Making the case for Cauchy transforms

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Prologue

Cauchy transform of a measure μ

$$\int \frac{d\mu(\zeta)}{\zeta - z}$$

Theorem (Cauchy - 1831)

If f is analytic on $\{|z| < 1\}$ and continuous on $\{|z| \leqslant 1\}$, then

$$f(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z} \frac{d\theta}{2\pi}.$$

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

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Privalov, Morera, Plemelj, and Sokhotski considered

$$(K\mu)(z) = \int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z} d\mu(\theta),$$

where μ is a measure on $[0, 2\pi]$. Actually, they considered

$$(K\mu)(z) = \int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z} dF(\theta), \quad F \in BV[0, 2\pi].$$

$$\mathcal{K} = \{K\mu : \mu \text{ is a measure on } [0, 2\pi]\}$$

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Example

$\mu = \delta_0$ (point mass at $\theta = 0$).

$$\delta_0(A) = \begin{cases} 1, & \text{if } 0 \in A; \\ 0, & \text{if } 0 \notin A. \end{cases}$$

$$(K\delta_0)(z) = \int_0^{2\pi} \frac{1}{1 - e^{-i\theta} z} d\delta_0 = \frac{1}{1 - z}$$

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Example

$\mu = \sum_{j=1}^{\infty} c_j \delta_{\theta_j}$ where $\sum_{j=1}^{\infty} |c_j| < \infty$. Then

$$K\mu(z) = \sum_{j=1}^{\infty} \frac{c_j}{1 - e^{-i\theta_j} z}.$$

What I am going to talk about:

- Boundary values of $K\mu$
- Mapping properties of the transformation $\mu \to K\mu$
- Which analytic functions f can be written as $f = K\mu$?
- Distribution function for $K\mu$, i.e., $m(|K\mu| > y)$

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Boundary values

$$(K\mu)(z) = \int_0^{2\pi} \frac{1}{1 - e^{-it} z} d\mu(t)$$

$$\lim_{r \to 1^-} (K\mu)(re^{i\theta}) = ??$$

$$\mu = c_1\delta_{\theta_1} + c_2\delta_{\theta_2} + \cdots + c_n\delta_{\theta_n}$$

$$\begin{aligned}(K\mu)(z) &= \int_0^{2\pi} \frac{1}{1 - e^{-i\theta}z}\,d\mu(\theta)\\ &= \frac{c_1}{1 - e^{-i\theta_1}z} + \frac{c_2}{1 - e^{-i\theta_2}z} + \cdots + \frac{c_n}{1 - e^{-i\theta_n}z}\end{aligned}$$

$$(1-r)(K\mu)(re^{it}) = \frac{(1-r)c_1}{1 - e^{-i\theta_1}re^{it}} + \frac{(1-r)c_2}{1 - e^{-i\theta_2}re^{it}} + \cdots + \frac{(1-r)c_n}{1 - e^{-i\theta_n}re^{it}}$$

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$$(1-r)(K\mu)(re^{it}) = \frac{(1-r)c_1}{1-re^{i(t-\theta_1)}} + \frac{(1-r)c_2}{1-re^{i(t-\theta_2)}} + \cdots + \frac{(1-r)c_n}{1-re^{i(t-\theta_n)}}$$

$$\lim_{r\to 1^-}(1-r)(K\mu)(re^{it}) = \begin{cases} c_j, & \text{if } t = \theta_j; \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\lim_{r\to 1^-}|(K\mu)(re^{it})| = +\infty \quad \text{if } t = \theta_j$$

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For a general measure μ,

$$\lim_{r \to 1^-} (1-r)(K\mu)(re^{it}) = \lim_{r \to 1^-} \int \frac{1-r}{1-re^{i(t-\theta)}} d\mu(\theta) = \mu(\{t\})$$

Thus, by taking $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{\theta_n}, \quad \theta_n$ dense in $[0, 2\pi]$

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$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{1 - e^{-i\theta_j} z}, \quad \sum_{j=1}^{\infty} |c_j| < \infty$$

In a way Poincaré knew about this behavior back in 1883 while creating non-continuable functions.

..... but needed a longer proof since the Lebesgue dominated convergence theorem was not discovered yet.

Theorem (Smirnov - 1929)

For almost every (Lebesgue measure) $t \in [0, 2\pi]$,

$$K\mu(e^{it}) = \lim_{r \to 1^-} (K\mu)(re^{it})$$

exists and is finite. Moreover, for each $0 < p < 1$,

$$\int_0^{2\pi} |K\mu(e^{it})|^p \, dt < \infty$$

and in fact

$$\int_0^{2\pi} |K\mu(e^{it})|^p \, dt = O\left(\frac{1}{1-p}\right).$$

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Theorem (Fatou - 1906)

For almost every (Lebesgue measure) $t \in [0, 2\pi]$,

$$\lim_{r \to 1^-} (K\mu)(re^{it}) - \lim_{r \to 1^-} (K\mu)(\frac{1}{r} e^{it}) = D\mu(t)$$

$$D\mu(t) = \lim_{h \to 0} \frac{\mu([t-h, t+h])}{2h}$$

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Theorem (Privalov - 1919)

For almost every (Lebesgue measure) $t \in [0, 2\pi]$,

$$\lim_{r \to 1^-} (K\mu)(re^{it}) + \lim_{r \to 1^-} (K\mu)(\frac{1}{r} e^{it}) = 2P.V. \int_0^{2\pi} \frac{1}{1 - e^{i(\theta - t)}} d\mu(\theta).$$

$$\int \frac{1}{\zeta - z} d\mu(\zeta), \quad |z| < 1$$

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Mapping properties

There are a multitude of results which talk about the mapping properties of $\mu \to K\mu$.

Theorem (M. Riesz - 1927)

If $1 < p < \infty$ *and* $g(\theta) \in L^p[0, 2\pi]$ *and*

$$f(z) = \int_0^{2\pi} \frac{g(\theta)}{1 - e^{-i\theta} z} \frac{d\theta}{2\pi},$$

then

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Theorem (Hollenbeck-Verbitski - 2000)

For $1 < p < \infty$,

$$\left(\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \right)^{1/p} \leqslant \frac{1}{\sin(\pi/p)} \|g\|_{L^p}.$$

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We began this talk with the Cauchy integral formula

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f analytic on $\{|z| < 1\}$ and continuous on $\{|z| \leqslant 1\}$.

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An extension of the CIF

$$f(z) = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z} \frac{d\theta}{2\pi}$$

Ul'yanov (1956), Aleksandrov (1981):

$$f(z) = \lim_{A \to \infty} \int_{|f|<A} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z} \frac{d\theta}{2\pi}.^{1}$$

[1] Some restrictions apply. Not available in all 50 states.

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Distribution function

We know that the function

$$(K\mu)(e^{i\theta}) = \lim_{r \to 1^-} (K\mu)(re^{i\theta})$$

is defined for almost every θ.

What can we say about the distribution function

$$y \to m(|K\mu| > y)?$$

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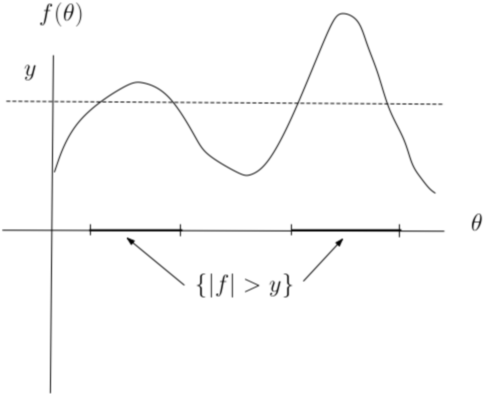
For any $f(\theta) \in L^1[0, 2\pi]$, we have

$$m(|f| > y) \leqslant \frac{1}{y} \int_0^{2\pi} |f(\theta)| \frac{d\theta}{2\pi}$$

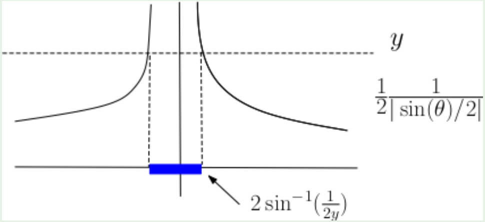
Note that

$$y\chi_{|f|>y} \leqslant |f|$$



Example

$$|K\delta_0(e^{i\theta})| = \frac{1}{|1 - e^{i\theta}|} = \frac{1}{2}\frac{1}{|\sin(\theta/2)|}$$



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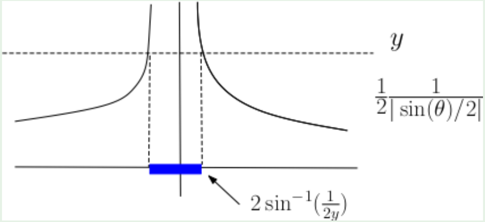

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Boole's idea:

Suppose $c_1, \cdots, c_n > 0$ and $a_1 < a_2 < \cdots < a_n$. Define

$$g(x) = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \cdots + \frac{c_n}{x - a_n}$$

$$m_1(g > y) = ?$$

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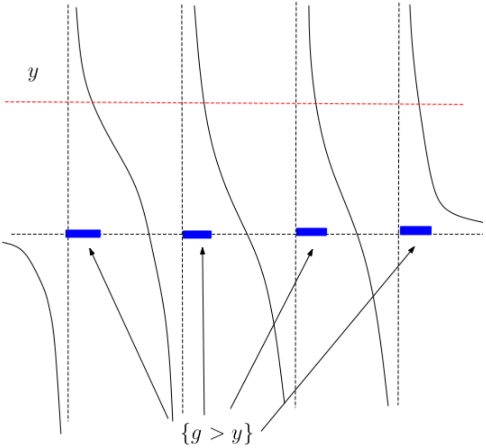


Theorem (Boole - 1857)

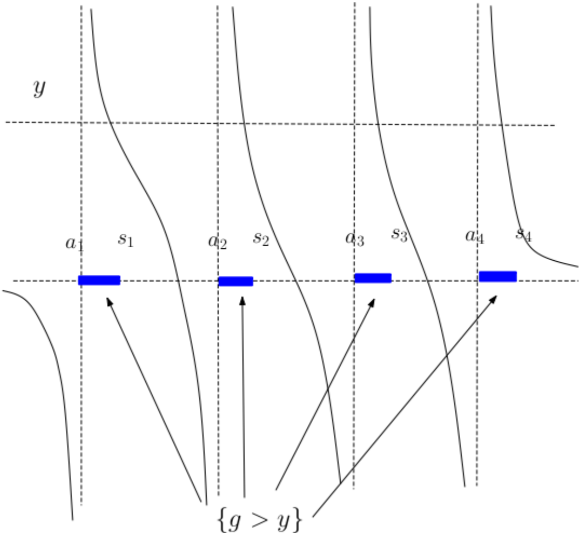
Suppose $c_1, \cdots, c_n > 0$, $a_1 < a_2 < \cdots < a_n$ *and*

$$g(x) = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \cdots + \frac{c_n}{x - a_n}.$$

Then

$$m_1(g > y) = \frac{1}{y}(c_1 + c_2 + \cdots + c_n)$$

Proof of Boole's theorem (not the original):



$$g(x) = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \cdots + \frac{c_n}{x - a_n}$$

$$m_1(g > y) = (s_1 - a_1) + (s_2 - a_2) + \cdots + (s_n - a_n)$$

Consider the polynomial

$$p(x) = \prod_{j=1}^{n} (x - a_j) \left(1 - \frac{g(x)}{y}\right)$$

The roots of this poly are $s_1, s_2, \cdots, s_n$.

$$p(x) = x^n - \left(\sum_{j=1}^{n} a_j + \frac{1}{y} \sum_{j=1}^{n} c_j\right) x^{n-1} + \cdots$$

$$\text{Viète's formula} \Rightarrow \sum_{j=1}^{n} s_j = \sum_{j=1}^{n} a_j + \frac{1}{y} \sum_{j=1}^{n} c_j$$

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Theorem (Kolmogorov - 1925)

$$m(|K\mu| > y) = O\left(\frac{1}{y}\right), \quad y \to \infty.$$

Kolmogorov says

$$\limsup_{y \to \infty} y m(|K\mu| > y) < \infty.$$

Theorem (Hruschev-Vinogradov - 1981)

$$\lim_{y \to \infty} y m(|K\mu| > y) = \frac{1}{\pi} \|\mu_s\|$$

Theorem (Poltoratski - 1996)

$$y \chi_{|K\mu| > y} \cdot m \to \frac{1}{\pi} \mu_s \quad \textit{weak-* as } y \to \infty$$

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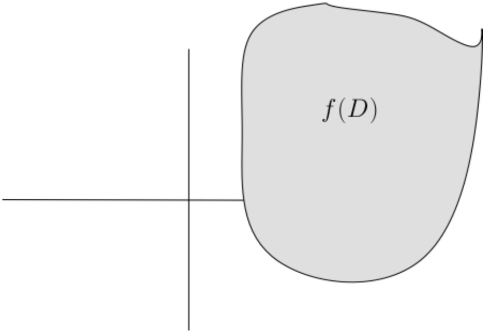
Geometric conditions

Q: If f is analytic on $\mathbb{D} = \{|z| < 1\}$. When is $f = K\mu$?

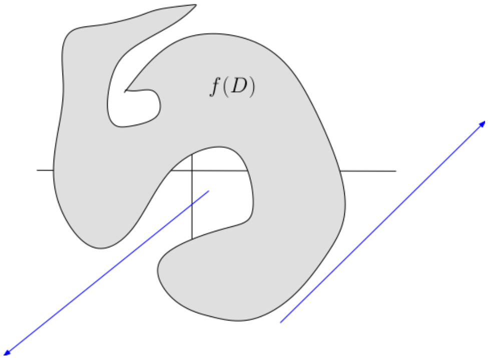
Theorem (Herglotz - 1911)

If $f(\mathbb{D})$ is contained in a half-plane, then $f = K\mu$.



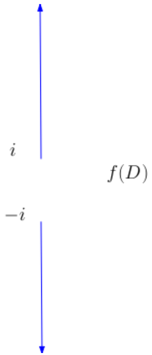
Theorem (Cima-Bourdon - 1986)

If $f(\mathbb{D})$ is contained in a region which omits two oppositely pointing rays, then $f = K\mu$.



Example

$$f(z) = \frac{2z}{1 - z^2} = \frac{1}{1 - z} - \frac{1}{1 + z} = K(\delta_0 - \delta_\pi)(z)$$



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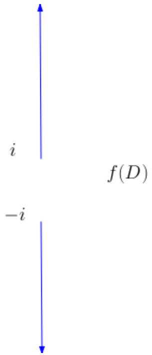

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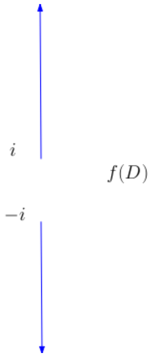

The normalized Cauchy transform

Consider the normalized Cauchy transform

$$\frac{K(fd\mu)}{K(d\mu)}$$

$$\lim_{r\to 1^-} \frac{K(fd\mu)(re^{i\theta})}{K(d\mu)(re^{i\theta})} = \lim_{r\to 1^-} \frac{(1-r)K(fd\mu)(re^{i\theta})}{(1-r)K(d\mu)(re^{i\theta})}$$

$$= \frac{f(e^{i\theta})\mu(\{\theta\})}{\mu(\{\theta\})}$$

$$= f(e^{i\theta})$$

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&= \frac{f(e^{i\theta})\mu(\{\theta\})}{\mu(\{\theta\})} \\
&= f(e^{i\theta})
\end{aligned}$$

Theorem (Poltoratski - 1993)

$$\lim_{r \to 1^-} \frac{K(f d\mu)(re^{i\theta})}{K(d\mu)(re^{i\theta})} = f(e^{i\theta}) \quad \mu_s\text{-a.e. } \theta.$$