Boundary values in the range of a truncated Toeplitz operator

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Blaschke products - classic theorem

Theorem (Blaschke, 1915)

Suppose

$$\{a_n\}_{n\geqslant 1} \subset \mathbb{D} \setminus \{0\}, \quad \sum_{n\geqslant 1}(1-|a_n|) < \infty.$$

Then

$$B(z) := \prod_{n\geqslant 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\overline{a_n}z}$$

is analytic on $\mathbb{D}$ *and* $B^{-1}(\{0\}) = \{a_n\}_{n\geqslant 1}$. *Moreover,*

$$|B(z)| < 1 \;\; (z \in \mathbb{D}) \qquad |B(\zeta)| = 1 \;\; (a.e.\ \zeta \in \partial\mathbb{D}).$$

Blaschke products - analytic continuation

Corollary

$$B(z) = \prod_{n \geqslant 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is analytic on

$$\mathbb{C} \setminus clos\left\{\frac{1}{\overline{a_n}} : n \geqslant 1\right\}.$$

Blaschke products - analytic continuation

$$\sigma(B) = \left\{ \lambda \in \operatorname{clos}(\mathbb{D}) : \varliminf_{z \to \lambda} |B(z)| = 0 \right\} = \operatorname{clos}\{a_n : n \geqslant 1\}$$

Corollary

B has an analytic continuation across $\partial\mathbb{D} \setminus \sigma(B)$.

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Blaschke products - non-tangential limits

Theorem (Frostman, 1939)

If $\zeta \in \partial\mathbb{D}$, then B, and all its sub-products, have non-tangential limits of modulus 1 at ζ $\Leftrightarrow$

$$\sum_{n \geqslant 1} \frac{1 - |a_n|}{|\zeta - a_n|} < \infty.$$

Blaschke products - angular derivatives

Definition

B has an *angular derivative* at $\zeta \in \partial\mathbb{D}$ if

$$\angle \lim_{z \to \zeta} B(z) = \eta \in \partial\mathbb{D}, \quad \angle \lim_{z \to \zeta} B'(z) \text{ exists.}$$

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B has an angular derivative at $\zeta \in \partial\mathbb{D}$ $\Leftrightarrow$

$$\sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty.$$

Moreover

$$\angle \lim_{z \to \zeta} B'(z) = \eta\overline{\zeta} \sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$

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Model spaces

$$(BH^2)^\perp = \{f \in H^2 : \langle f, Bh \rangle = 0, \forall h \in H^2\}$$

Note that

$$\left\langle Bh, \frac{1}{1 - \overline{a_j} z} \right\rangle = (Bh)(a_j) = 0, \quad h \in H^2$$

Proposition

$$(BH^2)^\perp = \bigvee \left\{ \frac{1}{1 - \overline{a_j} z} : n \geqslant 1 \right\}^a$$

[a] Powers needed when $\text{mult}(a_j) > 1$.

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A orthonormal basis for $(BH^2)^\perp$

$$\gamma_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} b_{a_k}(z).$$

$$b_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}$$

Theorem (Takanaka-Walsh)

$\{\gamma_n : n \geqslant 1\}$ *is an orthonormal basis for* $(BH^2)^\perp$.

Reproducing kernels

- $k_\lambda = \dfrac{1 - \overline{B(\lambda)}B(z)}{1 - \overline{\lambda}z}$
- $f(\lambda) = \langle f, k_\lambda \rangle, \quad f \in (BH^2)^\perp$
- $P_B : L^2 \to (BH^2)^\perp$
- $(P_B g)(\lambda) = \langle g, k_\lambda \rangle, \quad g \in L^2$

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Pseudocontinuations

Theorem (Douglas-Shapiro-Shields-1970)

For $f \in H^2$ the following are equivalent:

1. $f \in (BH^2)^\perp$.
2. f/B has a pseudocontinuation to an $F \in H^2(\mathbb{D}_e)$ with $F(\infty) = 0$.
3. $f \in H^2 \cap B\overline{H_0^2}$.

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Analytic continuation

$$\sigma(B) = \left\{ \lambda \in \text{clos}(\mathbb{D}) : \underline{\lim}_{z \to \lambda} |B(z)| = 0 \right\} = \text{clos}\{a_n : n \geqslant 1\}$$

Theorem (Livsic, Moeller, 1946)

1. *Every $f \in (BH^2)^\perp$ has an analytic continuation across $\partial\mathbb{D} \setminus \sigma(B)$*
2. *If $A_z f = P_B(zf)$, then $\sigma(A_z) = \sigma(B)$.*

$$\sigma_e(A_z) = \sigma(B) \cap \partial\mathbb{D}$$

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What happens on $\sigma(B)$?

Theorem (Ahern-Clark - 1970)

For $\zeta \in \partial\mathbb{D}$, TFAE:

1. *Every $f \in (BH^2)^\perp$ has a finite nt-limit at ζ.*
2. $P_B 1 \in Rng(I - \overline{\zeta} A_z)$
3. *B has a finite angular derivative at ζ.*
4. $$k_\zeta(z) := \frac{1 - \overline{B(\zeta)}B(z)}{1 - \overline{\zeta} z} \in H^2$$
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What happens on $\sigma(B)$ when $|B'(\zeta)| = \infty$?

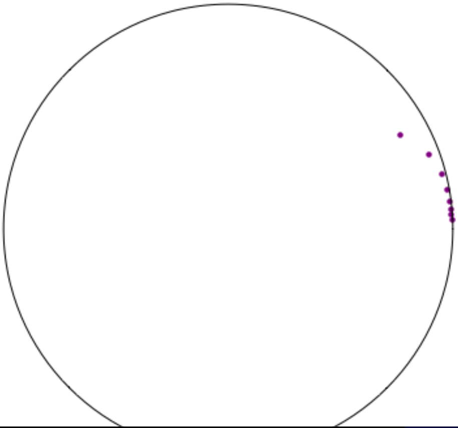
Recall Ahern-Clark: Every $f \in (BH^2)^\perp$ has a finite non-tangential limit at 1 $\Leftrightarrow$

$$|B'(1)| = \sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|1 - a_n|^2} < \infty.$$

$|B'(\zeta)| = \infty$

Let

$$a_n = r_n e^{i\theta_n}, \quad \theta_n = \frac{1}{2^n}, \quad 1 - r_n = x_n \theta_n^2, \quad x_n \to 0.$$



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Theorem (Hartmann-R, 2011)

1. $x_n = \frac{1}{n} \Rightarrow |f(r)| \lesssim \sqrt{\log\log\frac{1}{1-r}}, \quad f \in (BH^2)^\perp$
2. $x_n = \frac{1}{n\log n} \Rightarrow |f(r)| \lesssim \sqrt{\log\log\log\frac{1}{1-r}} \quad f \in (BH^2)^\perp$

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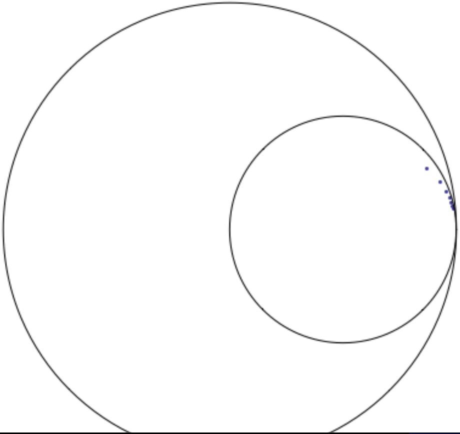
$|B'(\zeta)| = \infty$

Compare this to the well-known estimate

$$|f(r)| = o\left(\frac{1}{\sqrt{1-r}}\right), \quad f \in H^2.$$

$|B'(\zeta)| = \infty$

$$a_n = r_n e^{i\theta_n}, \quad \sum_{n \geqslant 1} \theta_n < \infty, \quad 1 - r_n = \theta_n^2$$



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Theorem (Hartmann-R, 2011)

1. $\theta_n = \dfrac{1}{n^\alpha}, \alpha > 1 \Rightarrow |f(r)| \lesssim \dfrac{1}{(1-r)^{1/2\alpha}}, \quad f \in (BH^2)^\perp$
2. $$\theta_n = 2^{-n^{1/\alpha}}, \alpha > 0 \Rightarrow |f(r)| \lesssim \left(\log \frac{1}{1-r}\right)^\alpha, \quad f \in (BH^2)^\perp$$

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Back to Ahern-Clark

Theorem (Ahern-Clark - 1970)

For $\zeta \in \partial\mathbb{D}$, TFAE:

1. *Every* $f \in (BH^2)^\perp$ *has a finite nt-limit at* ζ.
2. B *has a finite angular derivative at* ζ.
3. $$\sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty.$$
4. $$k_\zeta(z) := \frac{1 - \overline{B(\zeta)} B(z)}{1 - \overline{\zeta} z} \in H^2$$
5. $P_B 1 \in Rng(I - \overline{\zeta} A_z)$

The Cauchy integral formula

$$\begin{aligned}
\langle f, (I - \overline{\lambda} A_z)^{-1} P_B 1 \rangle &= \langle f, \sum_{n \geqslant 0} \overline{\lambda}^n A_z^n P_B 1 \rangle \\
&= \langle f, \sum_{n \geqslant 0} \overline{\lambda}^n A_{z^n} P_B 1 \rangle \quad (A_z^n = A_{z^n}) \\
&= \langle f, P_B (\frac{1}{1 - \overline{\lambda} z} P_B 1) \rangle \\
&= \langle f, \frac{1}{1 - \overline{\lambda} z} P_B 1 \rangle \\
&= \langle f, \frac{1}{1 - \overline{\lambda} z} (P_B 1 - 1) \rangle + \langle f, \frac{1}{1 - \overline{\lambda} z} 1 \rangle \\
&= f(\lambda)
\end{aligned}$$

The Cauchy integral formula

Conclusion:

$$k_\lambda = (I - \overline{\lambda} A_z)^{-1} P_B 1$$

So to determine when

$$f(\lambda) \to f(\zeta) \quad \text{for all } f \in (BH^2)^\perp$$

we need to prove that

$$(I - \overline{\lambda} A_z)^{-1} P_B 1 \to k_\zeta \ \text{ weakly}$$

The main driver of all this

Lemma (Ahern-Clark)

Suppose

1. $\xi \in \partial\mathbb{D}$
2. $L : \mathcal{H} \to \mathcal{H}$ *with* $\|L\| \leq 1$
3. $(I - \xi L)$ *is injective*
4. $\lambda_n \to \xi$ *non-tangentially (important!)*

Then, for a fixed $y \in \mathcal{H}$,

$$\sup_{n \geq 1} \|(I - \lambda_n L)^{-1} y\| < \infty \Leftrightarrow y \in Rng(I - \xi L),$$

in which case,

$$(I - \lambda_n L)^{-1} y \to (I - \xi L)^{-1} y \quad \text{weakly.}$$

An interpolation problem

So suppose $k_\lambda \to k_\zeta$ weakly. Then

$$k_\zeta = (I - \overline{\zeta} A_z)^{-1} P_B 1$$

$$\begin{aligned} P_B 1 &= (I - \overline{\zeta} A_z) k_\zeta \\ &= k_\zeta - \overline{\zeta} A_z k_\zeta \\ &= k_\zeta - \overline{\zeta} z k_\zeta + Bu \\ &= (1 - \overline{\zeta} z) k_\zeta + Bu \end{aligned}$$

An interpolation prblem

The above argument can be reversed. Thus

$$\angle \lim_{z \to \zeta} f(z) = f(\zeta) \quad \forall f \in (BH^2)^{\perp}$$

$$\Leftrightarrow$$

$$P_B 1 = (1 - \overline{\zeta} z)k + Bu, \quad k \in (BH^2)^{\perp}, \quad u \in H^2.$$

The Ahern-Clark solution

To solve:

$$P_B 1 = (1-\overline{\zeta}z)k + Bu, \quad k \in (BH^2)^\perp, \quad u \in H^2.$$

Set

$$k = \sum_{n \geqslant 1} \overline{\gamma_n(\zeta)}\gamma_n, \quad \text{(which is really } k_\zeta\text{)}$$

$$\gamma_n(z) := \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z} \prod_{k=1}^{n-1} b_{a_k}(z).$$

Our starting point

Look at

$$f(\lambda) = \langle f, (I - \overline{\lambda} A_z)^{-1} P_B 1 \rangle.$$

What happens when we change this to

$$\langle f, (I - \overline{\lambda} A_z)^{-1} P_B h \rangle, \quad h \in H^\infty?$$

$$
\begin{aligned}
\langle f, (I-\overline{\lambda}A_z)^{-1}P_B h\rangle &= \langle f, \sum_{n\geqslant 0}\overline{\lambda}^n A_{z^n} P_B h\rangle \\
&= \langle f, P_B(\frac{1}{1-\overline{\lambda}z}P_B h)\rangle \\
&= \langle f, \frac{1}{1-\overline{\lambda}z}P_B h\rangle \\
&= \langle f, \frac{1}{1-\overline{\lambda}z}(P_B h - h)\rangle + \langle f, \frac{1}{1-\overline{\lambda}z}h\rangle \\
&= \langle \overline{h}f, \frac{1}{1-\overline{\lambda}z}\rangle \\
&= \langle \overline{h}f, B(P_+\overline{B}\frac{1}{1-\overline{\lambda}z}) + k_\lambda\rangle \\
&= (P_B\overline{h}f)(\lambda) := (A_{\overline{h}}f)(\lambda)
\end{aligned}
$$

Conclusion:

$$k_\lambda^h = (I - \overline{\lambda} A_z)^{-1} P_B h$$

satisfies

$$\langle f, k_\lambda^h \rangle = (A_{\overline{h}} f)(\lambda).$$

So to determine when

$$\angle \lim_{\lambda \to \zeta} (A_{\overline{h}} f)(\lambda) \quad \text{exists for every } f \in (BH^2)^\perp$$

we need to determine when

$$k_\lambda^h \to k_\zeta^h \quad \text{weakly.}$$

Towards the main result

- $A_{\overline{h}} f = P_B(\overline{h} f)$
- $\langle f, (I - \overline{\lambda} A_z)^{-1} P_B h \rangle = (A_{\overline{h}} f)(\lambda)$
- $k_\lambda^h = (I - \overline{\lambda} A_z)^{-1} P_B h$
- When does $k_\lambda^h \to k_\zeta^h$ weakly?
- When can we solve

$$P_B h = (1 - \overline{\zeta} z) k + B u, \quad k \in (BH^2)^\perp, \quad u \in H^2?$$

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Towards the main result

- $A_{\overline{h}} f = P_B(\overline{h} f)$
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Our solution

To solve

$$P_B h = (1 - \overline{\zeta} z)k + Bu, \quad k \in (BH^2)^{\perp}, \quad u \in H^2$$

Set

$$k = \sum_{n \geqslant 1} \overline{(A_{\overline{h}} \gamma_n)(\zeta)} \gamma_n \quad \text{(which is } k_\zeta^h\text{)}$$

Note: $A_{\overline{h}} \gamma_n$ is a rational function so $(A_{\overline{h}} \gamma_n)(\zeta)$ is well defined.

The main result

Theorem (Hartmann-R, 2011)

Suppose B is a Blaschke product with zeros $\{a_n\}_{n \geqslant 1}$ and $h \in H^\infty$. Then every function in $Rng A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \partial \mathbb{D}$ if and only if

$$\sum_{n \geqslant 1} |(A_{\overline{h}} \gamma_n)(\zeta)|^2 < \infty.$$

Note then when $h = 1$, we get

$$\sum_{n \geqslant 1} \left| \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} \zeta} \prod_{k=1}^{n-1} b_{a_k}(\zeta) \right|^2 = \sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$

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A check

Note that

$$A_{\overline{h}}(BH^2)^{\perp} \subset (BH^2)^{\perp}$$

It is difficult to see that

$$\sum_{n \geqslant 1} |(A_{\overline{h}}\gamma_n)(\zeta)|^2 < \infty$$

is better than Ahern-Clark

$$\sum_{n \geqslant 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} < \infty.$$

Theorem (Hartmann-R, 2011)

Suppose B is an interpolating[a] *Blaschke product with zeros $\{a_n\}_{n \geqslant 1}$ and $h \in H^\infty$. Then every function in $Rng A_{\overline{h}}$ has a finite non-tangential limit at $\zeta \in \partial \mathbb{D}$ if and only if*

$$\sum_{n \geqslant 1} (1 - |a_n|^2) \left| \frac{h(a_n)}{\zeta - a_n} \right|^2 < \infty.$$

[a] Can fail if B is not interpolating

We can still work with non-interpolating sequences - on a case by case basis.

Theorem (Hartmann-R, 2011)

Suppose

$$a_n = 1 - \frac{1}{2^{n^\beta}}, \quad \beta > 1/2^{ab}$$

Suppose

$$h(z) = (1 - z)^{1-\epsilon}, \quad \epsilon < 1/2$$

Then every function in $Rng A_{\overline{h}}$ *has a finite nt limit at* 1.

[a] This sequence is not a finite union of interpolating sequences.
[b] $\angle \lim_{z \to 1} B(z) = 0$

General questions

Suppose Θ is inner and consider $(\Theta H^2)^{\perp}$

1. What happens when Θ is not a Blaschke product?
2. When happens when $A_{\overline{h}}$, $h \in H^{\infty}$, is replaced by A_h, $h \in L^{\infty}$?

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Genreal inner functions

Proposition

For inner Θ, $h \in H^\infty$, and $\zeta \in \partial\mathbb{D}$, TFAE

1. $\angle \lim_{\lambda \to \zeta} (A_{\overline{h}} f)(\lambda)$ exists for every $f \in (\Theta H^2)^\perp$.
2. $P_\Theta h \in \operatorname{Rng}(I - \overline{\zeta} A_z)$.
3. $k_\lambda^h = (I - \overline{\lambda} A_z)^{-1} P_\Theta h$ is norm bounded as $\lambda \to \zeta$ nt.
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An attempt to solve the interpolation problem

$$P_{\Theta}h = (1 - \overline{\zeta}z)k + \Theta u, \quad k \in (\Theta H^2)^{\perp}, \quad u \in H^2$$

Let $\{\phi_n : n \geqslant 1\}$ be an orthonormal basis for $(\Theta H^2)^{\perp}$.
Try

$$k = \sum_{n \geqslant 1} \overline{(A_{\overline{h}}\phi_n)(\zeta)}\phi_n.$$

Can you see the problem? (Hint: There are two of them!)

Evangelizing

For $\phi \in L^2$,

$$A_\phi f = P_\Theta(\phi f), \quad f \in (\Theta H^2)^\perp \cap H^\infty \ \text{(dense)}$$

$$\mathscr{T}_\Theta := \{A_\phi : \phi \in L^2, A_\phi \text{ is bounded}\}$$

- $\mathscr{T}_\Theta$ is weakly closed
- Clark unitary operators are in $\mathscr{T}_\Theta$
- $\mathscr{T}_\Theta$ are complex symmetric. Do they model every CSO?
- Weak and ultra-weak topologies on $\mathscr{T}_\Theta$ are the same
- Sometimes every $A \in \mathscr{T}_\Theta$ has a bounded symbol (but not always)
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