Reproducing kernels and symmetric operators

A. Aleman, R. Martin, and W. Ross

*

Lille 2013

Main question

When are two densely defined unbounded symmetric operators on a Hilbert space unitarily equivalent?

Note about the examples

Usually when one talks about unbounded symmetric operators, one talks about

- $Tf = -(pf')' + qf$ (Sturm-Liouville)
- $Tf = -f'' + Vf$ (Schrödinger)

We want unbounded symmetric Toeplitz operators T_φ on H^2.

Note about the examples

Usually when one talks about unbounded symmetric operators, one talks about

- $Tf = -(pf')' + qf$ (Sturm-Liouville)
- $Tf = -f'' + Vf$ (Schrödinger)

We want unbounded symmetric Toeplitz operators T_φ on H^2.

Note about the examples

Usually when one talks about unbounded symmetric operators, one talks about

- $Tf = -(pf')' + qf$ (Sturm-Liouville)
- $Tf = -f'' + Vf$ (Schrödinger)

We want unbounded symmetric Toeplitz operators T_φ on H^2.

Note about the examples

Usually when one talks about unbounded symmetric operators, one talks about

- $Tf = -(pf')' + qf$ (Sturm-Liouville)
- $Tf = -f'' + Vf$ (Schrödinger)

We want unbounded symmetric Toeplitz operators T_φ on H^2.

Note about the examples

Usually when one talks about unbounded symmetric operators, one talks about

- $Tf = -(pf')' + qf$ (Sturm-Liouville)
- $Tf = -f'' + Vf$ (Schrödinger)

We want unbounded symmetric Toeplitz operators T_φ on H^2.

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,
$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Bounded Toeplitz operators

- H^2 Hardy space of the open unit disk $\mathbb{D}$
- H^∞ bounded analytic functions on $\mathbb{D}$
- For $\varphi \in H^\infty$,

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

- When is $T_{\varphi_1} \cong T_{\varphi_2}$?
- When is $T_{\varphi_1} \sim T_{\varphi_2}$?

Cowen's work

First observation: Suppose that

$$B(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D}.$$

Then

$$T_{\varphi \circ B} \cong T_{\varphi}.$$

Proof: Let

$$U : H^2 \to H^2, \quad Ug := (g \circ B)\sqrt{B'}.$$

Then $UT_{\varphi}U^* = T_{\varphi \circ B}$

Cowen's work

First observation: Suppose that

$$B(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D}.$$

Then

$$T_{\varphi \circ B} \cong T_{\varphi}.$$

Proof: Let

$$U : H^2 \to H^2, \quad Ug := (g \circ B)\sqrt{B'}.$$

Then $UT_{\varphi}U^{*} = T_{\varphi \circ B}$

Cowen's work

First observation: Suppose that

$$B(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D}.$$

Then

$$T_{\varphi \circ B} \cong T_{\varphi}.$$

Proof: Let

$$U : H^2 \to H^2, \quad Ug := (g \circ B)\sqrt{B'}.$$

Then $UT_{\varphi}U^* = T_{\varphi \circ B}$

Cowen's work

First observation: Suppose that

$$B(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D}.$$

Then

$$T_{\varphi\circ B} \cong T_\varphi.$$

Proof: Let

$$U : H^2 \to H^2, \quad Ug := (g \circ B)\sqrt{B'}.$$

Then $UT_\varphi U^* = T_{\varphi\circ B}$

Cowen's work

First observation: Suppose that

$$B(z) = \frac{z - a}{1 - \overline{a}z}, \quad a \in \mathbb{D}.$$

Then

$$T_{\varphi \circ B} \cong T_\varphi.$$

Proof: Let

$$U : H^2 \to H^2, \quad Ug := (g \circ B)\sqrt{B'}.$$

Then $UT_\varphi U^* = T_{\varphi \circ B}$

Cowen's work

Proposition (C. Cowen)

Suppose B is any inner function with degree $n \in \mathbb{N} \cup \{\infty\}$ and $\varphi \in H^\infty$. Then

$$T_{\varphi \circ B} \cong T_\varphi \oplus T_\varphi \oplus \cdots \oplus T_\varphi.$$

Corollary

If B_1, B_2 are inner with same degree and $\varphi \in H^\infty$, then $T_{\varphi \circ B_1} \cong T_{\varphi \circ B_2}$.

Cowen's work

Proposition (C. Cowen)

Suppose B is any inner function with degree $n \in \mathbb{N} \cup \{\infty\}$ and $\varphi \in H^\infty$. Then

$$T_{\varphi \circ B} \cong T_\varphi \oplus T_\varphi \oplus \cdots \oplus T_\varphi.$$

Corollary

If B_1, B_2 are inner with same degree and $\varphi \in H^\infty$, then $T_{\varphi \circ B_1} \cong T_{\varphi \circ B_2}$.

Cowen's work

Theorem (Cowen)

Suppose $\varphi_1, \varphi_2 \in H^\infty$ and rational. Then TFAE:

1. $T_{\varphi_1} \cong T_{\varphi_2}$;
2. $T_{\varphi_1} \sim T_{\varphi_2}$;
3. *There is a (rational) $\psi \in H^\infty$ and BP B_1, B_2 of degree $n \in \mathbb{N}$ so that*

$$\varphi_1 = \psi \circ B_1, \quad \varphi_2 = \psi \circ B_2.$$

Cowen's work

Theorem (Cowen)

Suppose $\varphi_1, \varphi_2 \in H^\infty$ and rational. Then TFAE:

1. $T_{\varphi_1} \cong T_{\varphi_2}$;
2. $T_{\varphi_1} \sim T_{\varphi_2}$;
3. *There is a (rational) $\psi \in H^\infty$ and BP B_1, B_2 of degree $n \in \mathbb{N}$ so that*

$$\varphi_1 = \psi \circ B_1, \quad \varphi_2 = \psi \circ B_2.$$

Cowen's work

Theorem (Cowen)

Suppose $\varphi_1, \varphi_2 \in H^\infty$ and rational. Then TFAE:

1. $T_{\varphi_1} \cong T_{\varphi_2}$;
2. $T_{\varphi_1} \sim T_{\varphi_2}$;
3. *There is a (rational) $\psi \in H^\infty$ and BP B_1, B_2 of degree $n \in \mathbb{N}$ so that*

$$\varphi_1 = \psi \circ B_1, \quad \varphi_2 = \psi \circ B_2.$$

Cowen's work

Theorem (Cowen)

Suppose $\varphi_1, \varphi_2 \in H^\infty$ and rational. Then TFAE:

1. $T_{\varphi_1} \cong T_{\varphi_2}$;
2. $T_{\varphi_1} \sim T_{\varphi_2}$;
3. *There is a (rational) $\psi \in H^\infty$ and BP B_1, B_2 of degree $n \in \mathbb{N}$ so that*

$$\varphi_1 = \psi \circ B_1, \quad \varphi_2 = \psi \circ B_2.$$

Cowen's work

Theorem (Cowen)

Suppose $\varphi_1, \varphi_2 \in H^\infty$ and rational. Then TFAE:

1. $T_{\varphi_1} \cong T_{\varphi_2}$;
2. $T_{\varphi_1} \sim T_{\varphi_2}$;
3. *There is a (rational) $\psi \in H^\infty$ and BP B_1, B_2 of degree $n \in \mathbb{N}$ so that*

$$\varphi_1 = \psi \circ B_1, \quad \varphi_2 = \psi \circ B_2.$$

Unbounded Toeplitz operators

For $\varphi \in \mathcal{O}(\mathbb{D})$ define

$$T_\varphi : \mathscr{D}(T_\varphi) \subset H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

$$\mathscr{D}(T_\varphi) := \{f \in H^2 : \varphi f \in H^2\}$$

Proposition (Sarason)

$\mathscr{D}(T_\varphi) \neq \{0\} \Leftrightarrow \varphi \in N$ (Nevanlinna class).

Proposition

$\mathscr{D}(T_\varphi)$ is dense in $H^2 \Leftrightarrow \varphi \in N^+$ (Smirnov class).

Unbounded Toeplitz operators

For $\varphi \in \mathcal{O}(\mathbb{D})$ define

$$T_\varphi : \mathscr{D}(T_\varphi) \subset H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

$$\mathscr{D}(T_\varphi) := \{f \in H^2 : \varphi f \in H^2\}$$

Proposition (Sarason)

$\mathscr{D}(T_\varphi) \neq \{0\} \Leftrightarrow \varphi \in N$ (Nevanlinna class).

Proposition

$\mathscr{D}(T_\varphi)$ is dense in $H^2 \Leftrightarrow \varphi \in N^+$ (Smirnov class).

Unbounded Toeplitz operators

For $\varphi \in \mathcal{O}(\mathbb{D})$ define

$$T_\varphi : \mathscr{D}(T_\varphi) \subset H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

$$\mathscr{D}(T_\varphi) := \{f \in H^2 : \varphi f \in H^2\}$$

Proposition (Sarason)

$\mathscr{D}(T_\varphi) \neq \{0\} \Leftrightarrow \varphi \in N$ (Nevanlinna class).

Proposition

$\mathscr{D}(T_\varphi)$ is dense in $H^2 \Leftrightarrow \varphi \in N^+$ (Smirnov class).

Unbounded Toeplitz operators

For $\varphi \in \mathcal{O}(\mathbb{D})$ define

$$T_\varphi : \mathscr{D}(T_\varphi) \subset H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

$$\mathscr{D}(T_\varphi) := \{f \in H^2 : \varphi f \in H^2\}$$

Proposition (Sarason)

$\mathscr{D}(T_\varphi) \neq \{0\} \Leftrightarrow \varphi \in N$ (Nevanlinna class).

Proposition

$\mathscr{D}(T_\varphi)$ is dense in $H^2 \Leftrightarrow \varphi \in N^+$ (Smirnov class).

Unbounded Toeplitz operators

For $\varphi \in \mathcal{O}(\mathbb{D})$ define

$$T_\varphi : \mathscr{D}(T_\varphi) \subset H^2 \to H^2, \quad T_\varphi f = \varphi f.$$

$$\mathscr{D}(T_\varphi) := \{f \in H^2 : \varphi f \in H^2\}$$

Proposition (Sarason)

$\mathscr{D}(T_\varphi) \neq \{0\} \Leftrightarrow \varphi \in N$ (Nevanlinna class).

Proposition

$\mathscr{D}(T_\varphi)$ is dense in $H^2 \Leftrightarrow \varphi \in N^+$ (Smirnov class).

Unbounded Toeplitz operators

Theorem (Saranson)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in ball(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. *$\varphi = \dfrac{b}{a}$*

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Unbounded Toeplitz operators

Theorem (Sarason)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in ball(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. $\varphi = \dfrac{b}{a}$

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Unbounded Toeplitz operators

Theorem (Sarason)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in ball(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. *$\varphi = \dfrac{b}{a}$*

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Unbounded Toeplitz operators

Theorem (Sarason)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in ball(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. $\varphi = \dfrac{b}{a}$

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Unbounded Toeplitz operators

Theorem (Sarason)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in ball(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. $\varphi = \dfrac{b}{a}$

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Unbounded Toeplitz operators

Theorem (Sarason)

If $\varphi \in N^+ \setminus \{0\}$, then there are unique $a, b \in \mathrm{ball}(H^\infty)$ with

1. *a outer and $a(0) > 0$;*
2. *$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$ for m-almost every $\zeta \in \mathbb{T}$.*
3. $\varphi = \dfrac{b}{a}$

Corollary

If $\varphi \in N^+ \setminus \{0\}$, then T_φ is a closed densely defined operator on H^2 with $\mathscr{D}(T_\varphi) = aH^2$ and $\mathscr{D}(T_\varphi^*) = \mathscr{H}(b)$.

Real Smirnov functions

If p, q are inner with $p - q$ outer, then

$$\varphi := i\frac{p+q}{p-q} \in N^+$$

and $\varphi(\zeta) \in \mathbb{R}$ for m-almost every $\zeta \in \mathbb{T}$.

Definition

$N^+_{\mathbb{R}}$ is the set of N^+ functions with real boundary values.

Theorem (Helson)

$$\varphi \in N^+_{\mathbb{R}} \Leftrightarrow \varphi = i\frac{p+q}{p-q}$$

for inner p, q with $p - q$ outer. .

Real Smirnov functions

If p, q are inner with $p - q$ outer, then

$$\varphi := i\frac{p+q}{p-q} \in N^+$$

and $\varphi(\zeta) \in \mathbb{R}$ for m-almost every $\zeta \in \mathbb{T}$.

Definition

$N^+_{\mathbb{R}}$ is the set of N^+ functions with real boundary values.

Theorem (Helson)

$$\varphi \in N^+_{\mathbb{R}} \Leftrightarrow \varphi = i\frac{p+q}{p-q}$$

for inner p, q with $p - q$ outer. .

Real Smirnov functions

If p, q are inner with $p - q$ outer, then

$$\varphi := i\frac{p+q}{p-q} \in N^+$$

and $\varphi(\zeta) \in \mathbb{R}$ for m-almost every $\zeta \in \mathbb{T}$.

Definition

$N^+_{\mathbb{R}}$ is the set of N^+ functions with real boundary values.

Theorem (Helson)

$$\varphi \in N^+_{\mathbb{R}} \Leftrightarrow \varphi = i\frac{p+q}{p-q}$$

for inner p, q with $p - q$ outer. .

Real Smirnov functions

If p, q are inner with $p - q$ outer, then

$$\varphi := i\frac{p+q}{p-q} \in N^+$$

and $\varphi(\zeta) \in \mathbb{R}$ for m-almost every $\zeta \in \mathbb{T}$.

Definition

$N^+_{\mathbb{R}}$ is the set of N^+ functions with real boundary values.

Theorem (Helson)

$$\varphi \in N^+_{\mathbb{R}} \Leftrightarrow \varphi = i\frac{p+q}{p-q}$$

for inner p, q with $p - q$ outer. .

Saraso v. Helson

Note that if

$$\varphi = \frac{b}{a} \in N^+_{\mathbb{R}}$$

then

$$p = b + ia \quad \text{and} \quad q = b - ia \quad \text{are inner.}$$

$$p + q = 2b \quad \text{and} \quad p - q = 2ia \quad \text{is outer.}$$

Finally

$$\varphi = \frac{b}{a} = -i\frac{p+q}{p-q}.$$

Sarason v. Helson

Note that if

$$\varphi = \frac{b}{a} \in N^+_{\mathbb{R}}$$

then

$$p = b + ia \quad \text{and} \quad q = b - ia \quad \text{are inner.}$$

$$p + q = 2b \quad \text{and} \quad p - q = 2ia \quad \text{is outer.}$$

Finally

$$\varphi = \frac{b}{a} = -i\frac{p+q}{p-q}.$$

Sarason v. Helson

Note that if

$$\varphi = \frac{b}{a} \in N^+_{\mathbb{R}}$$

then

$$p = b + ia \quad \text{and} \quad q = b - ia \quad \text{are inner.}$$

$$p + q = 2b \quad \text{and} \quad p - q = 2ia \quad \text{is outer.}$$

Finally

$$\varphi = \frac{b}{a} = -i\frac{p+q}{p-q}.$$

Sarason v. Helson

Note that if

$$\varphi = \frac{b}{a} \in N^+_{\mathbb{R}}$$

then

$$p = b + ia \quad \text{and} \quad q = b - ia \quad \text{are inner.}$$

$$p + q = 2b \quad \text{and} \quad p - q = 2ia \quad \text{is outer.}$$

Finally

$$\varphi = \frac{b}{a} = -i\frac{p+q}{p-q}.$$

Sarason v. Helson

Note that if

$$\varphi = \frac{b}{a} \in N^+_{\mathbb{R}}$$

then

$$p = b + ia \quad \text{and} \quad q = b - ia \quad \text{are inner.}$$

$$p + q = 2b \quad \text{and} \quad p - q = 2ia \quad \text{is outer.}$$

Finally

$$\varphi = \frac{b}{a} = -i\frac{p+q}{p-q}.$$

Mapping properties

Proposition

If $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n,n), n \in \mathbb{N} \cup \{\infty\}$, then $\varphi(\mathbb{D}) \supset \mathbb{C} \setminus \mathbb{R}$ and $\varphi(\mathbb{D})$ is a connected slit domain with slits along the real axis.

Example

If

$$\varphi = i\frac{z + b_{1/2}}{z - b_{1/2}},$$

$$\varphi(\mathbb{D}) = \mathbb{C} \setminus \{(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)\}.$$

Mapping properties

Proposition

If $\varphi \in N^+_{\mathbb{R}}$ and $\operatorname{def}(T_\varphi) = (n, n), n \in \mathbb{N} \cup \{\infty\}$, then $\varphi(\mathbb{D}) \supset \mathbb{C} \setminus \mathbb{R}$ and $\varphi(\mathbb{D})$ is a connected slit domain with slits along the real axis.

Example

If

$$\varphi = i\frac{z + b_{1/2}}{z - b_{1/2}},$$

$$\varphi(\mathbb{D}) = \mathbb{C} \setminus \{(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)\}.$$

Mapping properties

Proposition

If $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n, n), n \in \mathbb{N} \cup \{\infty\}$, then $\varphi(\mathbb{D}) \supset \mathbb{C} \setminus \mathbb{R}$ and $\varphi(\mathbb{D})$ is a connected slit domain with slits along the real axis.

Example

If

$$\varphi = i\frac{z + b_{1/2}}{z - b_{1/2}},$$

$$\varphi(\mathbb{D}) = \mathbb{C} \setminus \{(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)\}.$$

Symmetric Toeplitz operators

Note that

$$\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g}dm, \quad f, g \in H^2.$$

Definition

T_φ is *symmetric* if

$$\langle T_\varphi f, g\rangle = \langle f, T_\varphi g\rangle, \quad f, g \in \mathcal{D}(T_\varphi).$$

Proposition

For $\varphi \in N^+$, T_φ is symmetric $\Leftrightarrow \varphi \in N^+_{\mathbb{R}}$.

Symmetric Toeplitz operators

Note that

$$\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g}dm, \quad f, g \in H^2.$$

Definition

T_φ is *symmetric* if

$$\langle T_\varphi f, g\rangle = \langle f, T_\varphi g\rangle, \quad f, g \in \mathcal{D}(T_\varphi).$$

Proposition

For $\varphi \in N^+$, T_φ is symmetric $\Leftrightarrow \varphi \in N^+_{\mathbb{R}}$.

Symmetric Toeplitz operators

Note that

$$\langle f, g \rangle = \int_{\mathbb{T}} f \overline{g} dm, \quad f, g \in H^2.$$

Definition

T_φ is *symmetric* if

$$\langle T_\varphi f, g \rangle = \langle f, T_\varphi g \rangle, \quad f, g \in \mathcal{D}(T_\varphi).$$

Proposition

For $\varphi \in N^+$, T_φ is symmetric $\Leftrightarrow \varphi \in N^+_{\mathbb{R}}$.

Symmetric Toeplitz operators

Note that

$$\langle f, g \rangle = \int_{\mathbb{T}} f\overline{g}\,dm, \quad f, g \in H^2.$$

Definition

T_φ is *symmetric* if

$$\langle T_\varphi f, g \rangle = \langle f, T_\varphi g \rangle, \quad f, g \in \mathcal{D}(T_\varphi).$$

Proposition

For $\varphi \in N^+$, T_φ is symmetric $\Leftrightarrow \varphi \in N^+_{\mathbb{R}}$.

Proposition

For $\varphi \in N^+_{\mathbb{R}}$,

- $n_+(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T^*_\varphi - \overline{\lambda} I)$ is constant on $\mathbb{C}_+$
- $n_-(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T^*_\varphi - \overline{\lambda} I)$ is constant on $\mathbb{C}_-$

Definition

The pair $\operatorname{def}(T_\varphi) := (n_+, n_-)$ is called the *deficiency indices* for T_φ

Proposition

When $n_+ = n_- = 0$, then T_φ is self-adjoint. When $n_+ = n_-$, then T_φ has self-adjoint extensions.

Proposition

For $\varphi \in N_{\mathbb{R}}^{+}$,

- $n_+(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T_\varphi^* - \overline{\lambda} I)$ is constant on $\mathbb{C}_+$
- $n_-(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T_\varphi^* - \overline{\lambda} I)$ is constant on $\mathbb{C}_-$

Definition

The pair $\operatorname{def}(T_\varphi) := (n_+, n_-)$ is called the *deficiency indices* for T_φ

Proposition

When $n_+ = n_- = 0$, then T_φ is self-adjoint. When $n_+ = n_-$, then T_φ has self-adjoint extensions.

Proposition

For $\varphi \in N^+_{\mathbb{R}}$,

- $n_+(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T_\varphi^* - \overline{\lambda} I)$ is constant on $\mathbb{C}_+$
- $n_-(T_\varphi) := \dim \operatorname{Rng}(T_\varphi - \lambda I)^\perp = \dim \ker(T_\varphi^* - \overline{\lambda} I)$ is constant on $\mathbb{C}_-$

Definition

The pair $\operatorname{def}(T_\varphi) := (n_+, n_-)$ is called the *deficiency indices* for T_φ

Proposition

When $n_+ = n_- = 0$, then T_φ is self-adjoint. When $n_+ = n_-$, then T_φ has self-adjoint extensions.

Model spaces

If

$$\varphi = i\frac{p+q}{p-q},$$

note that

$$\varphi - i = q\frac{2i}{p-q}, \quad \varphi + i = p\frac{2i}{p-q}$$

Thus

$$\mathrm{Rng}(T_\varphi - iI)^\perp = (qH^2)^\perp, \quad \mathrm{Rng}(T_\varphi + iI)^\perp = (pH^2)^\perp.$$

Corollary

For

$$\varphi = i\frac{p+q}{p-q}.$$

$$n_+(T_\varphi) = deg\, q, \quad n_-(T_\varphi) = deg\, p.$$

Model spaces

If

$$\varphi = i\frac{p+q}{p-q},$$

note that

$$\varphi - i = q\frac{2i}{p-q}, \quad \varphi + i = p\frac{2i}{p-q}$$

Thus

$$\mathrm{Rng}(T_\varphi - iI)^\perp = (qH^2)^\perp, \quad \mathrm{Rng}(T_\varphi + iI)^\perp = (pH^2)^\perp.$$

Corollary

For

$$\varphi = i\frac{p+q}{p-q}.$$

$$n_+(T_\varphi) = deg\, q, \quad n_-(T_\varphi) = deg\, p.$$

Model spaces

If

$$\varphi = i\frac{p+q}{p-q},$$

note that

$$\varphi - i = q\frac{2i}{p-q}, \quad \varphi + i = p\frac{2i}{p-q}$$

Thus

$$\text{Rng}(T_\varphi - iI)^\perp = (qH^2)^\perp, \quad \text{Rng}(T_\varphi + iI)^\perp = (pH^2)^\perp.$$

Corollary

For

$$\varphi = i\frac{p+q}{p-q}.$$

$$n_+(T_\varphi) = deg\, q, \quad n_-(T_\varphi) = deg\, p.$$

Model spaces

If

$$\varphi = i\frac{p+q}{p-q},$$

note that

$$\varphi - i = q\frac{2i}{p-q}, \quad \varphi + i = p\frac{2i}{p-q}$$

Thus

$$\operatorname{Rng}(T_\varphi - iI)^\perp = (qH^2)^\perp, \quad \operatorname{Rng}(T_\varphi + iI)^\perp = (pH^2)^\perp.$$

Corollary

For

$$\varphi = i\frac{p+q}{p-q}.$$

$$n_+(T_\varphi) = deg\, q, \quad n_-(T_\varphi) = deg\, p.$$

Model spaces

If

$$\varphi = i\frac{p+q}{p-q},$$

note that

$$\varphi - i = q\frac{2i}{p-q}, \quad \varphi + i = p\frac{2i}{p-q}$$

Thus

$$\mathrm{Rng}(T_\varphi - iI)^\perp = (qH^2)^\perp, \quad \mathrm{Rng}(T_\varphi + iI)^\perp = (pH^2)^\perp.$$

Corollary

For

$$\varphi = i\frac{p+q}{p-q}.$$

$$n_+(T_\varphi) = deg\, q, \quad n_-(T_\varphi) = deg\, p.$$

(n, n) Symmetric Toeplitz operators

Suppose that T_φ, $g \in N^+_{\mathbb{R}}$, with $\operatorname{def}(T_\varphi) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$. Note that

$$T^*_\varphi k_\lambda = \overline{\varphi(\lambda)} k_\lambda$$

Corollary

For $T_\varphi, \varphi \in N^+_{\mathbb{R}}$, with def$(T_\varphi) = (n, n)$. Then

$$\ker(T^*_\varphi - \lambda I) = \bigvee \{k_\eta : \varphi(\eta) = \overline{\lambda}\}.$$

(n, n) Symmetric Toeplitz operators

Suppose that T_φ, $g \in N^+_{\mathbb{R}}$, with $\operatorname{def}(T_\varphi) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$. Note that

$$T^*_\varphi k_\lambda = \overline{\varphi(\lambda)} k_\lambda$$

Corollary

For $T_\varphi, \varphi \in N^+_{\mathbb{R}}$, with def$(T_\varphi) = (n, n)$. Then

$$\ker(T^*_\varphi - \lambda I) = \bigvee \{k_\eta : \varphi(\eta) = \overline{\lambda}\}.$$

(n,n) Symmetric Toeplitz operators

Suppose that T_φ, $g \in N^+_{\mathbb{R}}$, with $\mathrm{def}(T_\varphi) = (n,n)$, $n \in \mathbb{N} \cup \{\infty\}$. Note that

$$T^*_\varphi k_\lambda = \overline{\varphi(\lambda)} k_\lambda$$

Corollary

For $T_\varphi, \varphi \in N^+_{\mathbb{R}}$, with def$(T_\varphi) = (n,n)$. Then

$$\ker(T^*_\varphi - \lambda I) = \bigvee \{k_\eta : \varphi(\eta) = \overline{\lambda}\}.$$

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^* \Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with $\operatorname{def}(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\mathrm{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \mathrm{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \mathrm{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\mathrm{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \mathrm{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \mathrm{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\text{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \text{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \text{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^*\Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Model for T_φ

$T : \mathscr{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is symmetric with $\operatorname{def}(T) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$

Let $\mathcal{K}$ be any Hilbert space of dimension n.

We say that Γ is a *model* for T if

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$ is co-analytic
- $\Gamma(\lambda) : \mathcal{K} \to \operatorname{Rng}(T - \lambda I)^\perp$ is invertible for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\Gamma(z)^* \Gamma(\lambda) : \mathcal{K} \to \mathcal{K}$ is invertible for all $\lambda, z \in \mathbb{C}_+$ or $\lambda, z \in \mathbb{C}_-$
- $\bigcap_{\Im\lambda \neq 0} \operatorname{Rng}\Gamma(\lambda) = \{0\}$

Theorem (Krein)

Every closed, densely defined, symmetric, simple T with def$(T) = (n, n)$ has a model.

Comment: Models are not unique. This is a good thing.

Comment: Models are not unique. This is a good thing.

Example

Suppose $\varphi \in N^+_{\mathbb{R}}$ with $\mathrm{def}(T_\varphi) = (1,1)$. Then φ is univalent.

$$\ker(T_\varphi^* - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}.$$

$$\gamma(\lambda) = \frac{1}{1 - \varphi^{-1}(\lambda) z}, \quad \Gamma(\lambda) = \gamma(\overline{\lambda}) \otimes 1$$

Example

- $\varphi \in N^+_{\mathbb{R}}$ and univalent with $\mathrm{def}(T_\varphi) = (1,1)$.
- Consider T_{φ^2} (which is $(2,2)$)
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2$.

Example

- $\varphi \in N^+_{\mathbb{R}}$ and univalent with $\operatorname{def}(T_\varphi) = (1,1)$.
- Consider T_{φ^2} (which is $(2,2)$)
- $\ker(T^*_{\varphi^2} - \bar{\lambda} I) = \bigvee \{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$

Example

- $\varphi \in N^+_{\mathbb{R}}$ and univalent with $\mathrm{def}(T_\varphi) = (1,1)$.
- Consider T_{φ^2} (which is $(2,2)$)
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2$.

Example

- $\varphi \in N^+_{\mathbb{R}}$ and univalent with $\operatorname{def}(T_\varphi) = (1, 1)$.
- Consider T_{φ^2} (which is $(2, 2)$)
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee \{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2$.

Example

- $\varphi \in N^+_{\mathbb{R}}$ and univalent with $\operatorname{def}(T_\varphi) = (1,1)$.
- Consider T_{φ^2} (which is $(2,2)$)
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2$.

Example

Suppose that $\varphi \in N^+_{\mathbb{R}}$ and $\operatorname{def}(T_\varphi) = (n, n)$. Then

$$\ker(T^*_\varphi - \overline{\lambda} I) = \bigvee \{k_\eta : \varphi(\eta) = \lambda\}.$$

Let

$$\{z_1(\lambda), \cdots, z_n(\lambda)\}$$

be the solutions to $\varphi(\eta) = \lambda$.
By Grauert's theorem, there are locally analytic $a_{i,j}(\lambda)$ so that

$$\gamma_j(\lambda) = \sum_{i=1}^{n} \overline{a_{i,j}(\lambda)} k_{z_i(\lambda)}$$

is co-analytic on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Gamma(\lambda) := \sum_{j=1}^{n} \gamma_j(\lambda) \otimes e_j.$$

Example

Suppose that $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n, n)$. Then

$$\ker(T_\varphi^* - \overline{\lambda} I) = \bigvee \{k_\eta : \varphi(\eta) = \lambda\}.$$

Let

$$\{z_1(\lambda), \cdots, z_n(\lambda)\}$$

be the solutions to $\varphi(\eta) = \lambda$.
By Grauert's theorem, there are locally analytic $a_{i,j}(\lambda)$ so that

$$\gamma_j(\lambda) = \sum_{i=1}^{n} \overline{a_{i,j}(\lambda)} k_{z_i(\lambda)}$$

is co-analytic on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Gamma(\lambda) := \sum_{j=1}^{n} \gamma_j(\lambda) \otimes e_j.$$

Example

Suppose that $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n, n)$. Then

$$\ker(T_\varphi^* - \overline{\lambda} I) = \bigvee \{k_\eta : \varphi(\eta) = \lambda\}.$$

Let

$$\{z_1(\lambda), \cdots, z_n(\lambda)\}$$

be the solutions to $\varphi(\eta) = \lambda$.

By Grauert's theorem, there are locally analytic $a_{i,j}(\lambda)$ so that

$$\gamma_j(\lambda) = \sum_{i=1}^{n} \overline{a_{i,j}(\lambda)} k_{z_i(\lambda)}$$

is co-analytic on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Gamma(\lambda) := \sum_{j=1}^{n} \gamma_j(\lambda) \otimes e_j.$$

Example

Suppose that $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n, n)$. Then

$$\ker(T_\varphi^* - \overline{\lambda} I) = \bigvee \{k_\eta : \varphi(\eta) = \lambda\}.$$

Let

$$\{z_1(\lambda), \cdots, z_n(\lambda)\}$$

be the solutions to $\varphi(\eta) = \lambda$.
By Grauert's theorem, there are locally analytic $a_{i,j}(\lambda)$ so that

$$\gamma_j(\lambda) = \sum_{i=1}^{n} \overline{a_{i,j}(\lambda)} k_{z_i(\lambda)}$$

is co-analytic on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Gamma(\lambda) := \sum_{j=1}^{n} \gamma_j(\lambda) \otimes e_j.$$

Example

Suppose that $\varphi \in N_{\mathbb{R}}^{+}$ and $\operatorname{def}(T_\varphi) = (n, n)$. Then

$$\ker(T_\varphi^* - \overline{\lambda} I) = \bigvee \{k_\eta : \varphi(\eta) = \lambda\}.$$

Let

$$\{z_1(\lambda), \cdots, z_n(\lambda)\}$$

be the solutions to $\varphi(\eta) = \lambda$.
By Grauert's theorem, there are locally analytic $a_{i,j}(\lambda)$ so that

$$\gamma_j(\lambda) = \sum_{i=1}^{n} \overline{a_{i,j}(\lambda)} k_{z_i(\lambda)}$$

is co-analytic on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Gamma(\lambda) := \sum_{j=1}^{n} \gamma_j(\lambda) \otimes e_j.$$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\mathrm{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\text{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Associated Hilbert space of analytic functions

- T symmetric with $\operatorname{def}(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\widehat{f}(\lambda) := \Gamma(\lambda)^* f, \quad f \in \mathcal{H}.$
- $\widehat{f}$ is a $\mathcal{K}$-valued analytic function on $\mathbb{C} \setminus \mathbb{R}$.
- $\mathcal{H}(\Gamma) := \{\widehat{f} : f \in \mathcal{H}\}$
- $\langle \widehat{f}, \widehat{g} \rangle_{\mathcal{H}(\Gamma)} := \langle f, g \rangle_{\mathcal{H}}.$
- $K_\lambda(z) := \Gamma(z)^* \Gamma(\lambda) \in \mathcal{B}(\mathcal{K}, \mathcal{K})$
- $\langle \widehat{f}(\lambda), a \rangle_{\mathcal{K}} = \langle \widehat{f}(\cdot), K_\lambda(\cdot) a \rangle_{\mathcal{H}(\Gamma)}, \quad a \in \mathcal{K}.$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

Proposition

If Γ_1 and Γ_1 are two models for T. Then there is a $B(\mathcal{K})$-valued analytic function W on $\mathbb{C} \setminus \mathbb{R}$ to that

$$f \mapsto Wf$$

is an isometric multiplier from $\mathcal{H}(\Gamma_1)$ onto $\mathcal{H}(\Gamma_2)$.

Example

- $\varphi \in N_{\mathbb{R}}^{+}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T_\varphi^* - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\mathrm{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\mathrm{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T_\varphi^* - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- $\ker(T^*_\varphi - \overline{\lambda} I) = \mathbb{C} k_{\varphi^{-1}(\lambda)}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\lambda)} \otimes 1$
- For $f \in \mathcal{H} = H^2$, $\widehat{f}(\lambda) := \Gamma(\lambda)^* f = \langle f, k_{\varphi^{-1}(\lambda)} \rangle = f(\varphi^{-1}(\lambda))$
- $\varphi : \mathbb{D} \to \mathbb{C} \setminus I$, $I \subset \mathbb{R}$.
- $\mathcal{H}(\Gamma) = H^2(\mathbb{C} \setminus I, \varphi(0))$
- $K_\lambda(z) = (1 - \overline{\varphi^{-1}(\lambda)} \varphi^{-1}(z))^{-1}$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee \{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N_{\mathbb{R}}^{+}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T_{\varphi^2}^* - \bar{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N_{\mathbb{R}}^{+}$ with $\mathrm{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T_{\varphi^2}^* - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee \{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_\varphi) = (1, 1)$
- Consider T_{φ^2}
- $\ker(T^*_{\varphi^2} - \overline{\lambda} I) = \bigvee\{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Example

- $\varphi \in N_{\mathbb{R}}^{+}$ with $\mathrm{def}(T_\varphi) = (1,1)$
- Consider T_{φ^2}
- $\ker(T_{\varphi^2}^* - \overline{\lambda} I) = \bigvee \{k_{\varphi^{-1}(\sqrt{\lambda})}, k_{\varphi^{-1}(-\sqrt{\lambda})}\}, \lambda \in \mathbb{C} \setminus \mathbb{R}$
- $\mathcal{K} = \mathbb{C}^2$
- $\Gamma(\lambda) = k_{\varphi^{-1}(\sqrt{\lambda})} \otimes e_1 + k_{\varphi^{-1}(-\sqrt{\lambda})} \otimes e_2.$
- $\widehat{f}(\lambda) = \Gamma(\lambda)^* f = (f(\varphi^{-1}(\sqrt{\lambda})), f(\varphi^{-1}(-\sqrt{\lambda}))), f \in H^2$
- $$K_\lambda(z) = \begin{pmatrix} \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(\sqrt{\lambda})}g^{-1}(-\sqrt{z})} \\ \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(\sqrt{z})} & \frac{1}{1-\overline{\varphi^{-1}(-\sqrt{\lambda})}\varphi^{-1}(-\sqrt{z})} \end{pmatrix}$$

Theorem (Aleman, Martin, R. (2013))

Suppose

- *T is symmetric and def$(T) = (n, n)$*
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- *$K_\lambda(z)$ is the kernel for $\mathcal{H}(\Gamma)$.*

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- *$V(z)$ is $\mathcal{B}(\mathcal{K})$-valued meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and analytic on $\mathbb{C}_+$ with $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.*

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R. (2013))

Suppose

- T *is symmetric and def*$(T) = (n, n)$
- $\dim \mathcal{K} = n$
- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{K}, \mathcal{H})$
- $\mathcal{H}(\Gamma) = \{\Gamma(\cdot)^* f : f \in \mathcal{H}\}$
- $K_\lambda(z)$ *is the kernel for* $\mathcal{H}(\Gamma)$.

Then

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

where

- $b(z) = (z - i)(z + i)^{-1}$
- $V(z)$ *is* $\mathcal{B}(\mathcal{K})$*-valued meromorphic on* $\mathbb{C} \setminus \mathbb{R}$ *and analytic on* $\mathbb{C}_+$ *with* $\|V(z)\| < 1$, $z \in \mathbb{C}_+$.

Theorem (Aleman, Martin, R (2013))

Furthermore, with

$$K_\lambda(z) = \Phi(z)\left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}}\right)\Phi(\lambda)^*.$$

- $$V(z) = b(z)K_i(i)^{1/2}K_i(z)^{-1}K_{-i}(z)K_{-i}(-i)^{-1/2}$$

- $T_1 \cong T_2$ *if and only if there are constant unitary operators* Q, R *on* $\mathcal{K}$ *so that*

$$V_1(z) = QV_2(z)R, \quad z \in \mathbb{C}_+.$$

Theorem (Aleman, Martin, R (2013))

Furthermore, with

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

- $$V(z) = b(z)K_i(i)^{1/2}K_i(z)^{-1}K_{-i}(z)K_{-i}(-i)^{-1/2}$$

- $T_1 \cong T_2$ *if and only if there are constant unitary operators* Q, R *on* $\mathcal{K}$ *so that*

$$V_1(z) = QV_2(z)R, \quad z \in \mathbb{C}_+.$$

Theorem (Aleman, Martin, R (2013))

Furthermore, with

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*.$$

- $$V(z) = b(z) K_i(i)^{1/2} K_i(z)^{-1} K_{-i}(z) K_{-i}(-i)^{-1/2}$$

- $T_1 \cong T_2$ *if and only if there are constant unitary operators* Q, R *on* $\mathcal{K}$ *so that*
$$V_1(z) = Q V_2(z) R, \quad z \in \mathbb{C}_+.$$

Example

Recall if T_φ, $\varphi \in N^+_{\mathbb{R}}$, and $\operatorname{def}(T_\varphi) = (1,1)$

$$K_\lambda(z) = \frac{1}{1 - \overline{\varphi^{-1}(\lambda)}\varphi^{-1}(z)}$$

and

$$V(z) = \frac{z-i}{z+i}\sqrt{\frac{1-|\varphi^{-1}(-i)|^2}{1-|\varphi^{-1}(i)|^2}}\frac{1-\overline{\varphi^{-1}(i)}\varphi^{-1}(z)}{1-\overline{\varphi^{-1}(-i)}\varphi^{-1}(z)}.$$

Theorem

Suppose $\varphi_j \in N_{\mathbb{R}}^{+}$ *and* $def(T_{\varphi_j}) = (1,1), j = 1,2$. *Then*

$$T_{\varphi_1} \cong T_{\varphi_2} \Leftrightarrow \varphi_1 = \varphi_2 \circ \omega$$

for some $\omega \in Aut(\mathbb{D})$.

Conjecture

For $\varphi_1, \varphi_2 \in N^+_{\mathbb{R}}$ with $\operatorname{def}(T_{\varphi_1}) = \operatorname{def}(T_{\varphi_2}) = (n, n), n \in \mathbb{N} \cup \{\infty\}$,

$$T_{\varphi_1} \cong T_{\varphi_2}$$

if and only if there is a $\psi \in N^+_{\mathbb{R}}$ and a inner functions ω_1, ω_2 of order n so that

$$\varphi_1 = \psi \circ \omega_1, \quad \varphi_2 = \psi \circ \omega_2.$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. *Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$

$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. *Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$

$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. Define

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$
$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. *Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$
$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$*. Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$

$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. *Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$

$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Theorem (Livsic (1960))

Let T be symmetric with def$(T) = (n, n)$, T' be any SA extension of T, and $\{u_1, \cdots, u_n\}$ be an ON basis for $\ker(T^* - iI)$. *Define*

$$w_T(z) = \frac{z-i}{z+i} B(z)^{-1} A(z), \quad z \in \mathbb{C}_+$$

$$A(z) = [\langle (T' + iI)(T' - zI)^{-1} u_j, u_k \rangle],$$
$$B(z) = [\langle (T' - iI)(T' - zI)^{-1} u_j, u_k \rangle].$$

$$T_1 \cong T_2 \Leftrightarrow w_{T_1}(z) = Q w_{T_2}(z) R, \quad z \in \mathbb{C}_+, \quad Q, R \in \mathscr{U}(n).$$

Recall for symmetric T with $\operatorname{def}(T) = (n, n)$ we have V is a unitary invariant as is Livsic's unitary invariant $w(T)$.

Theorem (Aleman, Martin, R. (2013))

There are unitary operators Q, R so that

$$V(z) = Q w_T(z) R, \quad z \in \mathbb{C}_+.$$

Recall for symmetric T with $\text{def}(T) = (n, n)$ we have V is a unitary invariant as is Livsic's unitary invariant $w(T)$.

Theorem (Aleman, Martin, R. (2013))

There are unitary operators Q, R so that

$$V(z) = Qw_T(z)R, \quad z \in \mathbb{C}_+.$$

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\mathrm{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*$$
$$= \Phi_1(z) \left(\frac{i}{2\pi} \frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}} \right) \Phi_1(\lambda)^*$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\mathrm{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*$$
$$= \Phi_1(z) \left(\frac{i}{2\pi} \frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}} \right) \Phi_1(\lambda)^*$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\mathrm{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$K_\lambda(z) = \Phi(z)\left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}}\right)\Phi(\lambda)^*$$

$$= \Phi_1(z)\left(\frac{i}{2\pi}\frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}}\right)\Phi_1(\lambda)^*$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\mathrm{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*$$
$$= \Phi_1(z) \left(\frac{i}{2\pi} \frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}} \right) \Phi_1(\lambda)^*$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\operatorname{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$K_\lambda(z) = \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^*$$
$$= \Phi_1(z) \left(\frac{i}{2\pi} \frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}} \right) \Phi_1(\lambda)^*$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

deBranges-Rovnyak spaces

Suppose that T is symmetric with $\operatorname{def}(T) = (n, n)$. Recall

- $\Gamma : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathbb{C}^n, \mathcal{H})$
- $\mathcal{H}(\Gamma) := \{\widehat{f}(\lambda) = \Gamma(\lambda)^* f : f \in \mathcal{H}\}$
- $T \cong M = M_z$ on $\mathcal{H}(\Gamma)$

$$
\begin{aligned}
K_\lambda(z) &= \Phi(z) \left(\frac{1 - V(z)V(\lambda)^*}{1 - b(z)\overline{b(\lambda)}} \right) \Phi(\lambda)^* \\
&= \Phi_1(z) \left(\frac{i}{2\pi} \frac{I - V(z)V(\lambda)^*}{z - \overline{\lambda}} \right) \Phi_1(\lambda)^*
\end{aligned}
$$

Theorem (Aleman, Martin, R. (2013))

If $V \in \mathscr{B}_{\mathbb{C}^n}$ is extreme then $T \cong M$ on $\mathscr{K}(V)$.

One can show that V is analytic on $\mathbb{C}_+$ and meromorphic on $\mathbb{C}_-$ and

$$V(z)V(\overline{z})^* = I$$

Corollary

If V has an analytic continuation across an arc $I \subset \mathbb{R}$, then V is extreme and so $T \cong M$ on $\mathscr{K}(V)$.

Corollary

If $\varphi \in N^+_{\mathbb{R}}$ with def(T_φ) $= (1,1)$, then

$$V(z) = \frac{z-i}{z+i}\sqrt{\frac{1-|\varphi^{-1}(-i)|^2}{1-|\varphi^{-1}(i)|^2}}\,\frac{1-\overline{\varphi^{-1}(i)}\varphi^{-1}(z)}{1-\overline{\varphi^{-1}(-i)}\varphi^{-1}(z)}$$

is extreme and so $T_\varphi \cong M$ on $\mathscr{K}(V)$.

One can show that V is analytic on $\mathbb{C}_+$ and meromorphic on $\mathbb{C}_-$ and

$$V(z)V(\overline{z})^* = I$$

Corollary

If V has an analytic continuation across an arc $I \subset \mathbb{R}$, then V is extreme and so $T \cong M$ on $\mathscr{K}(V)$.

Corollary

If $\varphi \in N^+_{\mathbb{R}}$ with def$(T_\varphi) = (1,1)$, then

$$V(z) = \frac{z-i}{z+i}\sqrt{\frac{1-|\varphi^{-1}(-i)|^2}{1-|\varphi^{-1}(i)|^2}}\,\frac{1-\overline{\varphi^{-1}(i)}\varphi^{-1}(z)}{1-\overline{\varphi^{-1}(-i)}\varphi^{-1}(z)}$$

is extreme and so $T_\varphi \cong M$ on $\mathscr{K}(V)$.

One can show that V is analytic on $\mathbb{C}_+$ and meromorphic on $\mathbb{C}_-$ and

$$V(z)V(\overline{z})^* = I$$

Corollary

If V has an analytic continuation across an arc $I \subset \mathbb{R}$, then V is extreme and so $T \cong M$ on $\mathscr{K}(V)$.

Corollary

If $\varphi \in N^+_{\mathbb{R}}$ with def$(T_\varphi) = (1,1)$, then

$$V(z) = \frac{z-i}{z+i}\sqrt{\frac{1-|\varphi^{-1}(-i)|^2}{1-|\varphi^{-1}(i)|^2}}\,\frac{1-\overline{\varphi^{-1}(i)}\varphi^{-1}(z)}{1-\overline{\varphi^{-1}(-i)}\varphi^{-1}(z)}$$

is extreme and so $T_\varphi \cong M$ on $\mathscr{K}(V)$.

Herglotz spaces

Theorem

$$K_\lambda(z) = \Phi_2(z)\left(\frac{\Omega(z)+\Omega(\lambda)^*}{\pi i(\overline{\lambda}-z)}\right)\Phi_2(\lambda)^*,$$

$$\Omega(z) = (I+iV(z))(I-iV(z))^{-1}.$$

Corollary

$T \cong M$ *on* $\mathscr{H}(\Omega)$.

Herglotz spaces

Theorem

$$K_\lambda(z) = \Phi_2(z) \left(\frac{\Omega(z) + \Omega(\lambda)^*}{\pi i(\overline{\lambda} - z)} \right) \Phi_2(\lambda)^*,$$

$$\Omega(z) = (I + iV(z))(I - iV(z))^{-1}.$$

Corollary

$T \cong M$ on $\mathscr{H}(\Omega)$.

Herglotz spaces

Theorem

$$K_\lambda(z) = \Phi_2(z) \left(\frac{\Omega(z) + \Omega(\lambda)^*}{\pi i(\overline{\lambda} - z)} \right) \Phi_2(\lambda)^*,$$

$$\Omega(z) = (I + iV(z))(I - iV(z))^{-1}.$$

Corollary

$T \cong M$ *on* $\mathscr{H}(\Omega)$.