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Analysis Seminar talk (Bordawick 2011)

I Toeplitz Matrices

$T_n = n \times n$ Toeplitz matrices.

Q Does T_n contain any algebras?

$S_a = \begin{bmatrix} 0 & & a \\ 1 & & \\ & \ddots & \\ & & 1 & 0 \end{bmatrix}$ $\mathbb{I}, S_a, S_a^2, S_a^3 \dots \in T_n$
 $(|a| \leq 1)$

Conclusion: $W(S_a) = \text{span}\{\mathbb{I}, S_a, S_a^2, \dots\}$
 is an algebra in T_n .

Cor: $W(S_b^*) = \text{span}\{\mathbb{I}, S_b^*, S_b^{*2}, \dots\}$
 $(|b| \leq 1)$ is also an algebra in T_n .

Thm (Sedlack 2009): $A, B \in T_n, A, B \in T_n$
 $\Rightarrow A, B \in W(S_a)$ or $A, B \in W(S_b^*)$

Cor: $W(S_a), W(S_b^*)$ are the only maximal
 algebras in T_n .

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Notes, (i) why $|a| \leq 1$

$$W(S_a) = W(S_{1/\bar{a}}^*)$$

$$(2) W(S_{a_1}) \cap W(S_{a_2}) = \mathbb{C} \mathbb{I}$$

$$(3) W(S_{a_1}) \cap W(S_{b_2}^*) = \mathbb{C} \mathbb{I}$$

Thm, (Garcia-Ruiz 2010)

$$\{S_a\}' = W(S_a)$$

(so maximal in another way)

(3)

II Spatial Isomorphisms

Still continuity with $W(S_a), W(S_a^*) \leq T_n$

$$U = \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

(unitary)

$$U S_a U^* = S_{\bar{a}}$$

Conclusion $W(S_a) \cong W(S_{\bar{a}}^*) \quad \forall |a| \leq 1$

$$U_{|z|=1} : \begin{bmatrix} 1 & & & \\ & \xi & & \\ & \xi^2 & & \\ & & \ddots & \\ & & & \xi^{n-1} \end{bmatrix} : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

(unitary)

$$U S_a U^* = S_{\bar{a}} \bar{z}^n$$

Conclusion $W(S_a) \cong W(S_{a|z|=1}) \quad \forall |a| \leq 1$
 $|z|=1$

Thm (García-R-Wogen - 2010)

$$W(S_a) \cong W(S_{\bar{a}}^*) \quad \forall |a| \leq 1$$

$$W(S_a) \cong W(S_{a'}) \iff |a| = |a'|$$

III Beyond Toeplitz matrices.

A new way to think about Toeplitz matrices

$$H^2, (z^n H^2)^\perp = \text{span}\{1, z, \dots, z^{n-1}\}.$$

$$\varphi \in L^\infty : A_\varphi : (z^n H^2)^\perp \rightarrow (z^n H^2)^\perp$$

$$A_\varphi f = \sum_{k=0}^{n-1} (\varphi f) z^k$$

$$[A_\varphi]_{\{1, z, z^2, \dots, z^{n-1}\}} = \begin{bmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & & \\ \hat{\varphi}(1) & & & \\ & \hat{\varphi}(1) & \hat{\varphi}(0) & \\ & & \hat{\varphi}(1) & \hat{\varphi}(0) \end{bmatrix}$$

(Toeplitz matrix)

Moreover,

$$S_a = A_z + a A_{\overline{z}}$$

$$\begin{bmatrix} 0 & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} & & & a \end{bmatrix}$$

This gives us an idea on how to generalize.

Θ = inner function.

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$$\Theta(z) = e^{i\gamma} z^k \prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z} \exp\left(-\frac{\frac{z + a_n}{z - a_n}}{\pi} \mu_n\right)$$

$$\boxed{\Theta(0) = 0}$$

$\mu \geq 0, \mu \perp m.$

Model space $(\Theta H^2)^\perp$.

$$\begin{aligned} A_\varphi : (\Theta H^2)^\perp &\rightarrow (\Theta H^2)^\perp \\ A_\varphi f &= P_\Theta(\varphi f) \end{aligned}$$

$$\boxed{T_\Theta = \text{bounded TTO}} \leftarrow \text{generalization of Top. mat.}$$

Thm (Sarason 2007): T_Θ is weakly closed (weakly = WOT).

Q, What are the subalgebras of T_Θ

Note Θ = fixed for most of talk

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$$\boxed{\text{Define } S_a = A_2 + a A\left(\frac{\Theta}{2}\right)}$$

Note: 1) when $\Theta = 2^\pi$, this is what we had before.

2) When $|a| = 1$, S_a is the "clock unitary".

$$\text{Thm (Sarason 2007)}: \{S_a\}' \subseteq T_\Theta$$

$$\text{Thm (Garcia-R-Wagen-2010)} \\ W(S_a) = \{S_a\}'$$

$$\text{Thm (Sedlack 2009)}: A, B \in T_\Theta \setminus \mathbb{C}I \\ A, B \in T_\Theta \Rightarrow A, B \in W(S_a) \quad |a| \leq 1 \\ A, B \in W(S_b^*) \quad |b| \leq 1.$$

Cor: $W(S_a), W(S_a^*), |a| \leq 1, |b| \leq 1$
are the maximal weakly closed algebras in T_Θ .

$$\text{Th (Sedlack)}: |a| < 1$$

$$W(S_a) = \left\{ \frac{A_\varphi}{1-a\bar{b}} : \varphi \in H^\infty \right\}$$

Notes : 1) $W(S_0)$ is familiar via
the H^2 functional calculus
for A_2

2) $W(S_0^*)$ etc - - -

(3) Every $W(S_a)$ $|a| < 1$
has a bis symbol.

(4) When $|a| = 1$ there is no
longer true.

$$(5) \frac{A_{\psi}}{1-a\bar{\theta}} \frac{A_{\psi}}{1-a\bar{\theta}} = A_{\frac{\psi\psi}{1-a\bar{\theta}}}$$

IV

Spatial Isomorphisms again

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Recall when $\Theta = \mathbb{Z}^n$

$$W(S_a) \cong W(S_a^*)$$

$$W(S_a) \cong W(S_{a'}) \iff |a| = |a'|$$

What happens for other inner Θ ,

Thm (Gauze-Rwogen)

Let $a_1, a_2 \in \mathbb{D}$.

$$1) W(S_{a_1}) \cong W(S_{a_2})$$

$$\iff$$

$$b_{a_1} \circ \Theta = \{ b_{a_2} \circ \Theta \circ \psi$$

$$2) W(S_{a_1}^*) \cong W(S_{a_2}^*)$$

$$\iff$$

$$b_{\bar{a}_1} \circ \Theta^\# = \{ b_{\bar{a}_2} \circ \Theta^\# \circ \psi$$

$$3) W(S_{a_1}) \cong W(S_{a_2}^*)$$

$$b_{a_1} \circ \Theta \cong \{ b_{\bar{a}_2} \circ \Theta^\# \circ \psi.$$

$$\text{Cor } W(A_2) \cong W(A_2^*)$$

(9) (9.50)

$$\Leftrightarrow \Theta = \sum \Theta^{\#} \phi$$

Ex $\Theta = \overline{B.P.}$, zeros have same argument.

$$\text{Cor } \Theta(\Theta) = e^{-\frac{1+z}{1-z}}$$

$$a_1, a_2 \in \mathbb{D}$$

$$W(S_{a_1}) \cong W(S_{a_2}) \Leftrightarrow \underline{a_1 = a_2}$$

We will see in a moment how when $|a_1| = |a_2| = 1$ the situation drastically changes.

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- What happens $W(S_{a_1}), W(S_{a_2})$
 $\underline{|a_1| < 1}, \quad \underline{|a_2| = 1}?$

Con $W(S_{a_1}) \not\cong W(S_{a_2})$.

Pf: $W(S_{a_1})$ contains no non-trivial normal operators, while S_{a_2} is unitary and hence $W(S_{a_2})$ only contains normal operators.

- What happens $W(S_{a_1}), W(S_{a_2})$?
 $|a_1| = |a_2| = 1$.

Th (Clark - 1970) For $|a| = 1$

$$S_a \cong (M_2, L^2(\mu_a)) \quad \left[\begin{array}{l} \{\theta = a\} \\ \text{carrier} \\ \text{for } \mu_a \end{array} \right]$$
$$P_c \left(\frac{a+\theta}{a-\theta} \right) = \int_{\partial \mathbb{D}} P_2 \zeta_{\mu_a}$$

(11)!!

$$W(S_a) \cong L^\infty(\mu_a) \leftarrow \begin{array}{l} \text{viewed as} \\ \text{mult ops} \\ \text{on } L^2(\mu_a) \end{array}$$



Cor $|a_1| = |a_2| = 1$

$$W(S_{a_1}) \cong W(S_{a_2}) \iff \underbrace{L^{\mathcal{H}}(\mu_{a_1}) \cong L^{\mathcal{H}}(\mu_{a_2})}_{\text{viewed as mult ops.}}$$

$$K(\mu) = (\varepsilon, n) \quad \varepsilon = \begin{cases} 0 & \text{no cont} \\ 1 & \text{cont} \end{cases}$$

$n = \# \text{ of atoms}$

Thm (Helmut Von Neumann)

$$L^{\mathcal{H}}(\mu_{a_1}) \cong L^{\mathcal{H}}(\mu_{a_2})$$



$$K(\mu_{a_1}) = K(\mu_{a_2}).$$

Ex: Θ any finite B.P.,

$$W(S_{a_1}) \cong W(S_{a_2}) \quad |a_1| = |a_2| = 1$$

Pl: Clark meas are limit- ε supp.

$\{\Theta = a_{i,j}\}$ n -atoms

$$\underline{\text{Ex}}: \Theta = e^{-\frac{1+\varepsilon}{1-\varepsilon}}$$

$$|a_1| = |a_2| = 1.$$

$$W(S_{a_1}) \cong W(S_{a_2}) \quad \text{Clark meas are discrete.}$$