Boundary values in the range of a truncated Toeplitz operator

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Theorem (Blaschke, 1915)

Suppose

\[ \{a_n\}_{n \geq 1} \subset \mathbb{D} \setminus \{0\}, \quad \sum_{n \geq 1} (1 - |a_n|) < \infty. \]

Then

\[ B(z) := \prod_{n \geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z} \]

is analytic on \( \mathbb{D} \) and \( B^{-1}(\{0\}) = \{a_n\}_{n \geq 1} \). Moreover,

\[ |B(z)| < 1 \quad (z \in \mathbb{D}) \quad |B(\zeta)| = 1 \quad (a.e. \ \zeta \in \partial \mathbb{D}). \]
Blaschke products - analytic continuation

Corollary

\[ B(z) = \prod_{n \geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z} \]

is analytic on

\[ \mathbb{C} \setminus \text{clos} \left\{ \frac{1}{a_n} : n \geq 1 \right\}. \]
Blaschke products - analytic continuation

\[ \sigma(B) = \left\{ \lambda \in \text{clos}(\mathbb{D}) : \lim_{z \to \lambda} |B(z)| = 0 \right\} = \text{clos}\{a_n : n \geq 1\} \]

Corollary

B has an analytic continuation across \( \partial \mathbb{D} \setminus \sigma(B) \).
Blaschke products - analytic continuation

$$\sigma(B) = \left\{ \lambda \in \text{clos}(\mathbb{D}) : \lim_{z \to \lambda} |B(z)| = 0 \right\} = \text{clos}\{a_n : n \geq 1\}$$

**Corollary**

$B$ has an analytic continuation across $\partial \mathbb{D} \setminus \sigma(B)$. 
Theorem (Frostman, 1939)

If $\zeta \in \partial \mathbb{D}$, then $B$, and all its sub-products, have non-tangential limits of modulus 1 at $\zeta \iff$

$$\sum_{n \geq 1} \frac{1 - |a_n|}{|\zeta - a_n|} < \infty.$$
Blaschke products - angular derivatives

Definition

\( B \) has an \textit{angular derivative} at \( \zeta \in \partial \mathbb{D} \) if

\[
\angle \lim_{z \to \zeta} B(z) = \eta \in \partial \mathbb{D}, \quad \angle \lim_{z \to \zeta} B'(z) \text{ exists.}
\]

Theorem (Frostman, 1939)

\( B \) has an angular derivative at \( \zeta \in \partial \mathbb{D} \) \( \iff \)

\[
\sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty.
\]

Moreover

\[
\angle \lim_{z \to \zeta} B'(z) = \eta \zeta \sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.
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Blaschke products - angular derivatives

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$B$ has an *angular derivative* at $\zeta \in \partial D$ if

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$B$ has an angular derivative at $\zeta \in \partial D$ $\iff$

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Moreover

$$\angle \lim_{z \to \zeta} B'(z) = \eta \overline{\zeta} \sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2}. $$
Model spaces

\[(BH^2)^\perp = \{ f \in H^2 : \langle f, Bh \rangle = 0, \forall h \in H^2 \}\]

Note that

\[
\langle Bh, \frac{1}{1 - a_j z} \rangle = (Bh)(a_j) = 0, \quad h \in H^2
\]

Proposition

\[(BH^2)^\perp = \bigvee \left\{ \frac{1}{1 - a_j z} : n \geq 1 \right\}^a
\]

\[^a\text{Powers needed when mult}(a_j) > 1.\]
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\(^a\text{Powers needed when } \text{mult}(a_j) > 1.\)
A orthonormal basis for $(BH^2)^\perp$

\[\gamma_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-a_n z} \prod_{k=1}^{n-1} b_{a_k}(z).\]

\[b_\lambda(z) = \frac{z-\lambda}{1-\lambda z}\]

**Theorem (Takanaka-Walsh)**

\[\{\gamma_n : n \geq 1\} \text{ is an orthonormal basis for } (BH^2)^\perp.\]
Reproducing kernels

\[ k_\lambda = \frac{1 - B(\lambda)B(z)}{1 - \overline{\lambda}z} \]

- \( f(\lambda) = \langle f, k_\lambda \rangle, \quad f \in (BH^2)^\perp \)
- \( P_B : L^2 \rightarrow (BH^2)^\perp \)
- \( (P_B g)(\lambda) = \langle g, k_\lambda \rangle, \quad g \in L^2 \)
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Theorem (Douglas-Shapiro-Shields-1970)

For $f \in H^2$ the following are equivalent:

1. $f \in (BH^2)^\perp$.
2. $f/B$ has a pseudocontinuation to an $F \in H^2(\mathbb{D}_e)$ with $F(\infty) = 0$.
3. $f \in H^2 \cap BH^2_0$. 
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Analytic continuation

\[ \sigma(B) = \left\{ \lambda \in \text{clos}(\mathbb{D}) : \lim_{z \to \lambda} |B(z)| = 0 \right\} = \text{clos}\{a_n : n \geq 1\} \]

Theorem (Livsic, Moeller, 1946)

1. Every \( f \in (BH^2)^\perp \) has an analytic continuation across \( \partial \mathbb{D} \setminus \sigma(B) \)
2. If \( A_z f = P_B(zf) \), then \( \sigma(A_z) = \sigma(B) \).

\[ \sigma_e(A_z) = \sigma(B) \cap \partial \mathbb{D} \]
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What happens on $\sigma(B)$?

Theorem (Ahern-Clark - 1970)

For $\zeta \in \partial \mathbb{D}$, TFAE:

1. Every $f \in (BH^2)\perp$ has a finite nt-limit at $\zeta$.
2. $P_B 1 \in \text{Rng}(I - \overline{\zeta} A_z)$
3. $B$ has a finite angular derivative at $\zeta$.
4. $k_\zeta(z) := \frac{1 - \overline{B(\zeta)}B(z)}{1 - \overline{\zeta}z} \in H^2$
5. $\sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty$.
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What happens on $\sigma(B)$ when $|B'(\zeta)| = \infty$?

Recall Ahern-Clark: Every $f \in (BH^2)^\perp$ has a finite non-tangential limit at $1 \iff$

$$|B'(1)| = \sum_{n \geq 1} \frac{1 - |a_n|^2}{|1 - a_n|^2} < \infty.$$
$|B'(\zeta)| = \infty$

Let

$$a_n = r_n e^{i\theta_n}, \quad \theta_n = \frac{1}{2^n}, \quad 1 - r_n = x_n \theta_n^2, \quad x_n \to 0.$$
\[ |B'(\zeta)| = \infty \]

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\[ |B'(1)| = \sum_{n \geq 1} \frac{1 - |a_n|^2}{1 - |a_n|^2} \asymp \sum_{n \geq 1} x_n. \]
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\[ \sum_{n \geq 1} \frac{1 - |a_n|^2}{|1 - a_n|^2} \ll \sum_{n \geq 1} x_n. \]

**Theorem (Hartmann-R, 2011)**

1. \( x_n = \frac{1}{n} \Rightarrow |f(r)| \lesssim \sqrt{\log \log \frac{1}{1 - r}}, \quad f \in (BH^2)^\perp \)

2. \( x_n = \frac{1}{n \log n} \Rightarrow |f(r)| \lesssim \sqrt{\log \log \log \frac{1}{1 - r}}, \quad f \in (BH^2)^\perp \)
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\[ |B'(\zeta)| = \infty \]

Compare this to the well-known estimate

\[ |f(r)| = o\left(\frac{1}{\sqrt{1-r}}\right), \quad f \in H^2. \]
\[ |B'(\zeta)| = \infty \]

\[ a_n = r_ne^{i\theta_n}, \quad \sum_{n \geq 1} \theta_n < \infty, \quad 1 - r_n = \theta_n^2 \]
\(|B'(\zeta)| = \infty\)

\[a_n = r_n e^{i \theta_n}, \quad \sum_{n \geq 1} \theta_n < \infty, \quad 1 - r_n = \theta_n^2\]

**Theorem (Hartmann-R, 2011)**

1. \(\theta_n = \frac{1}{n^\alpha}, \alpha > 1 \Rightarrow |f(r)| \lesssim \frac{1}{(1 - r)^{1/2\alpha}}, \quad f \in (BH^2)^\perp\)

2. \(\theta_n = 2^{-n^{1/\alpha}}, \alpha > 0 \Rightarrow |f(r)| \lesssim \left(\log \frac{1}{1 - r}\right)^\alpha, \quad f \in (BH^2)^\perp\)
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Theorem (Ahern-Clark - 1970)

For $\zeta \in \partial \mathbb{D}$, TFAE:

1. Every $f \in (BH^2)^\perp$ has a finite nt-limit at $\zeta$.
2. $B$ has a finite angular derivative at $\zeta$.
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   \[ \sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty. \]
4. 
   \[ k_\zeta(z) := \frac{1 - \overline{B(\zeta)}B(z)}{1 - \overline{\zeta}z} \in H^2 \]
5. $P_B 1 \in \text{Rng}(I - \overline{\zeta}A_z)$
The Cauchy integral formula

\[ \langle f, (I - \bar{\lambda}A_z)^{-1}P_B 1 \rangle = \langle f, \sum_{n \geq 0} \bar{\lambda}^n A_z^n P_B 1 \rangle \]

\[ = \langle f, \sum_{n \geq 0} \bar{\lambda}^n A_z^n P_B 1 \rangle \quad (A_z^n = A_{z^n}) \]

\[ = \langle f, P_B(\frac{1}{1 - \bar{\lambda}z})P_B 1 \rangle \]

\[ = \langle f, \frac{1}{1 - \bar{\lambda}z} P_B 1 \rangle \]

\[ = \langle f, \frac{1}{1 - \bar{\lambda}z} (P_B 1 - 1) \rangle + \langle f, \frac{1}{1 - \bar{\lambda}z} 1 \rangle \]

\[ = f(\lambda) \]
The Cauchy integral formula

Conclusion:

\[ k_\lambda = (I - \bar{\lambda}A_z)^{-1}P_B 1 \]

So to determine when

\[ f(\lambda) \to f(\zeta) \quad \text{for all } f \in (BH^2)\perp \]

we need to prove that

\[ (I - \bar{\lambda}A_z)^{-1}P_B 1 \to k_\zeta \quad \text{weakly} \]
The main driver of all this

Lemma (Ahern-Clark)

Suppose

1. $\xi \in \partial \mathbb{D}$
2. $L : \mathcal{H} \rightarrow \mathcal{H}$ with $\|L\| \leq 1$
3. $(I - \xi L)$ is injective
4. $\lambda_n \rightarrow \xi$ non-tangentially (important!)

Then, for a fixed $y \in \mathcal{H}$,

$$\sup_{n \geq 1} \|(I - \lambda_n L)^{-1} y\| < \infty \Leftrightarrow y \in Rng(I - \xi L),$$

in which case,

$$(I - \lambda_n L)^{-1} y \rightarrow (I - \xi L)^{-1} y \text{ weakly.}$$
An interpolation problem

So suppose \( k_\lambda \to k_\zeta \) weakly. Then

\[
k_\zeta = (I - \bar{\zeta} A_z)^{-1} P_B 1
\]

\[
P_B 1 = (I - \bar{\zeta} A_z) k_\zeta
= k_\zeta - \bar{\zeta} A_z k_\zeta
= k_\zeta - \bar{\zeta} z k_\zeta + B u
= (1 - \bar{\zeta} z) k_\zeta + B u
\]
The above argument can be reversed. Thus

$$\angle \lim_{z \to \zeta} f(z) = f(\zeta) \quad \forall f \in (BH^2)^\perp$$

$$\iff$$

$$P_B 1 = (1 - \bar{\zeta} z) k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2.$$
The Ahern-Clark solution

To solve:

\[ P_B 1 = (1 - \bar{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2. \]

Set

\[ k = \sum_{n \geq 1} \gamma_n(\bar{\zeta}) \gamma_n, \quad \text{(which is really } k_\zeta) \]

\[ \gamma_n(z) := \frac{\sqrt{1 - |a_n|^2}}{1 - \bar{a}_n z} \prod_{k=1}^{n-1} b_{a_k}(z). \]
Our starting point

Look at

\[ f(\lambda) = \langle f, (I - \overline{\lambda}A_z)^{-1}P_B 1 \rangle. \]

What happens when we change this to

\[ \langle f, (I - \overline{\lambda}A_z)^{-1}P_B h \rangle, \quad h \in H^\infty? \]
\[
\langle f, (I - \lambda A_z)^{-1} P_B h \rangle = \langle f, \sum_{n \geq 0} \frac{\lambda^n}{1 - \lambda z} A_z^n P_B h \rangle \\
= \langle f, P_B \left( \frac{1}{1 - \lambda z} P_B h \right) \rangle \\
= \langle f, \frac{1}{1 - \lambda z} P_B h \rangle \\
= \langle f, \frac{1}{1 - \lambda z} (P_B h - h) \rangle + \langle f, \frac{1}{1 - \lambda z} h \rangle \\
= \langle \bar{h} f, \frac{1}{1 - \lambda z} \rangle \\
= \langle \bar{h} f, B \left( P + \bar{B} \frac{1}{1 - \lambda z} \right) + k \lambda \rangle \\
= (P_B \bar{h} f)(\lambda) : = (A_{\bar{h}} f)(\lambda)
\]
Conclusion:

\[ k^h_\lambda = (I - \ov{\lambda} A_z)^{-1} P_B h \]

satisfies

\[ \langle f, k^h_\lambda \rangle = (A_{\ov{\lambda}} f)(\lambda). \]

So to determine when

\[ \angle \lim_{\lambda \to \zeta} (A_{\ov{\lambda}} f)(\lambda) \]

exists for every \( f \in (BH^2)^\perp \)

we need to determine when

\[ k^h_\lambda \to k^h_\zeta \text{ weakly.} \]
Towards the main result

- \( A_{\bar{h}}f = P_B(\bar{h}f) \)
- \( \langle f, (I - \bar{\lambda}A_z)^{-1}P_Bh \rangle = (A_{\bar{h}}f)(\lambda) \)
- \( k_{\chi}^{h} = (I - \bar{\lambda}A_z)^{-1}P_Bh \)
- When does \( k_{\chi}^{h} \rightarrow k_{\zeta}^{h} \) weakly?
- When can we solve

\[
P_Bh = (1 - \bar{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2?
\]
Towards the main result

- $A_{hf} = P_B(hf)$
- $\langle f, (I - \bar{\lambda}A_z)^{-1}P_B h \rangle = (A_{hf})(\lambda)$
- $k^h_{\lambda} = (I - \bar{\lambda}A_z)^{-1}P_B h$
- When does $k^h_{\lambda} \rightarrow k^h_{\xi}$ weakly?
- When can we solve

$$P_B h = (1 - \bar{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2?$$
Towards the main result

- $A_{\bar{h}}f = P_B(hf)$
- $\langle f, (I - \bar{\lambda}Az)^{-1}P_Bh \rangle = (A_{\bar{h}}f)(\lambda)$
- $k_\lambda^h = (I - \bar{\lambda}Az)^{-1}P_Bh$
- When does $k_\lambda^h \rightarrow k_\zeta^h$ weakly?
- When can we solve
  \[
P_Bh = (1 - \bar{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2?
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Towards the main result

- \( A_{\bar{h}} f = P_B(\bar{h} f) \)
- \( \langle f, (I - \bar{\lambda} A_z)^{-1} P_B h \rangle = (A_{\bar{h}} f)(\lambda) \)
- \( k_{\chi}^h = (I - \bar{\lambda} A_z)^{-1} P_B h \)
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P_B h = (1 - \bar{\zeta} z) k + B u, \quad k \in (BH^2)^\perp, \quad u \in H^2?
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Towards the main result

- \( A_{\overline{h}} f = P_B(\overline{h} f) \)
- \( \langle f, (I - \overline{\lambda}A_z)^{-1} P_B h \rangle = (A_{\overline{h}} f)(\lambda) \)
- \( k^h_\lambda = (I - \overline{\lambda}A_z)^{-1} P_B h \)
- When does \( k^h_\lambda \to k^h_\zeta \) weakly?
- When can we solve

\[
P_B h = (1 - \overline{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2?
\]
Our solution

To solve

\[ P_B h = (1 - \bar{\zeta} z)k + Bu, \quad k \in (B H^2)^\perp, \quad u \in H^2 \]

Set

\[ k = \sum_{n \geq 1} (A_{\bar{h} \gamma n})(\zeta) \gamma_n \quad (\text{which is } k_{\zeta}^h) \]

Note: \( A_{\bar{h} \gamma n} \) is a rational function so \( (A_{\bar{h} \gamma n})(\zeta) \) is well defined.
The main result

**Theorem (Hartmann-R, 2011)**

Suppose $B$ is a Blaschke product with zeros $\{a_n\}_{n \geq 1}$ and $h \in H^\infty$. Then every function in $\text{Rng} A_h$ has a finite non-tangential limit at $\zeta \in \partial \mathbb{D}$ if and only if

$$\sum_{n \geq 1} |(A_h \gamma_n)(\zeta)|^2 < \infty.$$ 

Note then when $h = 1$, we get

$$\sum_{n \geq 1} \left| \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} \zeta} \prod_{k=1}^{n-1} b_{a_k}(\zeta) \right|^2 = \sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$
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\]
Note that

$$A_{\overline{h}}(BH^2)^\perp \subset (BH^2)^\perp$$

It is difficult to see that

$$\sum_{n \geq 1} |(A_{\overline{h}\gamma_n})(\zeta)|^2 < \infty$$

is better than Ahern-Clark

$$\sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty.$$
Theorem (Hartmann-R, 2011)

Suppose $B$ is an interpolating\textsuperscript{a} Blaschke product with zeros \( \{a_n\}_{n \geq 1} \) and $h \in H^\infty$. Then every function in $\text{Rng}A_h$ has a finite non-tangential limit at $\zeta \in \partial \mathbb{D}$ if and only if

$$\sum_{n \geq 1} (1 - |a_n|^2) \left| \frac{h(a_n)}{\zeta - a_n} \right|^2 < \infty.$$ 

\textsuperscript{a}Can fail if $B$ is not interpolating
We can still work with non-interpolating sequences - on a case by case basis.

**Theorem (Hartmann-R, 2011)**

Suppose

\[ a_n = 1 - \frac{1}{2^n \beta}, \quad \beta > 1/2 \]

Suppose

\[ h(z) = (1 - z)^{1-\epsilon}, \quad \epsilon < 1/2 \]

Then every function in \( RngA_h \) has a finite nt limit at 1.

---

\(^a\) This sequence is not a finite union of interpolating sequences.

\(^b\) \( \angle \lim_{z \to 1} B(z) = 0 \)
Suppose $\Theta$ is inner and consider $(\Theta H^2)^\perp$

1. What happens when $\Theta$ is not a Blaschke product?
2. When happens when $A_{\overline{h}}$, $h \in H^\infty$, is replaced by $A_h$, $h \in L^\infty$?
General questions

Suppose $\Theta$ is inner and consider $(\Theta H^2)^\perp$

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Suppose $\Theta$ is inner and consider $(\Theta H^2)\perp$

1. What happens when $\Theta$ is not a Blaschke product?
2. When happens when $A_{\bar{h}}, \ h \in H^\infty$, is replaced by $A_h, \ h \in L^\infty$?
Proposition

For inner $\Theta$, $h \in H^\infty$, and $\zeta \in \partial \mathbb{D}$, TFAE

1. $\angle \lim_{\lambda \to \zeta} (A_{\overline{h}} f)(\lambda)$ exists for every $f \in (\Theta H^2)^\perp$.
2. $P_{\Theta} h \in \text{Rng}(I - \overline{\zeta} A_z)$.
3. $k^h_\lambda = (I - \overline{\lambda} A_z)^{-1} P_{\Theta} h$ is norm bounded as $\lambda \to \zeta$ nt.
4. There is a $k \in (\Theta H^2)^\perp$ and $u \in H^2$ so that

$$P_{\Theta} h = (1 - \overline{\zeta} z) k + \Theta u.$$
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An attempt to solve the interpolation problem

\[ P_\Theta h = (1 - \zeta z)k + \Theta u, \quad k \in (\Theta H^2)^\perp, \quad u \in H^2 \]

Let \( \{\phi_n : n \geq 1\} \) be an orthonormal basis for \( (\Theta H^2)^\perp \).

Try \[ k = \sum_{n \geq 1} (\overline{A_h \phi_n})(\zeta)\phi_n. \]

Can you see the problem? (Hint: There are two of them!)
Evangelizing

For $\phi \in L^2$, 

$$A_\phi f = P_\Theta (\phi f), \quad f \in (\Theta H^2) \perp \cap H^\infty \text{ (dense)}$$

$$\mathcal{T}_\Theta := \{A_\phi : \phi \in L^2, A_\phi \text{ is bounded}\}$$

- $\mathcal{T}_\Theta$ is weakly closed
- Clark unitary operators are in $\mathcal{T}_\Theta$
- $\mathcal{T}_\Theta$ are complex symmetric. Do they model every CSO?
- Weak and ultra-weak topologies on $\mathcal{T}_\Theta$ are the same
- Sometimes every $A \in \mathcal{T}_\Theta$ has a bounded symbol (but not always)
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