HELSINKI LECTURES ON MODEL SPACES

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ABSTRACT. These are some notes from a series of eight lectures I gave at the University of Helsinki during the summer of 2013.

1. Schedule of the Lectures

Lecture 1. Overview

Lecture 2. Inner functions

Lecture 3. Hardy spaces

Lecture 4. Model spaces

Lecture 5. Clark theory

Lecture 6. Aleksandrov's disintegration theorem

Lecture 7. Truncated Toeplitz operators

Lecture 8. Complex symmetric operators

2. Linear algebra warm up

For $n \in \mathbb{N}$, let $Q_n := \bigvee\{z^j : 0 \leq j \leq n-1\}$ be the polynomials of degree at most $n-1$. We will imagine $Q_n \subset L^2 = L^2(\mathbb{T}, m)$, where $\mathbb{T}$ is the circle and $m = \frac{d\theta}{2\pi}$ is normalized Lebesgue measure on $\mathbb{T}$. As such, Q_n inherits an inner product

$$\langle p, q \rangle = \int_{\mathbb{T}} p\overline{q} dm = \int_0^{2\pi} p(e^{i\theta})\overline{q(e^{i\theta})} \frac{d\theta}{2\pi}.$$

With this inner product note that

$$\langle z^m, z^n \rangle = \int_0^{2\pi} e^{im\theta} e^{-in\theta} \frac{d\theta}{2\pi} = \delta_{m,n}$$

and so $\{1, z, \cdots, z^{n-1}\}$ is an orthonormal basis for Q_n.

Define the operator

$$P_n : L^2 \to Q_n;$$

$$(P_n f)(z) = \int_{\mathbb{T}} f(\zeta)(1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta).$$

A computation will show that for $1 \leq k \leq n-1$

$$\begin{aligned} P_n z^k &= \int_{\mathbb{T}} \zeta^k (1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta) \\ &= \sum_{j=0}^{n-1} z^j \langle \zeta^k, \zeta^j \rangle \\ &= z^k \end{aligned}$$

and so $P_n Q_n = Q_n$. One can check that P_n is actually the orthogonal projection of L^2 onto Q_n.

For $\varphi \in L^\infty$, let

$$A_\varphi : Q_n \to Q_n, \quad A_\varphi f := P_n(\varphi f).$$

One can check that

$$\begin{aligned} (A_\varphi z^k)(z) &= \int_{\mathbb{T}} \varphi(\zeta) \zeta^k (1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta) \\ &= \sum_{j=0}^{n-1} \left(\int_{\mathbb{T}} \varphi(\zeta) \zeta^{k-j} dm(\zeta) \right) z^j \\ &= \sum_{j=0}^{n-1} \widehat{\varphi}(j-k) z^j, \end{aligned}$$

where, for $n \in \mathbb{Z}$,

$$\widehat{\varphi}(n) := \int_{\mathbb{T}} \varphi(\zeta) \overline{\zeta^n} dm(\zeta)$$

is the n-th Fourier coefficient of φ. This means that the matrix representation of A_ϕ with respect to the orthonormal bass $\{1, z, \cdots, z^{n-1}\}$ is a *Toeplitz matrix* (constant along the diagonals). In fact, every Toeplitz matrix can be thought of in this way.

One can easily see that $\mathcal{T}_n$, the $n \times n$ Toeplitz matrices do not form an algebra. However, for any $a \in \mathbb{C}$, the matrix

$$U_a := A_z + a(1 \otimes z^{n-1})$$

is indeed a Toeplitz matrix. Indeed a simple 3×3 example is

$$U_a = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

With this simple example notice that

$$U_a^2 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 1 & 0 & 0 \end{pmatrix}, \quad U_a^3 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

which are all Toeplitz matrices. Thus, in general,

$$\{I, U_a, U_a^2, \cdots\} \subset \mathcal{T}_n.$$

A theorem of Sedlock [12] will show that the maximal algebras in $\mathcal{T}_n$ are

$$\mathcal{W}(U_a) = \bigvee\{U_a^j : j \geq 0\}$$

or

$$\mathcal{W}(U_a^*) = \bigvee\{U_a^{*j} : j \geq 0\}.$$

Thus every algebra in $\mathcal{T}_n$ is contained in one of these maximal algebras.

One can also see that if $|a| = 1$, the matrix U_α is unitary. Furthermore if $\zeta \in \mathbb{T}$ satisfies $\zeta^n = \alpha$ then,

$$k_\zeta := 1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1},$$

satisfies $k_\zeta \in Q_n$ and a matrix calculation will show that

$$U_\alpha k_\zeta = \zeta k_\zeta.$$

Thus we have computed the spectral information for U_α, i.e., its eigenvalues and eigenvectors.

By the spectral theorem from functional analysis, one knows that any cyclic unitary operator is unitarily equivalent to the operator $f \mapsto \zeta f$ (multiplication by the independent variable) on $L^2(\sigma)$ for some finite positive Borel measure σ on $\mathbb{T}$. Generally, computing this measure can be somewhat difficult. For the cyclic (easily checked) unitary operator U_α, one can compute its spectral representation in the following very interesting way: Let σ_α be the measure on the circle defined by

$$d\sigma_\alpha = \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_j},$$

where $\{\zeta_1, \cdots, \zeta_n\}$ are the n roots of $z^n = \alpha$. Define

$$Z_\alpha : L^2(\sigma_\alpha) \to L^2(\sigma_\alpha), \quad (Z_\alpha f)(\zeta) = \zeta f(\zeta)$$

and note that if χ_{ζ_j} is the characteristic function for the one point set $\{\zeta_j\}$ then

$$Z\chi_{\zeta_j} = \zeta_j \chi_{\zeta_j}, \quad j = 1, \cdots, n.$$

That is to say, χ_{ζ_j} are eigenvectors for Z_α corresponding to the eigenvalues ζ_j. Furthermore, note that

$$\{\sqrt{n}\chi_{\zeta_1}, \cdots, \sqrt{n}\chi_{\zeta_j}\}$$

is an orthonormal basis for $L^2(\sigma_\alpha)$.

If we define

$$V_\alpha \sqrt{n}\chi_{\zeta_j} = \frac{\sqrt{n}}{n} k_{\zeta_j}$$

and extend by linearity we see that V_α is a bijective linear transformation from $L^2(\sigma_\alpha)$ onto Q_n. But since $\|k_\zeta\|_{L^2(m)} = \sqrt{n}$ (Parsevel's theorem) we see that V_α is actually an isometry. Finally, since χ_{ζ_j} are the eigenvectors for Z_α and k_{ζ_j} are the eigenvectors for U_α we see that

$$U_\alpha V_\alpha = V_\alpha Z_\alpha.$$

Thus we have a concrete realization of the spectral measure and spectral representation for U_α.

So why do we care? For one, this family of measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ will satisfy the following remarkable property:

$$\int_{\mathbb{T}} \sigma_\alpha(E) dm(\alpha) = m(E)$$

for every Borel subset E of $\mathbb{T}$. In other words, they disintegrate Lebesgue measure m. These types of disintegration theorems appear in mathematical physics for the rank-one perturbations of the Schrodinger operator. Moreover, we know that for a bounded Borel function we can compute $\varphi(U_\alpha)$ via the spectral theorem, i.e.,

$$\varphi(U_\alpha) = V_\alpha \varphi(Z_\alpha) V_\alpha^*,$$

where $\varphi(Z_\alpha) f = \varphi f$ is just multiplication by φ on $L^2(\sigma_\alpha)$. Coming full circle with our Toeplitz matrices we have, via the above disintegration theorem,

$$A_\varphi = \int_{\mathbb{T}} \varphi(U_\alpha) dm(\alpha).$$

Let us make another observation. The Toeplitz matrices are not self-adjoint. But they are quite close. If we define the following conjugation on Q_n by

$$C(a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}) = \overline{a_{n-1}} + \overline{a_{n-2}} z + \cdots + \overline{a_0} z^{n-1},$$

one can quickly check that C satisfies $C^2 = I$ is isometric as well as anti-linear. One can show that

$$C A_\varphi C = A_\varphi^*.$$

In other words a Toeplitz matrix is C-symmetric. The complex symmetric operators form a very interesting set of operators which has attracted quite a lot of attention lately [9, 11].

It will turn out that there are other types of conjugations C on $\mathbb{C}^n$ and matrices, which we think of as operators from $\mathbb{C}^n$ to itself, for which

$$CAC = A^*.$$

We want to realize, via unitary equivalence, these operators as operators of the form $A_\varphi = P(\varphi f)$, where P is the orthogonal projection from L^2 onto some finite dimensional subspace of L^2 of rational functions which are analytic on $\mathbb{D}$. As an example of what we mean here, let $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$ be points of the unit disk $\mathbb{D}$ and consider the n-dimensional space of rational functions

$$Q_\Lambda := \left\{ \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{(1 - \overline{\lambda_1} z) \cdots (1 - \overline{\lambda_n} z)} : a_j \in \mathbb{C} \right\}.$$

It will turn out that the analog of P_n here is the operator

$$(P_\Lambda f)(z) = \int_{\mathbb{T}} f(\zeta) \frac{1 - B(z)\overline{B(\zeta)}}{1 - \overline{\zeta} z} dm(\zeta),$$

where

$$B(z) = \prod_{j=1}^{n} \frac{z - \lambda_j}{1 - \overline{\lambda_j} z}.$$

That is to say, P_Λ is the orthogonal projection of L^2 onto Q_Λ. Again, one can define

$$A_\varphi f = P_\Lambda(\varphi f).$$

The operators (maybe think of them as matrices if you like) are the analogs of the Toeplitz matrices and there is a natural conjugation on Q_Λ

$$C\left(\frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{(1 - \overline{\lambda_1} z) \cdots (1 - \overline{\lambda_n} z)}\right) = \frac{\overline{a_0} z^{n-1} + \overline{a_1} z^{n-2} + \cdots + \overline{a_{n-1}}}{(1 - \overline{\lambda_1} z) \cdots (1 - \overline{\lambda_n} z)}$$

for which $C A_\varphi C = A_\varphi^*$. So the question now becomes: Is *every* complex symmetric matrix A, i.e., $CAC = A^*$ for some conjugation C of $\mathbb{C}^n$) unitarily equivalent to A_φ on some Q_Λ space?

Of course all of this has an infinite dimensional setting which will be worked out in these notes.

3. Hardy spaces

3.1. **Lebesgue spaces.** Let m denote Lebesgue measure on the unit circle $\mathbb{T}$, normalized so that $m(\mathbb{T}) = 1$. By this we mean that the m-measure on the arc $I(\theta_1, \theta_2)$ subtended by e^{θ_1} and $e^{i\theta_2}$, where θ_j are measured an radians, is

$$m(I(\theta_1, \theta_2)) = \frac{\theta_2 - \theta_1}{2\pi}.$$

One often writes $dm = d\theta/2\pi$.

Let $L^2 = L^2(\mathbb{T}, m)$ the space of m-measurable functions $f : \mathbb{T} \to \mathbb{C} \cup \{\infty\}$ such that

$$\int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) < \infty.$$

Standard measure theory and functional analysis says that L^2 becomes a Hilbert space with inner product

$$\langle f, g \rangle := \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} dm(\zeta).$$

The norm on L^2 is

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta)}.$$

We leave it to the reader to check the following:

Proposition 3.1. *The family of functions $\{\zeta \mapsto \zeta^n : n \in \mathbb{Z}\}$ is an orthonormal set in L^2.*

For $f \in L^2$ and $n \in \mathbb{Z}$ we can compute

$$\widehat{f}(n) := \langle f, \zeta^n \rangle = \int_{\mathbb{T}} f(\zeta) \overline{\zeta}^n dm(\zeta)$$

which the reader will recognize as the *Fourier coefficients* of f.

This well-known theorem relates the norm of an L^2 function with its Fourier coefficients.

Theorem 3.2 (Parseval's theorem)**.** *For each $f \in L^2$,*

$$\|f\|^2 = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2.$$

Furthermore

$$\lim_{N \to \infty} \left\| f - \sum_{n=-N}^{N} \widehat{f}(n) \zeta^n \right\| = 0.$$

The previous theorem says several things. First, it says that $\{\zeta^n : n \in \mathbb{Z}\}$ is not only an orthonormal set in L^2, it is complete in L^2 and is thus an orthonormal basis. Second, it says that the Fourier series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) \zeta^n$$

for f converges to f in the norm of L^2. It is important to note that, in general, the Fourier series of an L^2 function does not always converge pointwise. However, a deep theorem of Carleson says that it does converge pointwise almost everywhere to f [1]. We will not be using this fact but a beginning student should know such delicate matters exist.

3.2. **The Hardy space.** The classic texts for this material, together with complete proofs of the main results, are [8, 13, 14]. Consider the set of all power series

$$\sum_{n=0}^{\infty} a_n z^n$$

whose coefficients satisfy

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

For any fixed $z \in \mathbb{D}$ the Cauchy-Schwarz inequality shows that

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n||z|^n &\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |z|^{2n} \right)^{1/2} \\ &= \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \frac{1}{\sqrt{1 - |z|^2}}. \end{aligned}$$

This says that such power series converge uniformly on compact subsets of $\mathbb{D}$ and thus form analytic functions on $\mathbb{D}$. This allows us to make the following definition:

Definition 3.3. The *Hardy space* H^2 is the set of all analytic functions on $\mathbb{D}$ whose power series have square summable coefficients.

If we norm H^2 by

$$\left\| \sum_{n=0}^{\infty} a_n z^n \right\|_{H^2} := \sqrt{\sum_{n=0}^{\infty} |a_n|^2},$$

then one can show that H^2 is complete vector space – in fact a Hilbert space.

There is also this useful alternate definition of H^2 involving the integral means

$$\int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta), \quad 0 < r < 1.$$

Proposition 3.4. *An analytic function f on $\mathbb{D}$ belongs to H^2 if and only if*

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty.$$

Proof. Notice that using the orthogonality of the functions $\{\zeta^n : n \in \mathbb{Z}\}$ we see that

$$\int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} f(r\zeta)\overline{f(r\zeta)} dm(\zeta) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

One direction follows from Abel's theorem while the other direction follows from Fatou's lemma. □

Corollary 3.5. *Every bounded analytic function on $\mathbb{D}$ belongs to H^2.*

Not every H^2 function is bounded. Indeed $f(z) = \log(1 - z)$ is an unbounded analytic function on $\mathbb{D}$ with

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Observe how the power series coefficients of this function are square summable and so $f \in H^2$.

3.3. **Harmonic analysis interlude.** For a continuous function $u : \mathbb{T} \to \mathbb{C}$ recall the *Poisson integral*

$$(Pu)(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} dm(\zeta), \quad z \in \mathbb{D}.$$

Writing $z = r\zeta, r \in [0, 1), \zeta \in \mathbb{T}$, one can show that

$$(Pu)(r\zeta) = \sum_{n=-\infty}^{\infty} \widehat{u}(n) r^{|n|} \zeta^n. \tag{3.6}$$

Writing $\zeta = e^{i\theta}$ and using the polar representation of the Laplacian, one can show that Pu is harmonic on $\mathbb{D}$ and a somewhat delicate limiting argument will show that

$$\lim_{r \to 1^-} (Pu)(r\zeta) = u(\zeta), \quad \zeta \in \mathbb{T}.$$

For $u \in L^2$ (not necessarily continuous) we still have the same analysis as above except the limit result. The theorem here is called Fatou's theorem [8] and says the following:

Theorem 3.7 (Fatou's Theorem)**.** *For $u \in L^2$,*

$$\lim_{r \to 1^-} (Pu)(r\zeta) = u(\zeta), \quad m\text{-almost every } \zeta \in \mathbb{T}.$$

There is a more delicate version of this theorem involving Poisson integrals of measures on $\mathbb{T}$. We will get to this later in these notes.

3.4. **Boundary values.** Let us use Fatou's theorem to discuss the boundary behavior of Hardy space functions. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Parseval's theorem says that the function u_f whose Fourier series is

$$u_f \sim \sum_{n=0}^{\infty} a_n \zeta^n$$

defines an L^2 function. Furthermore by (3.6)

$$(Pu_f)(r\zeta) = \sum_{n=0}^{\infty} a_n r^n \zeta^n = f(r\zeta).$$

Now apply Fatou's theorem (Theorem 3.7) and Parsevel's theorem (Theorem 3.2) to obtain the following summary result.

Theorem 3.8. *For $f \in H^2$ we have the following:*

1. $\lim_{r \to 1^-} f(r\zeta) := f(\zeta)$ *exists for m-almost every $\zeta \in \mathbb{T}$.*
2. $\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) = \sum_{n=0}^{\infty} |a_n|^2.$

Remark 3.9. Notice how the norm of an H^2 function is equal to the L^2 norm of its boundary function.

This circle of ideas also produces a version of the maximum modulus theorem for the Hardy space.

Theorem 3.10 (Maximum Modulus Theorem)**.** *Suppose $f \in H^2$ and $|f(\zeta)| \leq M$ for almost every $\zeta \in \mathbb{T}$. Then $|f(z)| \leq M$ for all $z \in \mathbb{D}$.*

Proof. Note that by the Poisson integral formula we have

$$f(z) = \int_{\mathbb{T}} f(\zeta) P_z(\zeta) dm(\zeta), \quad z \in \mathbb{D},$$

and so

$$|f(z)| \leq K \int_{\mathbb{T}} P_z(\zeta) dm(\zeta) = K \cdot 1 = K.$$

In the above notice how we are using the well-known fact that the Poisson kernel integrates to 1 for each $z \in \mathbb{D}$. □

This section shows that there are three seemingly different definitions of H^2: (i) analytic functions on $\mathbb{D}$ whose power series coefficients are square summable; (ii) analytic functions f whose integral means

$$\int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta)$$

are bounded in r; (iii) functions in L^2 whose Fourier coefficients vanish for $n < 0$. Much of the success of Hardy space theory stems from viewing H^2 functions in all three ways.

It is worth pointing out a Hardy space version of the classical Cauchy integral formula.

Theorem 3.11 (Cauchy integral formula)**.** *For $f \in H^2$ and $z \in \mathbb{D}$,*

$$f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\overline{\zeta}} dm(\zeta).$$

Note how the above integral makes sense since $f(\zeta)$ is well defined for m-almost every $\zeta \in \mathbb{T}$ and $\zeta \mapsto f(\zeta)$ is an L^2 function. The kernel functions

$$k_\lambda(z) := \frac{1}{1 - \overline{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

are called *reproducing kernel functions* for H^2 since

$$f(\lambda) = \langle f, k_\lambda \rangle, \quad f \in H^2,$$

that is to say, these functions reproduce the values of f at λ.

3.5. **Beurling's theorem.** The observant reader has probably already noticed that the unitary operator

$$U : \ell^2 \to H^2, \quad U((a_n)_{n \geq 0}) = \sum_{n=0}^{\infty} a_n z^n.$$

Moreover, recalling that

$$S : \ell^2 \to \ell^2, \quad S(a_0, a_1, a_2, \cdots) = (0, a_0, a_1, a_2, \cdots),$$

is the unilateral shift operator, we note that if

$$M_z : H^2 \to H^2, \quad (M_z f)(z) = z f(z),$$

an easy computation will show that

$$US = M_z U.$$

Thus S is unitarily equivalent to M_z. Thus the problem of finding the S-invariant subspaces of ℓ^2 is equivalent to finding the M_z-invariant subspaces of H^2. This will involve the concept of an inner function.

Definition 3.12. A function $\Theta \in H^2$ is said to be *inner* if

$$\lim_{r \to 1^-} \Theta(r\zeta)$$

has modulus equal to one for almost every $\zeta \in \mathbb{T}$.

By our Hardy space version of the Maximum Modulus Theorem, every inner function Θ will also satisfy $|\Theta(z)| \leq 1$ on $\mathbb{D}$. A beginner in this subject will probably want some examples of inner functions.

Example 3.13. For $a \in \mathbb{D}$ consider the function

$$\Theta(z) = \frac{z-a}{1-\overline{a}z}.$$

One can see that this function is inner in two ways. One way is to show that

$$\Theta(e^{i\theta})\overline{\Theta(e^{i\theta})} = \frac{e^{i\theta}-a}{1-\overline{a}e^{i\theta}} \frac{e^{-i\theta}-\overline{a}}{1-ae^{-i\theta}} = 1, \quad \theta \in [0, 2\pi].$$

One can also use a more delicate analysis to see that Θ is a Mobius transformation which will map the unit circle bijectively to itself. Immediate from this is another class of inner functions which are finite (and even infinite) products of these types of Mobius transformations. More about this in the next section.

Example 3.14. Another type of inner function takes the form

$$\Theta(z) = e^{\frac{z+1}{z-1}}, \quad z \in \mathbb{D}.$$

Note that

$$\left|\exp\left(\frac{z+1}{z-1}\right)\right| = \exp\left(\Re\left(\frac{z+1}{z-1}\right)\right) = \exp\left(-\frac{1-|z|^2}{|1-z|^2}\right) < 1, \quad z \in \mathbb{D}.$$

Thus Θ is a bounded analytic function on $\mathbb{D}$, and hence belongs to H^2. From the previous identity one can easily check that $|\Theta(e^{i\theta})| = 1$ if $\theta \in (0, 2\pi)$. Thus Θ is an inner function.

Remark 3.15. (1) One needs to read the definition of inner function very carefully: $\Theta \in H^2$ is inner if the radial boundary values are unimodular almost everywhere. One can't forget to check the condition $\Theta \in H^2$. For instance, the reciprocal of the function from the previous example satisfies the property that it is analytic on $\mathbb{D}$ and has unimodular boundary values. However, this function does not belong to H^2 and hence does not qualify as an inner function.

(2) If $\alpha_1, \cdots, \alpha_n$ are positive numbers and $\theta_1, \cdots, \theta_n \in [0, 2\pi]$, one can also check that

$$\Theta(z) = \exp\left(\alpha_1 \frac{z + e^{i\theta_1}}{z - e^{i\theta_1}} + \cdots + \alpha_n \frac{z + e^{i\theta_n}}{z - e^{i\theta_n}}\right)$$

is also an inner function. We will talk more about such types of inner functions in a moment.

Why are inner functions important?

Theorem 3.16 (Beurling (1949)). *If $\mathcal{M}$ is a non-zero closed subspace of H^2 which is invariant under $S = M_z$, then there is an inner function Θ such that*

$$\mathcal{M} = \Theta H^2 = \{\Theta f : f \in H^2\}.$$

Proof. First we show that $\mathcal{M}$ is a closed subspace of H^2. Notice that since Θ is inner then for any $f \in H^2$ we have

$$\|\Theta f\|^2 = \int_{\mathbb{T}} |\Theta(\zeta) f(\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) = \|f\|^2$$

and so the operator $f \mapsto \Theta f$ is an isometry of H^2 onto ΘH^2. Thus ΘH^2 is a closed subspace of H^2. The S-invariance is clear.

Now comes the hard part: showing that every (non-zero) S-invariant subspace $\mathcal{M}$ of H^2 takes the form ΘH^2 for some inner Θ. First notice that

$$S\mathcal{M} \neq \mathcal{M}.$$

If this were not the case, then $f/z \in \mathcal{M}$ whenever $f \in \mathcal{M}$. Applying this k times we conclude that

$$\frac{f}{z^k} \in \mathcal{M} \quad \forall k \in \mathbb{N}.$$

But this would mean, since f/z^k must be analytic on $\mathbb{D}$, that $f \equiv 0$, a contradiction to the assumption that $\mathcal{M} \neq \{0\}$.

Second, since $S\mathcal{M} \neq \mathcal{M}$, one observes that

$$\mathcal{M} \cap (S\mathcal{M})^{\perp} \neq \{0\}$$

and so $\mathcal{M} \cap (S\mathcal{M})^{\perp}$ contains a non-trivial function Θ. We now argue that

$$|\Theta| = c$$

on a set of full measure in $\mathbb{T}$. Indeed,

$$\int_{\mathbb{T}} |\Theta(\zeta)|^2 \overline{\zeta}^n dm(\zeta) = \langle \Theta, S^n \Theta \rangle = 0 \quad \forall\, n \in \mathbb{N}.$$

Taking complex conjugates of both sides of the above equation, we also see that

$$\int_{\mathbb{T}} |\Theta(\zeta)|^2 \zeta^n dm(\zeta) = 0 \quad \forall\, n \in \mathbb{N}.$$

This means that the Fourier coefficients of $|\Theta|^2$ all vanish except for $n = 0$ and so $|\Theta|^2 = c$ almost everywhere on $\mathbb{T}$. Without loss of generality, we can assume that $|\Theta| = 1$ almost everywhere on $\mathbb{T}$ and so Θ is an inner function.

Third, let $[\Theta]$ denote the closed linear span of the functions
$$\Theta, S\Theta, S^2\Theta, \cdots$$
and observe that
$$[\Theta] = \Theta H^2.$$
To see this, notice that clearly $[\Theta] \subset \Theta H^2$. For the other containment, let $g = \Theta G \in \Theta H^2$ and let G_N be the N-th partial sum of the Taylor series of G. Notice that $\Theta G_N \in [\Theta]$ since G_N is a polynomial. From Parseval's theorem, $G_N \to G$ in H^2 and so, since Θ is a bounded function, ΘG_N converges to ΘG in H^2.

Finally, observe that
$$[\Theta] = \mathcal{M}.$$
Indeed, $\Theta \in \mathcal{M}$ and so $[\Theta] \subset \mathcal{M}$. Now suppose that $f \in \mathcal{M}$ and $f \perp [\Theta]$. Since $f \perp [\Theta]$,
$$\int_{\mathbb{T}} f(\zeta)\overline{\Theta(\zeta)\zeta^n}\, dm(\zeta) = \langle f, S^n\Theta\rangle = 0 \quad \forall n \in \mathbb{N}_0.$$
But since $\Theta \perp S\mathcal{M}$, we also know that
$$\int_{\mathbb{T}} f(\zeta)\overline{\Theta(\zeta)}\zeta^n\, dm(\zeta) = \langle S^n f, \Theta\rangle = 0 \quad \forall n \in \mathbb{N}.$$
The previous two equations say that all of the Fourier coefficients of $f\overline{\Theta}$ vanish and so $f\overline{\Theta} = 0$ almost everywhere on $\mathbb{T}$. But we have already shown that $|\Theta| = 1$ almost everywhere on $\mathbb{T}$ and so $f \equiv 0$. $\square$

3.6. **Factorization.** Due to the limited amount of time, I will not get into all of the details here. They are carefully worked out in standard texts [8, 13]. For a sequence of points $(a_n)_{n\geq 1}$ in $\mathbb{D} \setminus \{0\}$ which satisfy the condition
$$\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$$
one can show that the infinite product,
$$B(z) := \prod_{n=1}^{\infty} \frac{a_n}{|a_n|}\frac{z - a_n}{1 - \overline{a_n}z}$$
called a *Blaschke product*, converges uniformly on compact subsets of $\mathbb{D}$. Work of W. Blaschke shows the following:

Theorem 3.17 (Blaschke)**.** *For a sequence of points $(a_n)_{n\geq 1} \subset \mathbb{D} \setminus \{0\}$ satisfying $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$, the Blaschke product B is an inner function.*

Now consider a positive singular Borel measure μ on the circle, i.e., there are two Borel sets $A, B \subset \mathbb{T}$ with $A \cup B = \mathbb{T}$ and $m(A) = \mu(B) = 0$. We will not include a proof here but there is an old theorem of Fatou which says such a singular measure satisfies
$$\lim_{r\to 1^-} \int \frac{1 - r^2}{|\xi - r\zeta|^2} d\mu(\xi) = 0$$

for m-almost every $\zeta \in \mathbb{T}$. Notice how the above integral is the Poisson integral of μ. Fatou's theorem allows us to prove the following:

Theorem 3.18. *For a positive finite positive singular Borel measure μ on $\mathbb{T}$ the function*

$$s_\mu(z) = \exp\left(-\int \frac{\xi + z}{\xi - z} d\mu(\xi)\right), \quad z \in \mathbb{D},$$

is inner.

Proof. As was done previously one can show that

$$|s_\mu(z)| = \exp\left(\Re\left(-\int \frac{\xi + z}{\xi - z} d\mu(\xi)\right)\right) = \exp\left(-\int \frac{1 - |z|^2}{|z - \xi|^2} d\mu(\xi)\right) < 1.$$

Thus $s_\mu \in H^2$. Now use Fatou's theorem to show that s_μ has unimodular boundary values m-almost everywhere. □

Definition 3.19. The function s_μ from the previous theorem is called a *singular inner function.*

Clearly the functions

$$e^{i\gamma} z^n B(z) s_\mu(z),$$

where $\gamma \in \mathbb{R}$, $n \in \mathbb{N}_0$, B is a Blaschke product, and s_μ is a singular inner function. It turns out that these are all of them.

Theorem 3.20 (Nevanlinna)**.** *Any inner function Θ takes the form*

$$\Theta(z) = e^{i\gamma} z^n B(z) s_\mu(z).$$

The factors z^n, B, s_μ are unique. Furthermore, every $f \in H^2 \setminus \{0\}$ satisfies

$$\int_{\mathbb{T}} \log |f(\zeta)| dm(\zeta) > -\infty$$

and can be written as

$$f(z) = \Theta(z) \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |f(\zeta)| dm(\zeta)\right),$$

where Θ is inner.

Definition 3.21. For $f \in H^2 \setminus \{0\}$ the function

$$\exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |f(\zeta)| dm(\zeta)\right)$$

is called the *outer part* of f while Θ in the above theorem is called the *inner part* of f.

4. Model spaces

4.1. **Basic properties.** In the previous section we showed (Beurling's theorem) that the (non-trivial) invariant subspaces for the unilateral shift $S : H^2 \to H^2, Sf = zf$ are ΘH^2, where Θ is an inner function.

Proposition 4.1. *The Hilbert space adjoint of S is given by the formula*

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H^2.$$

Proof. For $f, g \in H^2$ notice that $\langle zf, c\rangle = 0$ for every $c \in \mathbb{C}$ and so

$$\langle Sf, g\rangle = \langle zf, g - g(0)\rangle = \langle f, \overline{z}(g - g(0))\rangle = \langle f, \frac{g - g(0)}{z}\rangle.$$

This says that S^* has the desired formula. □

For an inner function Θ form the *model space*

$$(\Theta H^2)^\perp = \{f \in H^2 : \langle f, \Theta z^n\rangle = 0, n \in \mathbb{N}_0\}$$

and notice, via Beurling's theorem and adjoints that every S^*-invariant subspace of H^2 takes the form $(\Theta H^2)^\perp$.

So why are they called model spaces? Well, they model things!

Theorem 4.2. *If T is a contraction on a separable Hilbert space $\mathcal{H}$ which satisfies*

$$\dim(I - T^*T)^{1/2}\mathcal{H} = \dim(I - TT^*)^{1/2}\mathcal{H} = 1;$$

$$\lim_{n\to\infty} \|T^{*n}x\| = 0, \quad x \in \mathcal{H},$$

then T is unitarily equivalent to $S_\Theta = P_\Theta S|(\Theta H^2)^\perp$, the compression of S to $(\Theta H^2)^\perp$ for some inner Θ.

There is plenty to talk about here. For example, how does one compute P_Θ, the orthogonal projection of L^2 onto $(\Theta H^2)^\perp$?

Theorem 4.3. *Define the kernel functions*

$$k_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

Then

(1) $k_\lambda^\Theta \in (\Theta H^2)^\perp$ *for every* $\lambda \in \mathbb{D}$;
(2) $\langle f, k_\lambda^\Theta\rangle = f(\lambda)$ *for every* $f \in (\Theta H^2)^\perp$;
(3) $(P_\Theta g)(\lambda) = \langle g, k_\lambda^\Theta\rangle$ *for all* $g \in L^2$.

Proof. For the proof of (1) note that for fixed $\lambda \in \mathbb{D}$ the function $k_\lambda^\Theta \in H^2$ since $z \mapsto k_\lambda^\Theta(z)$ is a bounded analytic function. To show that $k_\lambda \in (\Theta H^2)^\perp$ we need to verify that

$$\langle \Theta g, k_\lambda^\Theta\rangle = 0, \quad g \in H^2.$$

Indeed, recall the kernel function

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}$$

and its reproducing property

$$\langle f, k_\lambda \rangle = f(\lambda)$$

for $f \in H^2$. Then

$$\begin{aligned}
\langle \Theta g, k_\lambda^\Theta \rangle &= \langle \Theta g, k_\lambda \rangle - \Theta(\lambda) \langle \Theta g, \Theta k_\lambda \rangle \\
&= \Theta(\lambda) g(\lambda) - \Theta(\lambda) \langle g, k_\lambda \rangle \\
&= \Theta(\lambda) g(\lambda) - \Theta(\lambda) g(\lambda) \\
&= 0,
\end{aligned}$$

which proves (1).

The proof of (2) follows from

$$\begin{aligned}
\langle f, k_\lambda^\Theta \rangle &= \langle f, k_\lambda \rangle - \Theta(\lambda) \langle f, \Theta k_\lambda \rangle \\
&= f(\lambda) + 0 \\
&= f(\lambda).
\end{aligned}$$

The proof of (3) is a bit technical and will be omitted. □

A tangible example of some model spaces is in order here.

Example 4.4. Suppose $\Theta(z) = z^n$. Then $(z^n H^2)^\perp = \bigvee\{1, z, z^2, \cdots, z^{n-1}\}$. Furthermore.

$$k_\lambda^\Theta(z) = \frac{1 - \overline{\lambda}^n z^n}{1 - \overline{\lambda} z} = 1 + \overline{\lambda} z + \overline{\lambda}^2 z^2 + \cdots + \overline{\lambda}^{n-1} z^{n-1}.$$

Example 4.5. Suppose $\Theta = B$ is a Blaschke product with simple zeros. Then if $B(a) = 0$, note that

$$\langle B g, k_a \rangle = B(a) g(a) = 0, \quad g \in H^2$$

and so $k_a \in (BH^2)^\perp$. Moreover, if

$$g \perp k_a, \quad a \in B^{-1}(\{0\}),$$

then $g(a) = 0$ for all $a \in B^{-1}(\{0\})$ and so $g \in BH^2$. By annihilators,

$$\bigvee\{k_a : a \in B^{-1}(\{0\})\} = (BH^2)^\perp.$$

Furthermore, if B is a finite Blaschke product then $(BH^2)^\perp$ is finite dimensional with basis $\{k_a : a \in B^{-1}(\{0\})\}$.

Even further, if $B^{-1}(\{0\}) = \{a_n : 1 \leq k \leq \#B^{-1}(\{0\})$, then

$$T_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} z} \prod_{k=1}^{n-1} \frac{z - a_k}{1 - \overline{a_k} z},$$

then $\{T_n : n = 1, \cdots, \#B^{-1}(\{0\})\}$ is an orthonormal basis for $(BH^2)^\perp$. This basis is called the *Takenaka basis* [17].

4.2. **Continuation properties.** It turns out that there are some fascinating continuation properties of functions from $(\Theta H^2)^{\perp}$. In order to understand this, we need to recall that we can understand H^2 functions as analytic functions on $\mathbb{D}$ as well as the subspace of L^2 functions (on the circle) whose negative Fourier coefficients vanish. Here we choose this second rendition of H^2.

Theorem 4.6. *For an inner function* Θ*,*

$$(\Theta H^2)^{\perp} = H^2 \cap \Theta \overline{zH^2} = \{f \in H^2 : f = \Theta \overline{zh} \text{ a.e. on } \mathbb{T} \text{ for some } h \in H^2\}.$$

Proof. Suppose $f \in H^2 \cap \Theta \overline{zH^2}$. Then $f = \Theta \overline{zh}$ almost everywhere on $\mathbb{T}$ for some $h \in H^2$. Then, for any $g \in H^2$,

$$\langle f, \Theta g\rangle = \langle \Theta \overline{zh}, \Theta g\rangle = \langle \overline{zh}, g\rangle = \int_{\mathbb{T}} \overline{zgh} dm(\zeta) = 0.$$

The last identity follows from Cauchy's Theorem.

Conversely, suppose that $f \in (\Theta H^2)^{\perp}$. Then $\langle f, z^n \Theta\rangle = 0$ for all $n \in \mathbb{N}_0$. This can be written equivalently as $\widehat{v}(n) = 0$ for all $n \in \mathbb{N}$, where $v = fz\overline{\Theta}$. From here we see that $fz\overline{\Theta} \in \overline{H^2}$ and so $f \in \Theta \overline{zH^2}$. □

It turns out that we get quite a lot of information from this fact. If Θ is an inner function let

$$\Theta_c(z) = \frac{1}{\overline{\Theta(1/\overline{z})}}, \quad z \in \mathbb{C}_{\infty} \setminus \mathbb{D}^-.$$

Notice that Θ_c is a meromorphic function in $\mathbb{C}_{\infty} \setminus \mathbb{D}^-$ with poles at $\{1/\overline{a} : \Theta(a) = 0\}$, the reflected zeros of Θ across $\mathbb{T}$. To see this, just look at the poles of B, the Blaschke factor of $\Theta = B s_{\mu}$. By the Schwarz reflection principle, Θ_c and Θ are analytic continuations of each other across arcs of $\mathbb{T}$ where Θ is continuous and has unimodular boundary values. This will certainly take place at

$$\mathbb{T} \setminus (B^{-1}(\{0\})^- \cap \operatorname{supt}\mu).$$

More advanced analysis of inner functions will show that the above set is equal to

$$\sigma_b(\Theta) := \{\zeta \in \mathbb{T} : \liminf_{z \to \zeta} |\Theta(z)| = 0\}.$$

The set $\sigma_b(\Theta)$ is called the *boundary spectrum* of Θ. It is important to mention here that the boundary spectrum of Θ might be all of $\mathbb{T}$. For example, a Blaschke product whose zeros accumulate everywhere on $\mathbb{T}$ will have this property. So in this case Θ will not have an analytic continuation across any arc of $\mathbb{T}$.

If $f \in (\Theta H^2)^{\perp} = H^2 \cap \Theta \overline{zH^2}$ then $f = \Theta \overline{h}$, where $h \in zH^2$. Let

$$f_{\Theta}(z) := \Theta_c(z) \overline{h}(1/\overline{z}), \quad z \in \mathbb{C}_{\infty} \setminus \mathbb{D}^-.$$

As it stands now, f_{Θ} is a just a meromorphic function $\mathbb{C}_{\infty} \setminus \mathbb{D}^-$. Much more is true.

Corollary 4.7. *The function f_Θ satisfies the following properties;*

(1) *For almost every $\zeta \in \mathbb{T}$,*

$$\lim_{s \to 1^+} f_\Theta(s\zeta) = \lim_{r \to 1^-} f(r\zeta);$$

(2) *f_Θ is an analytic continuation of f across $\mathbb{T} \setminus \sigma_b(\Theta)$.*

The proof of this theorem uses Fatou's jump theorem and a version of Morera's theorem which is a bit too technical to get into here. The details are worked out in [5]. However, we would like to mention that the property that f_Θ, meromorphic on on the exterior disk, has the same, or almost everywhere the same, boundary values of f, is an important property called a *pseudo-continuation.* This property appears throughout analysis [6, 15]. Also important to emphasize here is that f_Θ might not have an analytic continuation across any arc of $\mathbb{T}$ (take, as before, Θ to be a Blaschke product whose zeros accumulate at every point of $\mathbb{T}$), but, nevertheless, this function will have a pseudo-continuation across $\mathbb{T}$. Pseudo-continuations are used to describe the cyclic vectors for S^*.

4.3. **Kernel functions and angular derivatives.** We have seen that every function in the model space has an analytic continuation across the complement of $\sigma_b(\Theta)$. If $\sigma_b(\Theta)$ omits an arc $I \subset \mathbb{T}$, then every f_Θ will be an analytic continuation of f across I for every $f \in (\Theta H^2)^\perp$. Moreover, a little analysis with the formula for k_λ^Θ (using the fact that Θ is analytic and unimodular on I), will show that

$$k_\zeta^\Theta \in H^2, \quad \zeta \in I;$$

$$f(\zeta) = \langle f, k_\zeta^\Theta \rangle, \quad f \in (\Theta H^2)^\perp.$$

This is saying that when $\zeta \in I$, then k_ζ is also the reproducing kernel for $(\Theta H^2)^\perp$.

But there is always this pesky case where $\sigma_b(\Theta) = \mathbb{T}$. Is there any nice behavior of functions in $(\Theta H^2)^\perp$ in this case? The answer is ... sometimes.

Theorem 4.8. *For inner Θ and $\zeta \in \mathbb{T}$, the following are equivalent:*

(1) *$k_\zeta^\Theta \in H^2$;*
(2) *Every $f \in (\Theta H^2)^\perp$ has a finite non-tangential limit at ζ;*
(3) *$\Theta(\zeta) \in \mathbb{T}$ and the non-tangential limit of $\Theta'(z)$ exists at ζ and is finite.*

If $\Theta = Bs_\mu$, it turns out (and this is somewhat technical to prove) that

$$|\Theta'(\zeta)| = |B'(\zeta)| + |s_\mu'(\zeta)|,$$

where $|\Theta'(\zeta)|$ denotes the modulus of the non-tangential limit of Θ' at ζ. Note that this quantity is the usual derivative of Θ when Θ is analytic in an neighborhood of ζ. Moreover

$$|B'(\zeta)| = \sum_{a \in B^{-1}(\{0\})} \frac{1 - |a|^2}{|\zeta - a|^2};$$

$$|s'_\mu(\zeta)| = \int \frac{d\mu(\xi)}{|\xi - \zeta|^2}.$$

4.4. **The compressed shift.** For an inner Θ, with $\Theta(0) = 0$ (This assumption is merely a technical one to make the formulas a bit easier to write. It does not change any of the results.), define

$$S_\Theta : (\Theta H^2)^\perp \to (\Theta H^2)^\perp, \quad S_\Theta = P_\Theta S|(\Theta H^2)^\perp.$$

The operator is called the *compressed shift*. Here are some known facts about S_Θ:

Theorem 4.9. (1) *The spectrum of S_Θ is equal to*

$$\{w \in \mathbb{D}^- : \liminf_{z \to w} |\Theta(z)| = 0\}.$$

(2) *Every invariant subspace of S_Θ takes the form*

$$\theta H^2 \cap (\Theta H^2)^\perp,$$

where θ is inner and $\Theta/\theta \in H^2$.

(3) *S_Θ is cyclic with cyclic vector* 1. *That is to say*

$$\bigvee\{S_\Theta^n 1 : n \in \mathbb{N}_0\} = (\Theta H^2)^\perp.$$

(4) *$S_\Theta^n = P_\Theta S^n|(\Theta H^2)^\perp$.*

(5) *$I - S_\Theta^* S_\Theta = \frac{\Theta}{z} \otimes \frac{\Theta}{z}$.*

Since we will be using this tensor notation $\otimes$ a lot in these lectures, let us officially define it. For $f, g \in (\Theta H^2)^\perp$, define

$$f \otimes g : (\Theta H^2)^\perp \to (\Theta H^2)^\perp, \quad (f \otimes g)(h) = \langle h, g \rangle f.$$

As easy exercise will show that the adjoint of $f \otimes g$ will be $g \otimes f$.

4.5. **Clark theory.** The Paley-Wiener approximation problem: Suppose $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and $\{y_n : n \in \mathbb{N}\}$ is a sequence in $\mathcal{H}$. If these sequences are close to each other (not quite defined yet), does $\{y_n : n \in \mathbb{N}\}$ span $\mathcal{H}$?

As an example of what we mean here, consider the standard orthonormal basis

$$\varphi_n(e^{i\theta}) = e^{in\theta}, \quad n \in \mathbb{Z},$$

for L^2. When does the sequence

$$\psi_n(e^{i\theta}) = e^{i\lambda_n\theta}, \quad n \in \mathbb{Z},$$

where $\lambda_n \in \mathbb{R}$, span L^2? A classical theorem of Paley-Weiner says this is indeed the case when

$$\max_{n \in \mathbb{Z}} |\lambda_n - n| < \frac{1}{\pi^2}.$$

So let $\Lambda \subset \mathbb{D}$ be a sequence in $\mathbb{D}$. When does

$$\bigvee\{k_\lambda^\Theta : \lambda \in \Lambda\} = (\Theta H^2)^\perp?$$

An easy exercise using the uniqueness theorem for analytic functions will show that if Λ has accumulation points in either the open unit disk or $\mathbb{T} \setminus \sigma_b(\Theta)$, then the set of kernel functions does indeed span. Hint: Use annihilators and the Hahn-Banach separation theorem. For other cases, things get a bit more complicated. What we want here is a orthonormal basis for $(\Theta H^2)^\perp$ consisting of (normalized) kernel functions!

So as to not get lost in the notation and the theorems, let's try an easy finite dimensional example. Consider the model space $(z^3 H^2)^\perp$. An orthonormal basis for this space is $\{1, z, z^2\}$. Compute the action of $S_3 := S_{z^3}$ on these basis elements as

$$S_3 1 = z, \quad S_3 z = z^2, \quad S_3 z^2 = 0.$$

Thus the matrix representation of S_3 with respect to the basis $\{1, z, z^2\}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For $\alpha \in \mathbb{T}$ consider the rank-one perturbation of S_3 defined by

$$U_\alpha = S_3 + \alpha(1 \otimes z^2).$$

Again, let's compute the action of U_α on the basis elements:

$$U_\alpha 1 = z, \quad U_\alpha z = z^2, \quad U_\alpha z^2 = \alpha.$$

Thus the matrix representation of U_α with respect to this basis is

$$\begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An easy matrix computation will show, using $\alpha \in \mathbb{T}$, that U_α is a unitary operator. The reproducing kernel function for $(z^3 H^2)^\perp$ is

$$k_\lambda^{z^3}(z) = \frac{1 - \overline{\lambda}^3 z^3}{1 - \overline{\lambda} z} = 1 + \overline{\lambda} z + \overline{\lambda}^2 z^2.$$

The characteristic polynomial for U_α is $\lambda^3 - \alpha$ and so the eigenvalues of U_α are the solutions to $\{z^3 = \alpha\}$. Furthermore, if $\beta^3 = \alpha$, then one can check, by thinking of $k_\beta^{z^3}(z)$ as the column vector $(1, \overline{\beta}, \overline{\beta}^2)^T$, that the eigenvector corresponding to β is indeed $k_\beta^{z^3}$. One can also see that each eigenvector $k_\beta^{z^3}$ has norm $\sqrt{3}$ which is precisely the square root of the modulus of the angular derivative of z^3 at β. Finally, since U_α is unitary, then the normalized eigenvectors form an orthonormal basis for U_α.

Let us summarize what we have here for $\Theta(z) = z^n$. For each $\alpha \in \mathbb{T}$ the operator

$$U_\alpha = S_\Theta + \alpha(1 \otimes \frac{\Theta}{z})$$

is a unitary with eigenvalues

$$\{\zeta : \Theta(\zeta) = \alpha\}.$$

Furthermore, the eigenvectors of U_α are

$$\{k_\zeta^{z^n} : \zeta^n = \alpha\}$$

and

$$\|k_\zeta^{z^n}\| = \sqrt{|\Theta'(\zeta)|}.$$

Thus these (normalized) elements form an orthonormal basis for $(z^n H^2)^\perp$.

This points to the following theorem (see [5] for a detailed proof):

Theorem 4.10. *For an inner function Θ with $\Theta(0) = 0$, the operator*

$$U_\alpha := S_\Theta + \alpha(1 \otimes \frac{\Theta}{z})$$

is a cyclic unitary operator whose spectral mesure is carried by the Borel set

$$\{\zeta \in \mathbb{T} : \lim_{r \to 1^-} \Theta(r\zeta) = \alpha\}.$$

The eigenvalues of U_α are the $\zeta \in \mathbb{T}$ so that $\Theta(\zeta) = \alpha$ and $|\Theta'(\zeta)| < \infty$. The corresponding eigenvector is k_ζ^Θ.

In order to discuss the spectral representation of U_α let us go back to our $(z^3 H^2)^\perp$ example. Let $\zeta_1, \zeta_2, \zeta_3$ be the three roots of $\Theta(z) = z^3 = \alpha$. Then the measure

$$d\sigma_\alpha = \frac{1}{3}\delta_{\zeta 1} + \frac{1}{3}\delta_{\zeta_2} + \frac{1}{3}\delta_{\zeta_3}$$

satisfies

$$\frac{1 + \overline{\alpha} z^3}{1 - \overline{\alpha} z^3} = \int \frac{\zeta + z}{\zeta - z} d\sigma_\alpha(\zeta).$$

Indeed from the Cauchy reside theorem one can can check that

$$\frac{1 + \overline{\alpha} z^3}{1 - \overline{\alpha} z^3} = \frac{1}{3}\frac{\zeta_1 + z}{\zeta_1 - z} + \frac{1}{3}\frac{\zeta_2 + z}{\zeta_2 - z} + \frac{1}{3}\frac{\zeta_3 + z}{\zeta_3 - z}.$$

Now define the operator

$$V_\alpha : L^2(\sigma_\alpha) \to (z^3 H^2)^\perp, \quad (V_\alpha f)(z) = (1 - \overline{\alpha} z^3) \int \frac{f(\zeta)}{1 - \overline{\zeta}} d\sigma_\alpha(\zeta).$$

To see that everything works out here note that if

$$f = c_1 \chi_{\zeta_1} + c_2 \chi_{\zeta_2} + c_3 \chi_{\zeta_3},$$

a typical element of $L^2(\sigma_\alpha)$, then

$$(Vf)(z) = (1 - \overline{\alpha} z^3)\left(\frac{1}{3} c_1 \frac{1}{1 - \overline{\zeta_1} z} + \frac{1}{3} c_2 \frac{1}{1 - \overline{\zeta_2} z} + \frac{1}{3} c_3 \frac{1}{1 - \overline{\zeta_3} z}\right).$$

Since $\zeta_j^3 = \alpha$, one can verify that the above expression is a polynomial of degree at most 2, which is precisely an element of $(z^3 H^2)^\perp$.

Furthermore, $\{\sqrt{3}\chi_{\zeta_1}, \sqrt{3}\chi_{\zeta_2}, \sqrt{3}\chi_{\zeta_3}\}$ is an orthonormal basis for $L^2(\sigma_\alpha)$ and

$$(V_\alpha \sqrt{3}\chi_{\zeta_j})(z) = \frac{\sqrt{3}}{3} k_{\zeta_j}^{z^3}(z).$$

But since

$$\{\frac{\sqrt{3}}{3} k_{\zeta_j}^{z^3}(z) : j = 1, 2, 3\}$$

is an orthonormal basis for $(z^3 H^2)^\perp$ we see that V_α is unitary.

Finally, if

$$Z_\alpha : L^2(\sigma_\alpha) \to L^2(\sigma_\alpha), \quad (Z_\alpha f)(\zeta) = \zeta f(\zeta),$$

when since

$$Z_\alpha \chi_{\zeta_j} = \zeta_j \chi_{\zeta_j}$$

we see that Z_α is unitary with eigenvalues ζ_j and corresponding eigenvectors χ_{ζ_j}. But by our earlier computation with V_α we see that

$$V_\alpha Z_\alpha = U_\alpha V_\alpha.$$

Thus we have identified the spectral representation of U_α.

Here is the general theorem:

Theorem 4.11. *For an inner function Θ with $\Theta(0) = 0$, let σ_α be the unique finite positive Borel measure on $\mathbb{T}$ satisfying*

$$\frac{1 + \overline{\alpha}\Theta(z)}{1 - \overline{\alpha}\Theta(z)} = \int \frac{\zeta + z}{\zeta - z} d\sigma_\alpha(\zeta).$$

Then the operator

$$(V_\alpha f)(z) = (1 - \overline{\alpha}\Theta(z)) \int \frac{f(\zeta)}{1 - \overline{\zeta} z} d\sigma_\alpha(\zeta)$$

is an isometric operator from $L^2(\sigma_\alpha)$ onto $(\Theta H^2)^\perp$. Furthermore if

$$Z_\alpha : L^2(\sigma_\alpha) \to L^2(\sigma_\alpha), \quad (Z_\alpha f)(\zeta) = \zeta f(\zeta),$$

then $V_\alpha Z_\alpha = U_\alpha V_\alpha$.

Corresponding to the inner function Θ is a family of measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$. This family is called the Clark measures corresponding to Θ named after Clark who first discovered them. Here are some basis properties of Clark measures.

Proposition 4.12. *For an inner function Θ with corresponding family of Clark measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$, The following hold:*

(1) *$\sigma_\alpha \perp m$ for all α.*
(2) *$\sigma_\alpha \perp \sigma_\beta$ for $\alpha \neq \beta$.*
(3) *σ_α has a point mass at ζ if and only if $\Theta(\zeta) = \alpha$ and $|\Theta'(\zeta)| < \infty$. Furthermore*

$$\sigma_\alpha(\{\zeta\}) = \frac{1}{|\Theta'(\zeta)|}.$$

Here is one more fascinating and useful gem about Clark measures.

Theorem 4.13. *For an inner function Θ with associated family of Clark measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ and $f \in C$,*

$$\int \left(\int f(\zeta) d\sigma_\alpha(\zeta) \right) dm(\alpha) = \int f(\xi) dm(\xi).$$

Proof. For a fixed $z \in \mathbb{D}$ notice that

$$\begin{aligned}
\int \left(\int P_z(\zeta) d\sigma_\alpha(\zeta) \right) dm(\alpha) &= \int \left(\frac{1 - |\Theta(z)|^2}{|\alpha - \Theta(z)|^2} \right) dm(\alpha) \\
&= \int P_{\Theta(z)}(\alpha) dm(\alpha) \\
&= 1 \\
&= \int P_z(\zeta) dm(\zeta).
\end{aligned}$$

Thus the theorem works for finite linear combinations of Poisson kernels.

To get the result for any $f \in C$, let $(f_n)_{n \geq 1}$ be a sequence of finite linear linear combinations of Poisson kernels which converge uniformly to f (indeed finite linear combinations of Poisson kernels are dense in C). Let us first note that for each n, the function

$$\alpha \mapsto \int f_n d\sigma_\alpha$$

is continuous on $\mathbb{T}$. To see this write

$$f_n = \sum_{j=1}^{N} c_j P_{z_j}$$

and, by the definition of the measure σ_α, observe that

$$\int f_n d\sigma_\alpha = \sum_{j=1}^{N} c_j P_{\Theta(z_j)}(\alpha)$$

which is continuous in the variable α. We can extend this to show that

$$\alpha \mapsto \int f d\sigma_\alpha$$

is also continuous. Indeed, it is easy to show that

$$B = \sup\{\sigma_\alpha(\mathbb{T}) : \alpha \in \mathbb{T}\} < \infty.$$

In fact, if $\Theta(0) = 0$ then each σ_α is a probability measure. Then

$$\left| \int f d\sigma_\alpha - \int f_n d\sigma_\alpha \right| \leq \|f - f_n\|_\infty \sigma_\alpha(\mathbb{T}) \leq \|f - f_n\|_\infty B$$

which means, via uniform convergence of continuous functions, that

$$\alpha \mapsto \int f d\sigma_\alpha$$

is continuous

Finally,

$$\begin{aligned}\int_{\mathbb{T}} f\,dm &= \lim_{n\to\infty}\int_{\mathbb{T}} f_n dm \quad \text{(uniform convergence)}\\ &= \lim_{n\to\infty}\int_{\mathbb{T}}\left(\int_{\mathbb{T}} f_n(\zeta)d\sigma_\alpha(\zeta)\right)dm(\alpha) \quad \text{(disintegration formula)}\\ &= \int_{\mathbb{T}}\left(\int_{\mathbb{T}} f(\zeta)d\sigma_\alpha(\zeta)\right)dm_\alpha \quad \text{(dominated convergence theorem).}\end{aligned}$$

This proves the result. □

Remark 4.14. Aleksandrov showed quite a bit more here in that the continuous functions C can be replaced by L^1 in the disintegration theorem. There are quite a few technical issues here. For example, the inner integrals

$$\int f(\zeta)d\sigma_\alpha(\zeta)$$

in the disintegration formula do not seem to be well defined for L^1 functions since indeed the measure σ_α can contain point masses on $\mathbb{T}$ while L^1 functions are defined merely m-almost everywhere. However, amazingly, the function

$$\alpha \mapsto \int f(\zeta)d\sigma_\alpha(\zeta)$$

is defined for m-almost every α and is integrable. An argument with the monotone class theorem is used to prove this more general result. See [4] for the details.

5. Truncated Toeplitz operators

Just so we have something to compare, let us say a very few words about classical Toeplitz operators. This is a vast and well studied field of analysis and we refer the reader to the text [7] for the basics and [2] for an encyclopedia on the subject. For $\varphi \in L^\infty$, define the *Toeplitz operator*

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = P_+(\varphi f),$$

where P_+ is the orthogonal projection from L^2 onto H^2. The operator P_+ operates on L^2 via their Fourier series in the following way

$$P_+ : \sum_{n=-\infty}^{\infty} \widehat{g}(n)\zeta^n \mapsto \sum_{n=0}^{\infty} \widehat{g}(n) z^n.$$

There is also the following integral formula

$$(P_+ g)(z) = \int_{\mathbb{T}} \frac{g(\zeta)}{1-\bar{\zeta}z} dm(\zeta).$$

One can show that T_φ is a bounded operator on H^2 with operator norm $\|T_\varphi\|$ satisfying

$$\|T_\varphi\| = \|\varphi\|_\infty.$$

Furthermore, where are no (non-zero) compact Toeplitz operators. Any Toeplitz operator T_φ satisfies the following identity

$$T_\varphi = S T_\varphi S^*,$$

where $S : H^2 \to H^2$ is the unilateral shift $Sf = zf$. An old result of Brown and Halmos says that the above identity actually characterizes the Toeplitz operators amongst the bounded operators on H^2: A bounded operator T on H^2 is a Toeplitz operator if and only $T = STS^*$. It is also easy to verify from the definition of T_φ that the matrix representation of T_φ with respect to the usual orthonormal normal basis $\{1, z, z^2, \cdots\}$ for H^2 is the matrix whose (j,k) entry is $\widehat{\varphi}(k-j)$ which is indeed an infinite Toeplitz matrix. Many other things are known about Toeplitz operators: their spectrum, essential spectrum, commutator ideals, algebras generated by them, etc. There are all thoroughly discussed in [2, 7].

For $\varphi \in L^2$ define the operator on bounded functions in $(\Theta H^2)^\perp$ (which are indeed dense in $(\Theta H^2)^\perp$) by

$$A_\varphi f = P_\Theta(\varphi f).$$

When A_φ can be extended to be bounded on all of $(\Theta H^2)^\perp$ we say A_φ is a bounded *truncated Toeplitz operator* and we set $\mathcal{T}_\Theta$ to be the set of all bounded truncated Toeplitz operators on $(\Theta H^2)^\perp$. This topic is growing and we will only mention a few basic results here. Two sources for this are [10, 16].

There are some similarities between truncated Toeplitz operators and Toeplitz operators. Certainly $T_\varphi^* = T_{\overline{\varphi}}$ along with $A_\varphi^* = A_{\overline{\varphi}}$. There is a version of the Brown-Halmos result: a bounded operator A on $(\Theta H^2)^\perp$ is a truncated Toeplitz operator if and only if $A_z A A_{\overline{z}} = A + R$, where R is a certain rank-two operator on $(\Theta H^2)^\perp$.

Now things get tricky. For one the symbol is not unique. In fact $A_\varphi = A_\psi$ if and and only if $\varphi - \psi \in \Theta H^2 + \overline{\Theta H^2}$. When φ is a bounded symbol, we know that A_φ is bounded. However, there are bounded truncated Toeplitz operators which can not be written with a bounded symbol. Furthermore, when φ is bounded, we have $\|A_\varphi\| \leq \|\varphi\|_\infty$ but equality need not hold. Despite these anomalies, there is some nice structure here in that $\mathcal{T}_\Theta$ is a weakly closed linear space.

There is a natural conjugation C on $(\Theta H^2)^\perp$. In order to understand it, we need to remind the reader of an alternate description of $(\Theta H^2)^\perp$ as $H^2 \cap \Theta \overline{z H^2}$ from Theorem 4.6. In other words, $f \in (\Theta H^2)^\perp$ if and only if $f \in H^2$ and $f(\zeta) = \Theta(\zeta)\overline{\zeta h(\zeta)}$ almost everywhere on $\mathbb{T}$ for some $h \in H^2$. It is important to think of all this as taking place on the circle and not on the disk.

Proposition 5.1. *For inner Θ the operator*

$$(Cf)(\zeta) = \overline{f(\zeta)\zeta}\Theta(\zeta), \quad \zeta \in \mathbb{T},$$

defines a mapping from $(\Theta H^2)^\perp$ onto itself satisfying:

(1) *C is conjugate linear:* $C(af+g) = \overline{a}Cf + Cg$,
(2) *C is involutive:* $C^2 = I$,
(3) $\langle Cf, Cg\rangle = \langle g, f\rangle$.

Proof. First we need to prove that $Cf \in (\Theta H^2)^\perp$ whenever $f \in (\Theta H^2)^\perp$. Indeed if $f \in (\Theta H^2)^\perp$. then $f = \Theta\overline{\zeta h}$ almost everywhere on $\mathbb{T}$ for some $h \in H^2$. We then obtain

$$Cf = \Theta\overline{f\zeta} = \Theta\overline{\Theta\overline{\zeta h}\zeta} = h \in H^2.$$

Thus $Cf \in H^2$. We need to now show that $Cf \perp \Theta H^2$. To see this observe that for any $g \in H^2$,

$$\langle Cf, \Theta g\rangle = \langle \overline{f\zeta}\Theta, \Theta g\rangle = \langle \overline{f\zeta}, g\rangle = \int_{\mathbb{T}} \overline{fg\zeta} dm(\zeta) = 0$$

by Cauchy's Theorem. Thus C maps $(\Theta H^2)^\perp$ to itself.

We leave it to the reader to easily verify (1) - (3). □

Example 5.2. When $\Theta(z) = z^n$, then $(z^n H^2)^\perp = \bigvee\{1, z, \cdots, z^{n-1}\}$ and the conjugation C becomes

$$C(a_0 + a_1 z + \cdots a_{n-1}z^{n-1}) = \overline{a_0}z^{n-1} + \overline{a_1}z^{n-2} + \cdots + \overline{a_{n-1}}.$$

Example 5.3. When

$$\Theta(z) = \prod_{j=1}^{n} \frac{z - \lambda_j}{1 - \overline{\lambda_j}z},$$

when λ_j are not necessarily distinct point of the disk, then

$$(\Theta H^2)^\perp = \left\{ \frac{a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}}{(1 - \overline{\lambda_1}z)\cdots(1 - \overline{\lambda_n}z)} : a_j \in \mathbb{C} \right\}.$$

Moreover the conjugation C becomes

$$C\left(\frac{a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}}{(1 - \overline{\lambda_1}z)\cdots(1 - \overline{\lambda_n}z)}\right) = \frac{\overline{a_0}z^{n-1} + \overline{a_1}z^{n-2} + \cdots + \overline{a_{n-1}}}{(1 - \overline{\lambda_1}z)\cdots(1 - \overline{\lambda_n}z)}.$$

Proposition 5.4. *For any $A_\varphi \in \mathcal{T}_\Theta$ we have*

$$CA_\varphi C = A_\varphi^*.$$

Proof. Recall that $Cf = \overline{\zeta f}\Theta$ on $\mathbb{T}$.

$$\begin{aligned}
\langle CA_\varphi Cf, g\rangle &= \langle Cg, A_\varphi Cf\rangle \\
&= \langle Cg, P_\Theta(\varphi Cf)\rangle \\
&= \langle Cg, \varphi Cf\rangle \\
&= \langle \overline{\zeta g}\Theta, \varphi\overline{\zeta f}\Theta\rangle \\
&= \langle \overline{g}, \varphi\overline{f}\rangle \\
&= \int \overline{g\varphi} f dm \\
&= \langle \overline{\varphi} f, g\rangle \\
&= \langle \overline{\varphi} f, P_\Theta g\rangle \\
&= \langle P_\Theta(\overline{\varphi} f), g\rangle \\
&= \langle A_{\overline{\varphi}} f, g\rangle \\
&= \langle A_\varphi^* f, g\rangle. \qquad \square
\end{aligned}$$

The previous result says that a truncated Toeplitz operator is C-symmetric. More about this in the last section.

Theorem 5.5. *If φ is a bounded Borel function and Θ is inner, then A_φ on $(\Theta H^2)^\perp$ can be written as*

$$A_\varphi = \int \varphi(U_\alpha) dm(\alpha).$$

Proof. Let $f, g \in (\Theta H^2)^\perp$. Then

$$\begin{aligned}
\langle A_\varphi f, g\rangle &= \langle P_\Theta(\varphi f), g\rangle \\
&= \langle \varphi f, P_\Theta g\rangle \\
&= \int \varphi f \overline{g} dm \\
&= \int \left(\int \varphi f \overline{g} d\sigma_\alpha\right) dm(\alpha) \quad \text{(disintegration theorem)} \\
&= \int \left(\int \varphi V_\alpha f \overline{V_\alpha g} d\sigma_\alpha\right) dm(\alpha) \quad \text{(Poltoratski)} \\
&= \int \langle M_\varphi V_\alpha f, V_\alpha g\rangle_{L^2(\sigma_\alpha)} dm(\alpha) \quad \text{(Clark)} \\
&= \int \langle V_\alpha^* M_\varphi V_\alpha f, V_\alpha^* V_\alpha g\rangle_{H^2} dm(\alpha) \quad \text{(Clark)} \\
&= \int \langle \varphi(U_\alpha) f, g\rangle_{H^2} dm(\alpha).
\end{aligned}$$

Thus we have shown the formula holds weakly. $\square$

6. Complex symmetric operators

A bounded operator A on a Hilbert space $\mathcal{H}$ is *complex symmetric* if there is a conjugation C on $\mathcal{H}$ so that

$$CAC = A^*.$$

Here a *conjugation* C on $\mathcal{H}$ is a mapping satisfying: (i) C is anti-linear, (ii) involutive $C^2 = I$; (iii) $\langle Cx, Cy\rangle = \langle y, x\rangle$. We will only present a few ideas here but a more thorough treatment of complex symmetric operators can be found in [9].

Example 6.1. Any $n \times n$ Toeplitz matrix T is C-symmetric with respect to the conjugation $C : \mathbb{C}^n \to \mathbb{C}^n$, defined by

$$C(c_1, \cdots, c_n) = (\overline{c_n}, \cdots, \overline{c_1}).$$

Example 6.2. The classical Volterra operator

$$V : L^2[0,1] \to L^2[0,1], \quad (Vf)(x) = \int_0^x f(t)dt$$

is C-symmetric with respect to the conjugation $C : L^2[0,1] \to L^2[0,1]$, $(Cf)(x) = \overline{f(1-x)}$. Indeed a quick integral computation will show that

$$(CVCf)(x) = \int_x^1 f(t)dt = (V^*f)(x),$$

Example 6.3. Every normal operator is complex symmetric. To see this note from the spectral theorem that a normal operator is unitarily equivalent to $(M_\phi, L^2(\mu))$. Here μ is a positive measure on $\mathbb{C}$, $\phi \in L^\infty(\mu)$ and M_ϕ is multiplication by ϕ on $L^2(\mu)$. The conjugation here is $Cf = \overline{f}$. One can check that $CM_\phi C = M_{\overline{\phi}} = M_\phi^*$.

Here is a non-example.

Example 6.4. The unilateral shift S on H^2 is *not* complex symmetric. Indeed, if there were a conjugation C on H^2 for which $CSC = S^*$, then

$$I = S^*S = S^*CCS = CSS^*C$$

which would imply that $SS^* = I$. This last statement is clearly not true since $SS^*1 = 0$.

Question 6.5. Is every complex symmetric operator unitarily equivalent to a truncated Toeplitz operator A_φ on some model space $(\Theta H^2)^\perp$?

For example, complex symmetric operators such as 2×2 matrices, normal operators, the Voltera operator, k-fold inflation of Tyeplitz matrices, rank-one operators, are all complex symmetric and are all known to be unitarily equivalent to truncated Toeplitz operators. This seems to give some evidence that there is a positive answer to this conjecture.

Example 6.6. One can indeed show that every normal operator is unitarily equivalent to a truncated Toeplitz operator [3]. Let us first focus on a relatively non-technical example of a normal *matrix* N. Indeed: By the Spectral Theorem, we know that N is unitarily equivalent to the diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of N, repeated according to their multiplicity. Select a Clark unitary operator $U = U_\alpha$ corresponding to $\Theta(z) = z^n$. Note that that U along with is $p(U)$ for any polynomial $p(z)$ are truncated Toeplitz operators (equivalently Toeplitz matrices). This last statement can be easily verified with some simple matrix computations. Also note that the eigenvalues $\zeta_1, \zeta_2, \ldots, \zeta_n$ of U all have multiplicity one. Thus, there exists an (interpolating) polynomial $p(z)$ such that $p(\zeta_i) = \lambda_i$ for $i = 1, 2, \ldots, n$. It follows that $p(U)$ is unitarily equivalent to $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and hence to N itself.

Here is a sketch of the proof of the general result [3]: A (general) normal operator N is unitarily equivalent to $(M_\phi, L^2(\mu))$ for some positive measure on $\mathbb{C}$ and $\phi \in L^\infty(\mu)$. By a very deep result of Halmos and vonNeumann, there is a singular measure ν on $\mathbb{T}$ (with respect to m) and a $\psi \in L^\infty(\nu)$ so that $(M_\phi, L^2(\mu))$ is unitarily equivalent to $(M_\psi, L^2(\nu))$. The measure ν is the Clark measure σ_1 for some inner function Θ (Every positive singular measure on $\mathbb{T}$ is the Clark measure for some inner function). The Clark unitary U_1 is unitarily equivalent to $(M_z, L^2(\nu))$. The operators U_1 as well as $\psi(U_1)$ are truncated Toeplitz operators on $(\Theta H^2)^\perp$ [16] and

$$\psi(U_1) \cong (M_\psi, L^2(\nu)) \cong (M_\phi, L^2(\nu)) \cong N.$$

If one is willing to settle for similarity, one can use the facts that any $n \times n$ matrix is similar to its Jordan form and that each Jordan block is a C-symmetric matrix to define a new conjugation on $\mathbb{C}^n$ to eventually show that any $n \times n$ matrix is similar to a complex symmetric matrix [9]. Working a bit harder along these lines, one can show that every $n \times n$ matrix is similar to a truncated Toeplitz operator on $(\Theta H^2)^\perp$, where Θ is a finite Blaschke product [3].

References

1. Juan Arias de Reyna, *Pointwise convergence of Fourier series*, Lecture Notes in Mathematics, vol. 1785, Springer-Verlag, Berlin, 2002.
2. Albrecht Böttcher and Bernd Silbermann, *Analysis of Toeplitz operators*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, Prepared jointly with Alexei Karlovich.
3. J. A. Cima, S. R. Garcia, W. T. Ross, and W. R. Wogen, *Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity*, Indiana U. Math. J. **59** (2010), no. 2, 595–620.
4. J. A. Cima, A. L. Matheson, and W. T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006.
5. J. A. Cima and W. T. Ross, *The backward shift on the Hardy space*, Mathematical Surveys and Monographs, vol. 79, American Mathematical Society, Providence, RI, 2000.

6. R. G. Douglas, H. S. Shapiro, and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator.*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 37–76.
7. Ronald G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 49.
8. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
9. S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic).
10. S. R. Garcia and William T. Ross, *Recent progress on truncated Toeplitz operators*, Blaschke products and their aplications, Fields Institute Communications, vol. 65, Springer-Verlag, New York, 2013.
11. S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077.
12. Stephan Ramon Garcia, William T. Ross, and Warren R. Wogen, *Spatial isomorphisms of algebras of truncated Toeplitz operators*, Indiana Univ. Math. J. **59** (2010), no. 6, 1971–2000.
13. J. Garnett, *Bounded analytic functions*, first ed., Graduate Texts in Mathematics, vol. 236, Springer, New York, 2007.
14. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall Inc., Englewood Cliffs, N. J., 1962.
15. W. T. Ross and H. S. Shapiro, *Generalized analytic continuation*, University Lecture Series, vol. 25, American Mathematical Society, Providence, RI, 2002.
16. D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526.
17. S. Takenaka, *On the orthonormal functions and a new formula of interpolation*, Jap. J. Math. **2** (1925), 129 – 145.

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