \right| \\ &\geqslant \left| \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \overline{\zeta}^n \chi_E \zeta^{n-1} \, d\zeta \right| \\ &= \frac{|E|}{2\pi} \end{aligned}$$

The difficult case: $n \geqslant 1$

Theorem (Chalendar, Garcia, R, Timotin - 2013)

If $E \subset \partial\mathbb{D}$, $0 < |E| < 2\pi$, then

$$\max\left(\frac{1}{2}, \frac{|E|}{2\pi}\right) \leqslant \mathit{dist}(\overline{\zeta}^n \chi_E, H^\infty) < 1, \quad n \geqslant 1.$$

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A better estimate

Theorem (Chalendar, Garcia, R, Timotin - 2013)

If $E \subset \partial\mathbb{D}$ and $n \geqslant 1$ then

$$dist(\overline{\zeta}^n \chi_E, H^\infty) \geqslant \int_{-\pi}^{\pi} \chi_E(e^{ix}) \frac{\sin^2(nx/2)}{n \sin^2(x/2)} \frac{dx}{2\pi}.$$

Proof.

Truncated Toeplitz operators and complex symmetric operators. □

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A better estimate - examples

Corollary

If $\alpha \in (0, \pi)$ *and* $I_\alpha = \widehat{(e^{-i\alpha}, e^{i\alpha})}$*, then*

1. $dist(\chi_{I_\alpha}, H^\infty) = \frac{1}{2}$
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Compare this to the previous (crude) estimate

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A better estimate - examples

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A precise answer when $n = 1$

Theorem (Chalendar, Garcia, R, Timotin - 2013)

If $\alpha \in (0, \pi)$ *and* $I_\alpha = \widehat{(e^{-i\alpha}, e^{i\alpha})}$*, then*

$$dist(\bar{\zeta}\chi_{I_\alpha}, H^\infty) = \frac{1}{2}\sec(\frac{\pi\alpha}{\pi + 2\alpha}).$$

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If $\alpha \in (0, \pi)$ *and* $I_\alpha = \widehat{(e^{-i\alpha}, e^{i\alpha})}$, *then*

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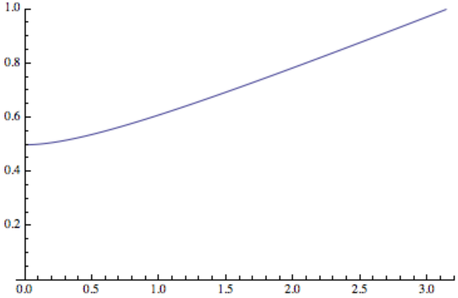

Figure : Plot of $\frac{1}{2}\sec(\frac{\pi\alpha}{\pi+2\alpha})$, $0 < \alpha < \pi$

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If $\alpha \in (0, \pi)$ *and* $I_\alpha = \widehat{(e^{-i\alpha}, e^{i\alpha})}$, *then*

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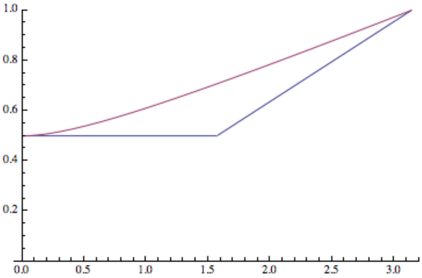

Figure : $\frac{1}{2}\sec(\frac{\pi\alpha}{\pi+2\alpha})$ (purple) versus $\max(1/2, \alpha/\pi)$ (blue)

Corollary

If $E \subset \partial\mathbb{D}$ *with* $0 < |E| < 2\pi$ *and* $|\partial E| = 0$*, then*

$$dist(\bar{\zeta}\chi_E, H^\infty) = \frac{1}{2}\sec\left(\frac{\pi|E|/2}{\pi + |E|}\right).$$

How we got there

Fix $n \geqslant 1$, $\alpha \in (0, \pi)$, and $I := I_\alpha = \widehat{(e^{-i\alpha}, e^{i\alpha})}$.

By means of a conformal mapping argument, there is a $\phi \in H^\infty$ and an $0 < r < 1$ such that

$$\overline{\zeta}^n \chi_I - \phi \quad \text{has constant modulus } r \text{ on } \partial\mathbb{D}$$

Then

$$\operatorname{dist}(\overline{\zeta}^n \chi_I, H^\infty) = \operatorname{dist}(\overline{\zeta}^n \chi_I - \phi, H^\infty)$$

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$$\begin{aligned} \operatorname{dist}(\overline{\zeta}^n \chi_I, H^\infty) &= \operatorname{dist}(\overline{\zeta}^n \chi_I - \phi, H^\infty) \\ &= \|\mathfrak{H}_{\overline{\zeta}^n \chi_I - \phi}\| \quad \text{(Nehari)} \\ &= \|\overline{\zeta}^n \chi_I - \phi\|_\infty \quad \text{(constant modulus)} \\ &= r \end{aligned}$$

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Then

$$\begin{aligned} \operatorname{dist}(\overline{\zeta}^n \chi_I, H^\infty) &= \operatorname{dist}(\overline{\zeta}^n \chi_I - \phi, H^\infty) \\ &= \|H_{\overline{\zeta}^n \chi_I - \phi}\| \quad \text{(Nehari)} \\ &= \|\overline{\zeta}^n \chi_I - \phi\|_\infty \quad \text{(constant modulus)} \\ &= r \end{aligned}$$

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\end{aligned}$$

The creation of a $\phi \in H^\infty$ and $0 < r < 1$ such that

$$\overline{\zeta}^n \chi_I - \phi \quad \text{has constant modulus } r \text{ on } \partial\mathbb{D}$$

needs the creation of a $\Phi \in A(\mathbb{D})$ and an $0 < r < 1$ with

1. $\Phi(\partial\mathbb{D} \setminus I) \subset r\partial\mathbb{D}$
2. $\Phi(I) \subset 1 + r\partial\mathbb{D}$
3. Φ has a zero of order n at 0.

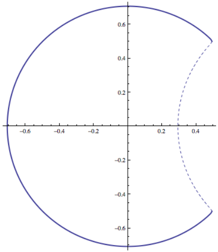


Figure : $\Phi(\partial\mathbb{D} \setminus I) \subset r\partial\mathbb{D}$ (solid) and $\Phi(I) \subset 1 + r\partial\mathbb{D}$ (dashed).

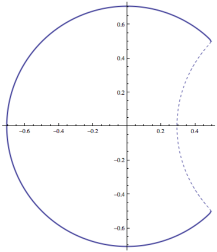
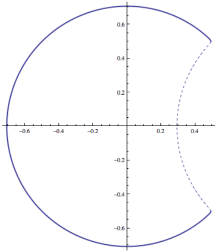


Figure : $\Phi(\partial\mathbb{D} \setminus I) \subset r\partial\mathbb{D}$ (solid) and $\Phi(I) \subset 1 + r\partial\mathbb{D}$ (dashed).

Notice that $\overline{\zeta}^n(\Phi - \chi_I)$ has constant modulus r on $\partial\mathbb{D}$. Then

$$\overline{\zeta}^n(\Phi - \chi_I) = \overline{\zeta}^n(\zeta^n\phi - \chi_I) = \phi - \overline{\zeta}^n\chi_I.$$

When $n = 1$ we can actually compute r and Φ!



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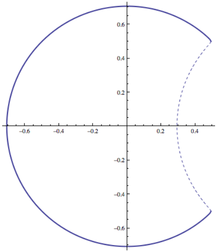

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We need dent domains: For fixed $n \geqslant 1$ and $r \in [\frac{1}{2}, 1]$ define

$$\mathfrak{D}_{n,r} := \mathbb{D} \setminus \left\{ \left| z^n - \frac{1}{r} \right| \leqslant 1, -\frac{\pi}{2n} \leqslant \arg z \leqslant \frac{\pi}{2n} \right\}.$$

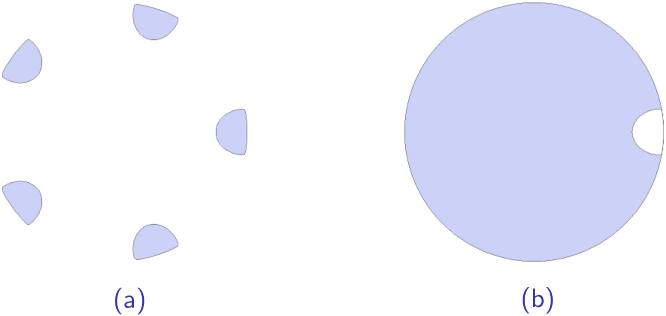

Figure : (a) The components (shaded) of the pre-image of $D(1/r, 1)$ of the mapping $z \mapsto z^n$. (a) The region (shaded) $\mathfrak{D}_{n,r}$.

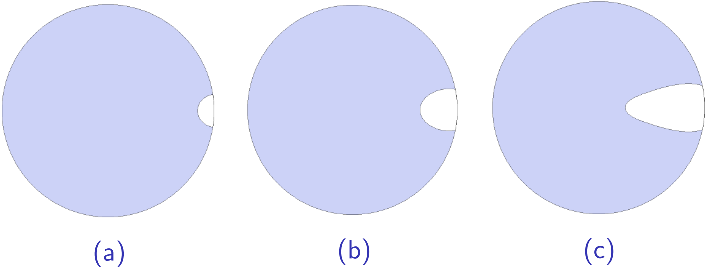

Figure : (a) $\mathfrak{D}_{5,.07}$, (a) $\mathfrak{D}_{5,0.9}$, (c) $\mathfrak{D}_{5,0.999}$

Adjust the $r = r_{n,\alpha}$ so that there is a conformal map $\varphi : \mathbb{D} \to \mathfrak{D}_{n,r}$ which maps $e^{-i\alpha}$, $e^{i\alpha}$ to the "corners" of $\mathfrak{D}_{n,r}$. Then

$$\Phi := r\varphi^n.$$

Thanks for coming and listening!