LENS LECTURES ON ALEKSANDROV-CLARK MEASURES

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1. Introduction

In this series of three 90 minute lectures I will give a gentle introduction to the topic of Aleksandrov-Clark measures which turn out to have an uncanny way of appearing in various areas of analysis. In short, for an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ is the open unit disk, there is a family $\{\mu_\alpha : |\alpha| = 1\}$ of positive finite measures on the unit circle $\mathbb{T}$ associated with φ by the formula

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu_\alpha(\xi), \quad z \in \mathbb{D}.$$

These measures, called Aleksandrov-Clark measures, appear as the spectral representing measures for a certain important unitary operator (the rank-one unitary perturbation of the compressed shift) (Clark's theorem). They disintegrate Lebeague measure on the circle (Aleksandrov's theorem). The help us compute adjoints and essential norms of composition operators on the Hardy space. The list goes on and on.

This course is intended for graduate students with just the basics of real analysis (measure theory, Lebesgue theory), functional analysis (Riesz representation theorem, Hahn-Banach theorem, spectral theorem), complex analysis (Cauchy theory), and of course, the basics of linear algebra. I will build everything from the ground up or provide references for the technical details.

Lecture 1. Self maps and measure theory.

Lecture 2. Aleksandrov-Clark measures.

Lecture 3. Clark theory, Aleksandrov operators, composition operators

2. A motivational example

Let $\mathcal{H}$ be a complex separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and A be a bounded cyclic self-adjoint operator on $\mathcal{H}$ with cyclic vector φ. Here *cyclic* means that

$$\bigvee \{A^n \varphi : n \in \mathbb{N}_0\} = \mathcal{H}.$$

The symbol $\bigvee$ will denote the closed linear span in $\mathcal{H}$. The easiest example to think of here is

$$M_x : L^2[0, 1] \to L^2[0, 1], \quad M_x f = xf, \quad \varphi \equiv 1.$$

For the self-adjoint operator A, cyclic vector φ, and $\lambda \in \mathbb{R}$ let

$$A_\lambda := A + \lambda(\varphi \otimes \varphi),$$

where $\varphi \otimes \varphi$ is the rank-one operator on $\mathcal{H}$ defined by

$$(\varphi \otimes \varphi)(v) = \langle v, \varphi \rangle \varphi, \quad v \in \mathcal{H}.$$

Clearly A_λ is self-adjoint (since λ is real and the rank-one tensor $\varphi \otimes \varphi$ is self adjoint). Moreover, A_λ is also cyclic with cyclic vector φ. Indeed

$$A_\lambda \varphi = A\varphi + \lambda \|\varphi\|^2 \varphi$$

and so

$$A_\lambda \varphi \in \operatorname{span}\{\varphi, A\varphi\}.$$

Continuing in this fashion we see that

$$\bigvee \{A_\lambda^n \varphi : n \in \mathbb{N}_0\} = \bigvee \{A^n \varphi : n \in \mathbb{N}_0\} = \mathcal{H}.$$

By the spectral theorem for bounded cyclic self-adjoint operators we know that

$$A_\lambda \cong (M_x, L^2(\mu_\lambda))$$

for some positive finite Borel measure on $\mathbb{R}$. An interesting result here is the following disintegration theorem.

Theorem 2.1. *For the situation above*

$$\left(\int d\mu_\lambda\right) d\lambda = dx$$

in the sense that for all $f \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t) d\mu_\lambda(t)\right) d\lambda = \int_{\mathbb{R}} f(t) dt.$$

Example 2.2. If $A = M_x$ on $L^2[0,1]$ and $\varphi \equiv 1$, then

$$A_\lambda = M_x + \lambda(1 \otimes 1).$$

More specifically,

$$A_\lambda f = xf + \lambda \int_0^1 f(t) dt, \quad f \in L^2[0,1].$$

We won't give the details here but the spectral measures μ_λ turns out to be

$$d\mu_\lambda = \frac{\chi_{[0,1]}}{(1 + \lambda \log(\frac{1-x}{x}))^2 + \lambda^2 \pi^2} dx + \frac{e^{-\frac{1}{\lambda}}}{\lambda^2 (1 - e^{-\frac{1}{\lambda}})} \delta_{(1 - e^{-\frac{1}{\lambda}})^{-1}}.$$

Notice that μ_λ has both absolutely continuous and singular parts (with respect to Lebesgue measures on $\mathbb{R}$).

Example 2.3. Let $A : \mathbb{C}^2 \to \mathbb{C}^2$ be represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The linear transformation A is certainly cyclic with cyclic (column) vector $v = \frac{1}{\sqrt{2}}(1,1)^T$. The eigenvalues of $A_\lambda = A + \lambda(v \otimes v)$ turn out to be

$$\lambda_1 = \frac{1}{2}(1 + \lambda - \sqrt{1+\lambda^2}), \quad \lambda_2 = \frac{1}{2}(1 + \lambda + \sqrt{1+\lambda^2}).$$

The spectral measures for A_λ turn out to be

$$\mu_\lambda = \frac{-\lambda + \sqrt{1+\lambda^2}}{2\sqrt{1+\lambda^2}}\delta_{\lambda_1} + \frac{\lambda + \sqrt{1+\lambda^2}}{2\sqrt{1+\lambda^2}}\delta_{\lambda_2}.$$

So why did we work this example? We wanted to review the spectral theorem as well as showing a student that the material we are about to present in the context of analytic functions on the disk, appear in a much broader context.

3. Self-maps of the disk

These notes will focus on a class of measures associated with analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$. Such maps are often called *analytic self-maps of the disk*. In the study of such measures is the boundary behavior of the associated map φ. On old theorem of Fatou [4] says the following:

Theorem 3.1 (Fatou's theorem)**.** *For an analytic self-map φ, the radial limit*

$$\lim_{r \to 1^-} \varphi(r\zeta)$$

exists for m-almost every $\zeta \in \mathbb{T}$.

We often write $\varphi(\zeta)$ for this limit whenever it exists (and is finite!).

Proposition 3.2. *The set $\{\zeta \in \mathbb{T} : \varphi(\zeta) \text{ exists}\}$ is a Borel set.*

Actually it turns out that the above limit also exists for m-almost every ζ but the radial limit above is replaced by a *non-tangential limit*. By this we mean that $\varphi(z) \to \varphi(\zeta)$ not only as $z \to \zeta$ along the radius connecting 0 and ζ but as $z \to \zeta$ inside any Stolz domain

$$\left\{ z \in \mathbb{D} : \frac{|z - \zeta|}{1 - |z|} < \alpha \right\}, \quad \alpha \in (1, \infty).$$

This is a triangular shaped region with vertex at ζ. We often write

$$\angle \lim_{z \to \zeta} \varphi(z)$$

for the non-tangential limit.

Definition 3.3. A self-map φ is said to be *inner* if $|\varphi(\zeta)| = 1$ for m-almost every $\zeta \in \mathbb{T}$.

Example 3.4. For $a \in \mathbb{D}$ consider the function

$$\varphi(z) = \frac{z-a}{1-\overline{a}z}, \quad z \in \mathbb{D}.$$

Note that

$$\varphi(e^{i\theta})\overline{\varphi(e^{i\theta})} = \frac{e^{i\theta}-a}{1-\overline{a}e^{i\theta}} \frac{e^{-i\theta}-\overline{a}}{1-ae^{-i\theta}} = 1, \quad \theta \in [0, 2\pi],$$

and so, by the maximum modulus theorem, φ is an analytic self-map. The above calculation shows that φ has unimodular boundary values and so φ is inner.

Example 3.5. Consider

$$\varphi(z) = e^{\frac{z+1}{z-1}}, \quad z \in \mathbb{D}.$$

Note that

$$\left|\exp\left(\frac{z+1}{z-1}\right)\right| = \exp\left(\Re\left(\frac{z+1}{z-1}\right)\right) = \exp\left(-\frac{1-|z|^2}{|1-z|^2}\right) < 1, \quad z \in \mathbb{D}.$$

Thus φ is an analytic self-map. From the previous identity one can easily check that $|\varphi(e^{i\theta})| = 1$ if $\theta \in (0, 2\pi)$. Thus φ is an inner function. If $\alpha_1, \cdots, \alpha_n$ are positive numbers and $\theta_1, \cdots, \theta_n \in [0, 2\pi]$, one can also check that

$$\varphi(z) = \exp\left(\alpha_1 \frac{z+e^{i\theta_1}}{z-e^{i\theta_1}} + \cdots + \alpha_n \frac{z+e^{i\theta_n}}{z-e^{i\theta_n}}\right)$$

is also an inner function.

Due to the limited amount of time, I will not get into all of the details of inner functions here. They are carefully worked out in standard texts [4, 5].

For a sequence of points $(a_n)_{n\geq 1}$ in $\mathbb{D} \setminus \{0\}$ which satisfy the condition

$$\sum_{n=1}^{\infty}(1-|a_n|) < \infty,$$

one can show that the infinite product

$$B(z) := \prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \frac{z-a_n}{1-\overline{a_n}z},$$

called a *Blaschke product*, converges uniformly on compact subsets of $\mathbb{D}$. Work of Blaschke shows the following:

Theorem 3.6 (Blaschke)**.** *For a sequence of points $(a_n)_{n\geq 1} \subset \mathbb{D} \setminus \{0\}$ satisfying $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$, the Blaschke product B is an inner function.*

There is another basic type of inner function associated with a singular measure. See the next section for a reminder of the definition of a singular measure.

Theorem 3.7. *For a positive finite positive singular Borel measure μ on $\mathbb{T}$ the function*

$$s_\mu(z) := \exp\left(-\int \frac{\xi+z}{\xi-z} d\mu(\xi)\right), \quad z \in \mathbb{D},$$

is inner.

Proof. As was done previously one can show that

$$|s_\mu(z)| = \exp\left(\Re\left(-\int \frac{\xi+z}{\xi-z} d\mu(\xi)\right)\right) = \exp\left(-\int \frac{1-|z|^2}{|z-\xi|^2} d\mu(\xi)\right) < 1.$$

Thus $s_\mu \in H^2$. Now use Proposition 4.14 (below) to show that s_μ has unimodular boundary values m-almost everywhere. □

Definition 3.8. The function s_μ from the previous theorem is called a *singular inner function.*

Notice how the singular inner function from Example 3.5 is formed from $\mu = \delta_1$, the point mass at one.

Clearly the functions

$$e^{i\gamma} z^n B(z) s_\mu(z),$$

where $\gamma \in \mathbb{R}$, $n \in \mathbb{N}_0$, B is a Blaschke product, and s_μ is a singular inner function. It turns out that these are all of them.

Theorem 3.9 (Nevanlinna)**.** *Any inner function φ takes the form*

$$\varphi(z) = e^{i\gamma} z^n B(z) s_\mu(z).$$

The factors z^n, B, s_μ are unique.

The definition of an inner function says that φ has unimodular boundary values m-almost everywhere. There is on old theorem of Frostman (and expended by Ahern and Clark) which says when an inner function has unimodular boundary values at a particular point.

Theorem 3.10 (Frostman-Ahern-Clark)**.** *An inner function $\varphi = Bs_\mu$ and all of its inner divisors have limits which are unimodular at $\zeta \in \mathbb{T}$ if and only if*

$$\sum_{\lambda \in B^{-1}(\{0\})} \frac{1-|\lambda|^2}{|\zeta-\lambda|} + \int \frac{d\mu(\xi)}{|\xi-\zeta|} < \infty.$$

The next concept here is the notion of an angular derivative.

Definition 3.11. For an analytic self-map φ and $\zeta \in \mathbb{T}$ we say that φ has an *angular derivative* at ζ if

$$\angle \lim_{z \to \zeta} \varphi(z) \in \mathbb{T};$$

$$\angle \lim_{z \to \zeta} \varphi'(z) \text{ exists.}$$

Note that if φ has an analytic continuation to a neighborhood of ζ and $|\varphi(\zeta)| = 1$, then the angular derivative exists.

For inner functions, there is this following definitive criterion for the existence of angular derivatives due to Ahern and Clark.

Theorem 3.12 (Frostman-Ahern-Clark)**.** *An inner function $\varphi = Bs_\mu$ has a finite angular derivative at $\zeta \in \mathbb{T}$ if and only if*

$$\sum_{\lambda \in B^{-1}(\{0\})} \frac{1-|\lambda|^2}{|\zeta - \lambda|^2} + \int \frac{d\mu(\xi)}{|\xi - \zeta|^2} < \infty.$$

Remark 3.13. Much of the results of this section are discussed in detail in [1, 4, 5].

4. Some measure theory and harmonic analysis reminders

Let M denote the space of complex Borel measures on $\mathbb{T}$. We will use $m = \frac{d\theta}{2\pi}$ to denote normalized Lebesgue measure on $\mathbb{T}$. Here is a reminder of some definitions [8].

Definition 4.1.
(1) For $\mu \in M$ we say that μ is *absolutely continuous* with respect to m, written $\mu \ll m$, if $\mu(A) = 0$ whenever A is a Borel subset of $\mathbb{T}$ with $m(A)$ =0.
(2) We say that μ is *singular* with respect to m, written $\mu \perp m$, if there are disjoint Borel sets A and B with $A \cup B = \mathbb{T}$ and $\mu(A) = m(B) = 0$.
(3) A measure μ is *positive*, written $\mu \in M_+$, if $\mu(E) \geq 0$ for every Borel subset E of $\mathbb{T}$.

Theorem 4.2 (Jordan decomposition theorem)**.** *Every $\mu \in M$ can be written uniquely as*

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4), \quad \mu_j \in M_+.$$

Theorem 4.3 (Radon-Nikodym theorem)**.** *A measure $\mu \in M$ is absolutely continuous if and only if $d\mu = f dm$ for some $f \in L^1(m)$, i.e.,*

$$\mu(E) = \int_E f dm$$

for all Borel subset E of $\mathbb{T}$.

Theorem 4.4 (Lebesuge decomposition theorem)**.** *Every $\mu \in M$ can be written uniquely as $\mu = \mu_a + \mu_s$, where $\mu_a \ll m$ and $\mu_s \perp m$.*

Theorem 4.5 (Riesz Representation Theorem)**.** *If C is the Banach space of continuous functions on $\mathbb{T}$ normed with the usual supremum norm, then the dual C^* of C is isometrically isomorphic to M via the dual pairing*

$$\langle g, \mu \rangle = \int f d\mu.$$

Corollary 4.6. *If*

$$\widehat{\mu}(n) := \int \overline{\xi}^n d\mu(\xi), \quad n \in \mathbb{Z},$$

the n-th Fourier coefficient of $\mu \in M$, is equal to zero for all $n \in \mathbb{Z}$, then $\mu \equiv 0$.

Proof. Suppose that $\widehat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, then

$$\int f d\mu = 0$$

for every trigonometric polynomial f. Since such functions are dense in C, we see that the above holds for all $f \in C$. By the Riesz Representation Theorem (Theorem 4.5) $\mu \equiv 0$. □

Definition 4.7.

$$P_z(\zeta) := \frac{1 - |z|^2}{|\zeta - z|^2}, \quad z \in \mathbb{D}, \zeta \in \mathbb{T},$$

is the standard *Poisson kernel*.

Observe that

$$P_z(\zeta) = \Re\left(\frac{1 + \overline{\zeta}z}{1 - \overline{\zeta}z}\right)$$

and so $P_z(\zeta)$ is a positive harmonic function of z on $\mathbb{D}$ (being the real part of an analytic function). Furthermore, if $z = r\xi$, where $\xi \in \mathbb{T}$ and $r \in (0, 1)$, then writing

$$\frac{1 + \overline{\zeta}z}{1 - \overline{\zeta}z} = 1 + 2\frac{\overline{\zeta}r\xi}{1 - \overline{\zeta}r\xi},$$

using geometric series to expand the last term as a series, and then taking real parts, we see that

$$P_{r\xi}(\zeta) = \sum_{n=-\infty}^{\infty} r^{|n|}\overline{\zeta}^n \xi^n. \tag{4.8}$$

From here we can us the fact that

$$\int_{\mathbb{T}} \overline{\zeta}^n dm(\zeta) = \delta_{n,0}$$

to see that

$$\int_{\mathbb{T}} P_z(\zeta) dm(\zeta) = 1, \quad z \in \mathbb{D}.$$

Definition 4.9. For $\mu \in M$, let

$$(P\mu)(z) := \int P_z(\zeta) d\mu(\zeta)$$

be the *Poisson integral* of μ.

Using (4.8) we see that

$$(P\mu)(r\xi) = \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) r^{|n|} \xi^{n}. \tag{4.10}$$

This following fact will be used several times in these notes.

Proposition 4.11. *The set $\{P_z : z \in \mathbb{D}\}$ has dense linear span in C.*

Proof. Suppose $\mu \perp P_z$ for all $z \in \mathbb{D}$. Then

$$0 = \langle P_z, \mu \rangle = (P\mu)(z), \quad z \in \mathbb{D}.$$

But by formula (4.10) we have

$$0 = \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) r^{|n|} \xi^{n}, \quad r\xi \in \mathbb{D}.$$

Since $|\widehat{\mu}(n)| \leq \|\mu\|$ one can see that for fixed $r \in (0,1)$ the above series converges uniformly in ξ. This means that for each $k \in \mathbb{Z}$

$$0 = \int_{\mathbb{T}} \overline{\xi^{k}} \sum_{n=-\infty}^{\infty} \widehat{\mu}(n) r^{|n|} \xi^{n} dm(\xi) = \sum_{n=-\infty}^{\infty} r^{|n|} \widehat{\mu}(n) \int_{\mathbb{T}} \xi^{n-k} dm = r^{|k|} \widehat{\mu}(k).$$

Since $\widehat{\mu}(k) = 0$ for all $k \in \mathbb{Z}$, one can apply Corollary 4.6 to see that $\mu \equiv 0$. An application of the Hahn-Banach separation theorem completes the proof. □

For $\mu \in M_{+}$, the Poisson integral $P\mu$ is a positive harmonic function on $\mathbb{D}$ (differentiating under the integral). A classical theorem of Herglotz says these are the only ones.

Theorem 4.12 (Herglotz)**.** *Suppose u is a positive harmonic function on $\mathbb{D}$. Then $u = P\mu$ for some unique $\mu \in M_{+}$.*

Proof. Let $u_r(\zeta) := u(r\zeta)$ and note that the family of measures

$$\{u_r dm : 0 < r < 1\}$$

satisfy

$$\|u_r dm\| = \int_{\mathbb{T}} u_r(\zeta) dm(\zeta) = u(0)$$

and are thus uniformly bounded in total variation norm $\|\cdot\|$ on M. Notice how we used the positivity of u_r as well as the mean-value theorem for harmonic functions. By the Banach-Alaoglu theorem, $u_{r_n} dm \to d\mu$ weak-$*$ for some sequence $r_n \to 1$ and some $\mu \in M_{+}$. That is to say

$$\int_{\mathbb{T}} g u_{r_n} dm \to \int_{\mathbb{T}} g d\mu, \quad g \in C.$$

Apply this to $g = P_z$ to get

$$\begin{aligned}(P\mu)(z) &= \lim_{r_n \to 1} \int_{\mathbb{T}} P_z(\zeta) u_r(\zeta) dm(\zeta) \\ &= \lim_{r_n \to 1} u(r_n z) \\ &= u(z).\end{aligned}$$

So $u = P\mu$ for some measure $\mu \in M_+$. For uniqueness, suppose $u = P\mu_1 = P\mu_2$. Then $P(\mu_1 - \mu_2) \equiv 0$. By Proposition 4.11, $\mu_1 = \mu_2$. □

Let's talk about derivatives of measures.

Definition 4.13. For $\mu \in M_+$, define, for each $\zeta \in \mathbb{T}$,

$$(\underline{D}\mu)(\zeta) := \liminf_{t \to 0^+} \frac{\mu(I(t,\zeta))}{m(I(t,\zeta)};$$

$$(\overline{D}\mu)(\zeta) := \limsup_{t \to 0^+} \frac{\mu(I(t,\zeta))}{m(I(t,\zeta)},$$

where, for $\zeta \in \mathbb{T}$ and $t > 0$, $I(t, \zeta)$ is the arc of $\mathbb{T}$ subtended by the points $e^{-it}\zeta$ and $e^{it}\zeta$. The above derivatives are called the lower and upper *symmetric derivatives* of μ.

Here are some technical (standard) facts we will not prove since they are carefully worked out in [8].

Proposition 4.14. *For $\mu \in M_+$. we have*

1. $\underline{D}\mu = \overline{D}\mu$ *m-almost everywhere.*
2. *For every $\zeta \in \mathbb{T}$,*

$$(\underline{D}\mu)(\zeta) \leq \liminf_{r \to 1} (P\mu)(r\zeta) \leq \limsup_{r \to 1} (P\mu)(r\zeta) \leq (\overline{D}\mu)(\zeta).$$

3. *If $D\mu$ is the m-almost everywhere defined function $D\mu = \underline{D}\mu = \overline{D}\mu$, then if $(D\mu)(\zeta)$ exists, then $(D\mu)(\zeta) = \lim_{r \to 1} (P\mu)(r\zeta)$.*
4. *If $\mu = \mu_a + \mu_s$, where $\mu_a \ll m$ and $\mu_s \perp m$, is the Lebesgue decomposition then $D\mu_s = 0$ and $D\mu = D\mu_a$ m-almost everywhere.*
5. *μ_s is carried by $\{\underline{D}\mu = \infty\}$.*
6. *μ_s is carried by $\{0 < \underline{D}\mu < \infty\}$.*

We used the word *carrier* in the above theorem. Let us by quite precise about this.

Remark 4.15. For $\mu \in M$, consider the union $\mathcal{U}$ of all open subsets $U \subset \mathbb{T}$ for which $\mu(U) = 0$. The set $\mathbb{T} \setminus \mathcal{U}$ is a closed set called the *support* of μ. A Borel set $H \subset \mathbb{T}$ for which $\mu(H \cap A) = \mu(A)$ for all Borel sets $A \subset \mathbb{T}$ is called a *carrier* for μ. Certainly the support is a carrier but a carrier need not be the support. In fact, it need not even be closed. For example, if f is continuous on $\mathbb{T}$ and $d\mu = f dm$, then a carrier of μ is $\mathbb{T} \setminus f^{-1}(\{0\})$ (which is open) while the support of μ is the closure of this set.

5. The basics of Aleksandrov-Clark measures

For an analytic self-map φ and $|\alpha| = 1$ consider the function u_α on $\mathbb{D}$ defined by

$$u_\alpha(z) := \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = P_{\varphi(z)}(\alpha).$$

Since φ is analytic and $z \mapsto P_z(\alpha)$ is harmonic, then $z \mapsto P_{\varphi(z)}(\alpha)$ is harmonic. Clearly u_α is also positive on $\mathbb{D}$ and so by the Herglotz theorem (Theorem 4.12)

$$u_\alpha(z) = (P\mu_\alpha)(z)$$

for some unique $\mu_\alpha \in M_+$. Denote the family

$$\mathcal{A}_\varphi := \{\mu_\alpha : |\alpha| = 1\}$$

as the *Aleksandrov-Clark measures* (often written as AC measures) associated with φ. Why two names? When φ is an inner function then this family of measures, along with an associated family of unitary operators (more about this later in the notes), was first studied by Clark. General self-maps φ were later studied by Aleksandrov.

Example 5.1. Let us compute the family of AC measures for $\varphi(z) = z^n$. Here, for each $\alpha \in \mathbb{T}$ we want to find a $\mu_\alpha \in M_+$ so that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \frac{1 - |z|^{2n}}{|\alpha - z|^2} = \int \frac{1 - |z|^2}{|\xi - z|^2} d\mu_\alpha(\xi), \quad z \in \mathbb{D}.$$

To give a student an idea of the thinking that goes on here (and to foreshadow later results), let us try to derive the measure rather that just having me give it to you for you to verify. Let $z = r\zeta$, where $0 < r < 1$ and $|\zeta| = 1$. Then the above identity becomes

$$\frac{1 - r^{2n}}{|\alpha - r^n \zeta^n|^2} = \int \frac{1 - r^2}{|\xi - r\zeta|^2} d\mu_\alpha(\xi). \tag{5.2}$$

When $\zeta^n \neq \alpha$, then the LHS of the previous identity goes to zero as $r \to 1$. Thus the RHS must do the same. By Proposition 4.14 this says that μ_α is placing no mass on $\mathbb{T} \setminus \{\zeta : \zeta^n = \alpha\}$. Let $\{\zeta_j : 1 \leq j \leq n\}$ denote the n solutions to $\zeta^n = \alpha$. Note that these are equally spaced points on $\mathbb{T}$. The above analysis says that

$$\mu_\alpha = \sum_{j=1}^{n} c_j \delta_{\zeta_j}.$$

Thus we are looking to find positive constants c_j so that

$$\frac{1 - r^{2n}}{|\alpha - r^n \zeta^n|^2} = \sum_{j=1}^{n} c_j \frac{1 - r^2}{|\zeta_j - r\zeta|^2}, \quad r \in (0,1), \zeta \in \mathbb{T}.$$

Fix k so that $1 \leq k \leq n$ and let $\zeta = \zeta_k$ in the above formula. This gives us (since $\zeta_k^n = \alpha$)

$$\frac{1 - r^{2n}}{(1 - r^n)^2} = \sum_{j=1}^{n} c_j \frac{1 - r^2}{|\zeta_j - r\zeta_k|^2}.$$

Write the above expression (after a little simplification) as

$$\frac{1 + r^n}{1 - r^n} = c_k \frac{1 + r}{1 - r} + \sum_{j \neq k} c_j \frac{1 - r^2}{|\zeta_j - r\zeta_k|^2}.$$

Now multiply the above expression through by $1 - r^n$ and do a little algebra to get

$$1 + r^n = c_k(1 + r)(1 + r + r^2 + \cdots + r^{n-1}) + \sum_{j \neq k} c_j \frac{(1 - r^2)(1 - r^n)}{|\zeta_j - r\zeta_k|^2}.$$

Now let $r \to 1$ (the finite sum goes to zero) to get

$$c_k = \frac{1}{n}.$$

We have just computed our first AC measure

$$\mu_\alpha = \frac{1}{n} \sum_{j=1}^{n} \delta_{\zeta_j},$$

where $\{\zeta, \cdots, \zeta_n\}$ are the solutions to $\varphi(\zeta) = \alpha$.

Example 5.3. Suppose

$$\varphi(z) = \exp\left(\frac{z + 1}{z - 1}\right).$$

One can check that

$$\left|\exp\left(\frac{z + 1}{z - 1}\right)\right| = \exp\left(\Re\left(\frac{z + 1}{z - 1}\right)\right) = \exp\left(-\frac{1 - |z|^2}{|1 - z|^2}\right) < 1, \quad z \in \mathbb{D},$$

and so φ is an analytic self-map of $\mathbb{D}$. Again, for each $\alpha \in \mathbb{T}$ we want to find a $\mu_\alpha \in M_+$ so that

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int \frac{1 - |z|^2}{|\xi - z|^2} d\mu_\alpha(\xi), \quad z \in \mathbb{D}.$$

As in the previous example, we let $z = r\zeta, |\zeta| = 1, r \in (0, 1)$ and, as in the previous example, we argue that the LHS approaches zero as $r \to 1$ except for solutions to the equation $\varphi(\zeta) = \alpha$. Solutions to this equations are a discrete set of points

$$\zeta_n = \frac{i \arg \alpha + 2\pi i n + 1}{i \arg \alpha + 2\pi i n - 1}, \quad n \in \mathbb{Z},$$

which accumulate at 1. As in the previous example this says that

$$\mu_\alpha = \sum_{n \in \mathbb{Z}} c_n \delta_{\zeta_n}.$$

Now we need to compute positive constants c_n which satisfy

$$\frac{1-|\varphi(r\zeta)|^2}{|\alpha-\varphi(r\zeta)|^2} = \sum_{n\in\mathbb{Z}} c_n \frac{1-r^2}{|\zeta_n - r\zeta|^2}, \quad \zeta \in \mathbb{T}, r \in (0,1).$$

Let $\zeta = \zeta_k$ and observe that the previous equation becomes

$$\frac{1-|\varphi(r\zeta_k)|^2}{|\alpha-\varphi(r\zeta_k)|^2} = c_k \frac{1-r^2}{(1-r)^2} + \sum_{n\neq k} \frac{1-r^2}{|\zeta_n - r\zeta_k|^2}.$$

Multiply both sides by

$$\frac{|\alpha-\varphi(r\zeta_k)|^2}{1-r^2}$$

to get

$$\frac{1-|\varphi(r\zeta_k)|^2}{1-r^2} = c_k \left|\frac{\alpha-\varphi(r\zeta_k)}{1-r}\right|^2 + \sum_{n\neq k} \frac{|\alpha-\varphi(r\zeta_k)|^2}{|\zeta_n - r\zeta_k|^2}.$$

Now take limits as $r \to 1$ and argue (since φ is analytic in a neighborhood of each ζ_k) that

$$|\varphi'(\zeta_k)| = c_k|\varphi'(\zeta_k)|^2 + 0$$

and hence $c_k = |\varphi'(\zeta_k)|^{-1}$. In summary,

$$d\mu_\alpha = \sum_{n\in\mathbb{Z}} \frac{1}{|\varphi'(\zeta_n)|}\delta_{\zeta_n}.$$

Example 5.4. Suppose that φ is an analytic self-map of $\mathbb{D}$ which maps $\mathbb{D}$ to a compact subset of $\mathbb{D}$ (something like $\varphi(z) = z/2$). Then we have

$$\frac{1-|\varphi(r\zeta)|^2}{|\alpha-\varphi(r\zeta)|^2} = \int_{\mathbb{T}} P_{r\zeta}(\xi) d\mu_\alpha(\xi).$$

Taking limits as $r \to 1$ we use our earlier discussion of Poisson integrals (Proposition 4.14) to see that

$$\frac{d\mu_\alpha}{dm}(\zeta) = \frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2}$$

for m-almost every $\zeta \in \mathbb{T}$. Since the limits never vanish, we see that there is no singular part to μ_α and so indeed

$$d\mu_\alpha = \frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2} dm.$$

At this point, the reader might think that AC measures have to be very special and form an exclusive club. They don't.

Proposition 5.5. *If $\mu \in M_+$, then there exists an analytic self-map φ of $\mathbb{D}$ so that $\mu \in \mathcal{A}_\varphi$.*

Proof. Define
$$H_\mu(z) := \int \frac{\zeta + z}{\zeta - z} d\mu(\zeta), \quad z \in \mathbb{D},$$
and note that
$$\Re(H_\mu(z)) = \int \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta) = (P\mu)(z) > 0.$$
Thus H_μ maps $\mathbb{D}$ onto the right half plane $\{z : \Re z > 0\}$. The map
$$w \mapsto \frac{w-1}{w+1}$$
maps $\Re z$ onto $\mathbb{D}$ and so
$$\varphi(z) := \frac{H_\mu(z) - 1}{H_\mu(z) + 1}.$$
is an analytic self-map of $\mathbb{D}$.

A little algebra yields
$$H_\mu(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)}$$
and so
$$(P\mu)(z) = \Re(H_\mu(z)) = \frac{1 - |\varphi(z)|^2}{|1 - \varphi(z)|^2}$$
which means $\mu = \mu_1 \in \mathcal{A}_\varphi$. □

One last basic fact about AC measures is the total mass.

Proposition 5.6. *If $\mu_\alpha \in \mathcal{A}_\varphi$, then*
$$\mu_\alpha(\mathbb{T}) = \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2}.$$

Proof. Since P_0 is the constant function one, we have
$$\mu_\alpha(\mathbb{T}) = \int 1 d\mu_\alpha = \int P_0(\zeta) d\mu_\alpha(\zeta) = (P\mu_\alpha)(0) = \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2}. \quad \square$$

Corollary 5.7. *For fixed φ we have*

(1) *Each μ_α is a probability measure if and only if $\varphi(0) = 0$.*
(2) $\sup\{\mu_\alpha(\mathbb{T}) : \alpha \in \mathbb{T}\} < \infty$.

6. The Aleksandrov disintegration theorem

Before getting bogged down in all of the technical details about AC measures, let's discuss a fascinating result which is the analog of Theorem 2.1 discussed earlier.

Theorem 6.1 (Aleksndrov's disintegration theorem)**.** *For a self map φ with associated AC measures $\mathcal{A}_\varphi$ and $f \in C$,*
$$\int \left(\int f(\zeta) d\mu_\alpha(\zeta) \right) dm(\alpha) = \int f(\xi) dm(\xi).$$

Proof. For a fixed $z \in \mathbb{D}$ notice that

$$\begin{aligned}\int \left(\int P_z(\zeta) d\mu_\alpha(\zeta) \right) dm(\alpha) &= \int \left(\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} \right) dm(\alpha) \\ &= \int P_{\varphi(z)}(\alpha) dm(\alpha) \\ &= 1 \\ &= \int P_z(\zeta) dm(\zeta).\end{aligned}$$

Thus the disintegration theorem works for finite linear combinations of Poisson kernels.

To get the result for any $f \in C$, let $(f_n)_{n \geq 1}$ be a sequence of finite linear combinations of Poisson kernels which converge uniformly to f (Proposition 4.11). Let us first note that for each n, the function

$$\alpha \mapsto \int f_n d\mu_\alpha$$

is continuous on $\mathbb{T}$. To see this write

$$f_n = \sum_{j=1}^{N} c_j P_{z_j}$$

and, by the definition of an AC measure, observe that

$$\int f_n d\mu_\alpha = \sum_{j=1}^{N} c_j P_{\varphi(z_j)}(\alpha)$$

which is continuous in the variable α. We can extend this to show that

$$\alpha \mapsto \int f d\mu_\alpha$$

is also continuous. Indeed, by Corollary 5.7 we know that

$$B = \sup\{\mu_\alpha(\mathbb{T}) : \alpha \in \mathbb{T}\} < \infty.$$

Then

$$\left| \int f d\mu_\alpha - \int f_n d\mu_\alpha \right| \leq \|f - f_n\|_\infty \mu_\alpha(\mathbb{T}) \leq \|f - f_n\|_\infty B$$

which means, via uniform convergence of continuous functions, that

$$\alpha \mapsto \int f d\mu_\alpha$$

is continuous

Finally,

$$\begin{aligned}\int_{\mathbb{T}} f dm &= \lim_{n\to\infty} \int_{\mathbb{T}} f_n dm \quad \text{(uniform convergence)}\\ &= \lim_{n\to\infty} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f_n(\zeta) d\mu_\alpha(\zeta) \right) dm(\alpha) \quad \text{(disintegration formula)}\\ &= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(\zeta) d\mu_\alpha(\zeta) \right) dm_\alpha \quad \text{(dominated convergence theorem).}\end{aligned}$$

This proves the result. □

Remark 6.2. Aleksandrov showed quite a bit more here in that the continuous functions C can be replaced by L^1 in the disintegration theorem. There are quite a few technical issues here. For example, the inner integrals

$$\int f(\zeta) d\mu_\alpha(\zeta)$$

in the disintegration formula do not seem to be well defined for L^1 functions since indeed AC measures can be point masses and L^1 functions are defined m-almost everywhere. However, amazingly, the function

$$\alpha \mapsto \int f(\zeta) d\mu_\alpha(\zeta)$$

is defined for m-almost every α and is integrable. An argument with the monotone class theorem is used to prove this more general result. See [1] for the details.

A careful reading of the proof of the disintegration theorem will yield the following:

Corollary 6.3. *For a self-map φ, the function $\alpha \mapsto \mu_\alpha$ is continuous from $\mathbb{T}$ to the $(M, *)$, the space of measures endowed with the weak-$*$ topology.*

7. Carriers of AC measures

Recall from Proposition 4.14 that for $\mu = \mu_a + \mu_s \in M_+$, where $\mu_a \ll m$ and $\mu_s \perp m$, a carrier for μ_a is $\{0 < \underline{D}\mu < \infty\}$ and a carrier of μ_s is $\{\underline{D}\mu = \infty\}$. Suppose $\mu_\alpha \in \mathcal{A}_\varphi$ let us write the Lebesgue decomposition of μ_α as

$$d\mu_\alpha = h_\alpha dm + d\sigma_\alpha,$$

where $h_\alpha \in L^1(m)$.

Proposition 7.1. *The set $E_\alpha = \{\zeta \in \mathbb{T} : \varphi(\zeta) = \alpha\}$ is a Borel set and is a carrier for σ_α.*

Proof. Let

$$u_\alpha(z) := \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2}$$

and note that by Proposition 4.14

$$\{\zeta \in \mathbb{T} : (\underline{D}\mu_\alpha)(\zeta) = \infty\} \subset \{\zeta \in \mathbb{T} : u_\alpha(\zeta) = \infty\} \subset E_\alpha,$$

where $u_\alpha(\zeta)$ denotes the radial limit. Again by Proposition 4.14 $\{\underline{D}\mu_\alpha = \infty\}$ is a carrier for the singular part of μ_α, i.e., σ_α, we see that E_α is a carrier for σ_α. The proof that E_α is a Borel sets is a bit more technical and can be found in [1]. □

Corollary 7.2. *For an analytic self map φ the following are true:*

(1) $\sigma_\alpha \perp \sigma_\beta$ *when* $\alpha \neq \beta$.
(2) $\varphi(\zeta) = \alpha$ *for* σ_α*-almost every* $\zeta \in \mathbb{T}$.
(3) $\mu_\alpha = \sigma_\alpha$ *for all* α *if and only if* φ *is inner*
(4) *For a Borel subset* $B \subset \mathbb{T}$, *let* $\varphi^{-1}(B)$ *be the set of* $\zeta \in \mathbb{T}$ *such that* $\varphi(\zeta)$ *exists and* $\varphi(\zeta) \in B$. *Then*

$$\varphi^{-1}(B) = \bigcup_{\alpha \in B} E_\alpha.$$

8. Measure preserving

A nice application of the disintegration theorem is this theorem about measure preserving inner functions. Recall for an inner function φ and a Borel set $A \subset \mathbb{T}$, $\varphi^{-1}(A)$ is the set of $\zeta \in \mathbb{T}$ such that $\varphi(\zeta)$ exists and $\varphi(\zeta) \in B$.

Theorem 8.1. *Suppose φ is inner with $\varphi(0) = 0$. Then for any Borel set $A \subset \mathbb{T}$, $m(A) = m(\varphi^{-1}(A))$.*

Proof. Apply the disintegration theorem to the function $f = \chi_{\varphi^{-1}(A)}$, noting also that $\mu_\alpha = \sigma_\alpha$, since φ is inner, to get

$$\int_{\mathbb{T}} \left(\int_{\mathbb{T}} \chi_{\varphi^{-1}(A)}(\zeta) d\sigma_\alpha(\zeta) \right) dm(\alpha) = \int_{\mathbb{T}} \chi_{\varphi^{-1}(A)}(\zeta) dm(\zeta).$$

The above identity becomes

$$\int_{\mathbb{T}} \sigma_\alpha(\varphi^{-1}(A)) dm(\alpha) = m(\varphi^{-1}(A)).$$

Apply Corollary 7.2 to the LHS to get

$$\int_{\mathbb{T}} \sigma_\alpha \left(\bigcup_{\beta \in A} E_\beta \right) dm(\alpha) = m(\varphi^{-1}(A)).$$

Apply Corollary 7.2 (E_α is a carrier for σ_α) to get

$$\int_A \sigma_\alpha(E_\alpha) dm(\alpha) = m(\varphi^{-1}(A)).$$

Since we are assuming that $\varphi(0) = 0$ we know that σ_α is a probability measure carried by E_α and so $\sigma_\alpha(E_\alpha) = 1$ for all α. Thus the above becomes

$$m(A) = m(\varphi^{-1}(A))$$

and our proof is complete. □

9. Point masses and AC measures

We know from Proposition 7.1 that for $\mu_\alpha \in \mathcal{A}_\varphi$, the measure σ_α is carried by E_α. When is $\zeta \in E_\alpha$ a point mass for μ_α?

Theorem 9.1. *An AC measure $\mu_\alpha \in \mathcal{A}_\varphi$ has a point mass at $\zeta \in \mathbb{T}$ if and only if $\varphi(\zeta) = \alpha$ and φ has a finite angular derivative at ζ.*

Proof. This proof will be a bit skimpy on the details. By the definition of an AC measure we have

$$\frac{1 - |\varphi(r\zeta)|^2}{|\alpha - \varphi(r\zeta)|^2} = \int \frac{1 - r^2}{|\xi - r\zeta|^2} d\mu_\alpha(\xi).$$

Multiply both sides of the previous identity by

$$\frac{(1-r)^2}{1-r^2}$$

to get

$$\left|\frac{1-r}{\alpha - \varphi(r\zeta)}\right|^2 \frac{1 - |\varphi(r\zeta)|^2}{1-r^2} = \int \frac{(1-r)^2}{|\xi - r\zeta|^2} d\mu_\alpha(\xi).$$

Now take limits as $r \to 1^-$ to get

$$\frac{1}{|\varphi'(\zeta)|^2} \cdot |\varphi'(\zeta)| = \mu_\alpha(\{\zeta\}). \quad \square$$

Corollary 9.2. *For an analytic self-map φ and $\alpha \in \mathbb{T}$ the set*

$$P_\alpha := \{\zeta \in \mathbb{T} : \varphi(\zeta) = \alpha, |\varphi'(\zeta)| < \infty\}$$

is discrete and the pure point part of $\mu_\alpha \in \mathcal{A}_\varphi$ is equal to

$$\sum_{\zeta \in P_\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta.$$

10. Clark measures as spectral measures

It turns out that the AC measures are the spectral measures for a certain unitary operator, the Clark unitary operator. In fact, this is the real reason they were discovered in the first place. Let us present this in a purely linear algebra way so as to point to the general result without getting tied up in all of the technical definitions.

For $n \in \mathbb{N}$, let Q_n be polynomials of degree at most $n-1$. We will imagine $Q_n \subset L^2$. As such Q_n, inherits an inner product

$$\langle p, q \rangle = \int_{\mathbb{T}} p\overline{q} dm = \int_0^{2\pi} p(e^{i\theta})\overline{q(e^{i\theta})} \frac{d\theta}{2\pi}.$$

With this inner product note that

$$\langle z^m, z^n \rangle = \int_0^{2\pi} e^{im\theta} e^{-in\theta} \frac{d\theta}{2\pi} = \delta_{m,n}$$

and so $\{1, z, \cdots, z^{n-1}\}$ is an orthonormal basis for Q_n.

Define the operator

$$P_n : L^2 \to Q_n;$$
$$(P_n f)(z) = \int_{\mathbb{T}} f(\zeta)(1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta).$$

A computation will show that for $0 \leq k \leq n-1$

$$P_n z^k = \int_{\mathbb{T}} \zeta^k (1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta) = z^k$$

and so $P_n Q_n = Q_n$. With a little be more work, one can check that P_n is actually the orthogonal projection of L^2 onto Q_n.

For $\varphi \in L^\infty$, let

$$A_\varphi : Q_n \to Q_n, \quad A_\phi f = P_n(\varphi f).$$

One can check that

$$\begin{aligned}(A_\varphi z^k)(z) &= \int_{\mathbb{T}} \varphi(\zeta) \zeta^k (1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1}) dm(\zeta) \\ &= \sum_{j=1}^{n-1} \left(\int_{\mathbb{T}} \varphi(\zeta) \zeta^{k-j} dm(\zeta) \right) z^j \\ &= \sum_{j=1}^{n-1} \widehat{\varphi}(j-k) z^j.\end{aligned}$$

This means that the matrix representation of A_φ with respect to the orthonormal bass $\{1, z, \cdots, z^{n-1}\}$ is a Toeplitz matrix. In fact, every Toeplitz matrix can be thought of in this way.

Define, for $|\alpha| = 1$,

$$U_\alpha := A_z + \alpha(1 \otimes z^{n-1}).$$

This operator is called the Clark unitary operator. One can also see that if $|a| = 1$, U_α is unitary. Indeed a simple 3×3 example is

$$U_a = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The general $n \in \mathbb{N}$ case follows a similar pattern (1s down the sub-diagonal, α in the right corner, zeros elsewhere).

Furthermore if $\zeta^n = \alpha$ one can check that if

$$k_\zeta(z) = 1 + \overline{\zeta} z + \overline{\zeta}^2 z^2 + \cdots \overline{\zeta}^{n-1} z^{n-1},$$

then certainly $k_\zeta \in Q_n$ and a matrix calculation will show that

$$U_\alpha k_\zeta = \zeta k_\zeta$$

and so we have computed the spectral information for U_α, i.e., its eigenvalues and eigenvectors.

By the spectral theorem from functional analysis, one knows that any cyclic unitary operator is unitarily equivalent to the operator $f \mapsto \zeta f$ (multiplication by the independent variable) on $L^2(\sigma)$ for some finite positive Borel measure σ on $\mathbb{T}$. Computing this measure can be somewhat difficult. For the cyclic (easily checked) unitary operator U_α, one can compute its spectral representation in the following very interesting way: Let σ_α be the measure on the circle defined by

$$d\sigma_\alpha = \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_j},$$

where $\{\zeta_1, \cdots, \zeta_n\}$ are the n roots of $z^n = \alpha$. Do you recognize σ_α as the AC measure for $\varphi(z) = z^n$?

Define

$$Z_\alpha : L^2(\sigma_\alpha) \to L^2(\sigma_\alpha), \quad (Z_\alpha f)(\zeta) = \zeta f(\zeta)$$

and note that

$$Z\chi_{\zeta_j} = \zeta_j \chi_{\zeta_j}, \quad j = 1, \cdots, n.$$

That is to say, χ_{ζ_j} are eigenvectors for Z_α corresponding to the eigenvalues ζ_j. Furthermore, note that

$$\{\sqrt{n}\chi_{\zeta_1}, \cdots, \sqrt{n}\chi_{\zeta_j}\}$$

is an orthonormal basis for $L^2(\sigma_\alpha)$.

If we define

$$V_\alpha \sqrt{n}\chi_{\zeta_j} = \frac{\sqrt{n}}{n} k_{\zeta_j}$$

and extend by linearity we see that V_α is an bijective linear transformation from $L^2(\sigma_\alpha)$ onto Q_n. But since $\|k_\zeta\|_{L^2(m)} = \sqrt{n}$ (Parsevel's theorem) we see that V_α is actually an isometry. Finally, since χ_{ζ_j} are the eigenvectors for Z_α and k_{ζ_j} are the eigenvectors for U_α we see that

$$U_\alpha V_\alpha = V_\alpha Z_\alpha.$$

Thus we have a concrete realization of the spectral measure and spectral representation for U_α.

It turns out that this greatly generalizes to model spaces $(\Theta H^2)^\perp$, where Θ is inner function and H^2 is the Hardy space [4, 5]. We will just state the results. The interested reader can find all of the details worked out in [1].

But first we need a crash course in Hardy spaces. The classics texts for this material, together with complete proofs of the main results, are [4, 5, 6]. Consider the set of all power series

$$\sum_{n=0}^{\infty} a_n z^n$$

whose coefficients satisfy

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

For any fixed $z \in \mathbb{D}$ the Cauchy-Schwarz inequality shows that

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n||z|^n &\leq \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{1/2} \\ &= \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} \frac{1}{\sqrt{1-|z|^2}}.\end{aligned}$$

This says that such power series converge uniformly on compact subsets of $\mathbb{D}$ and thus form analytic functions on $\mathbb{D}$. This allows us to make the following definition:

Definition 10.1. The *Hardy space* H^2 is the set of all analytic functions on $\mathbb{D}$ whose power series have square summable coefficients.

If we norm H^2 by

$$\left\|\sum_{n=0}^{\infty} a_n z^n\right\|_{H^2} := \sqrt{\sum_{n=0}^{\infty} |a_n|^2},$$

then one can show that H^2 is complete vector space – in fact a Hilbert space.

There is also this useful alternate definition of H^2 involving the integral means

$$\int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta), \quad 0 < r < 1.$$

Proposition 10.2. *An analytic function f on $\mathbb{D}$ belongs to H^2 if and only if*

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty.$$

Theorem 10.3. *For $f \in H^2$ we have the following:*

(1) $\lim_{r \to 1^-} f(r\zeta) := f(\zeta)$ *exists for m-almost every $\zeta \in \mathbb{T}$.*

(2) $\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) = \sum_{n=0}^{\infty} |a_n|^2.$

Remark 10.4. Notice how the norm of an H^2 function is equal to the L^2 norm of its boundary function.

It is worth pointing out a Hardy space version of the classical Cauchy integral formula.

Theorem 10.5 (Cauchy integral formula)**.** *For $f \in H^2$ and $z \in \mathbb{D}$,*

$$f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\overline{\zeta}} dm(\zeta).$$

Note how the above integral makes sense since $f(\zeta)$ is well defined for m-almost every $\zeta \in \mathbb{T}$ and $\zeta \mapsto f(\zeta)$ is an L^2 function. The kernel functions

$$k_\lambda(z) := \frac{1}{1-\overline{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

are called *reproducing kernel functions* for H^2 since

$$f(\lambda) = \langle f, k_\lambda \rangle, \quad f \in H^2,$$

that is to say, these functions reproduce the values of f at λ.

After that brief interlude on the basics of Hardy spaces, it back to our discussion of Clark theory.

Theorem 10.6. *For an inner function Θ with $\Theta(0) = 0$, the operator*

$$U_\alpha := S_\Theta + \alpha \left(1 \otimes \frac{\Theta}{z} \right),$$

where $S_\Theta = P_\Theta S|(\Theta H^2)^\perp$ is the compression of the unilateral shift $S : H^2 \to H^2, Sf = zf$, is a cyclic unitary operator on $(\Theta H^2)^\perp$. The eigenvalues of U_α are the $\zeta \in \mathbb{T}$ so that $\Theta(\zeta) = \alpha$ and $|\Theta'(\zeta)| < \infty$. The corresponding eigenvector is

$$k_\zeta^\Theta := \frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z}.$$

This next theorem says that the spectral measure of U_α (which exists by the spectral theorem for cyclic unitary operators) is indeed the AC measure σ_α.

Theorem 10.7. *For an inner function Θ with $\Theta(0) = 0$, let $\sigma_\alpha \in \mathcal{A}_\Theta$. Then the operator*

$$(V_\alpha f)(z) = (1-\overline{\alpha}\Theta(z)) \int \frac{f(\zeta)}{1-\overline{\zeta}z} d\sigma_\alpha(\zeta)$$

is an isometric operator from $L^2(\sigma_\alpha)$ onto $(\Theta H^2)^\perp$. Furthermore if

$$Z_\alpha : L^2(\sigma_\alpha) \to L^2(\sigma_\alpha), \quad (Z_\alpha f)(\zeta) = \zeta f(\zeta),$$

then $V_\alpha Z_\alpha = U_\alpha V_\alpha$.

11. A connection to the Paley-Wiener approximation probem

The Paley-Wiener approximation problem: Suppose $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and $\{y_n : n \in \mathbb{N}\}$ is a sequence in $\mathcal{H}$. If these sequences are 'close' to each other (not quite defined yet), does $\{y_n : n \in \mathbb{N}\}$ span $\mathcal{H}$?

As an example of what we mean here, consider the standard orthonormal basis

$$\varphi_n(e^{i\theta}) = e^{in\theta}, \quad n \in \mathbb{Z},$$

for L^2. When does the sequence

$$\psi_n(e^{i\theta}) = e^{i\lambda_n\theta}, \quad n \in \mathbb{Z},$$

where $\lambda_n \in \mathbb{R}$, span L^2? A classical theorem of Paley-Wiener says this is indeed the case when

$$\max_{n \in \mathbb{Z}} |\lambda_n - n| < \frac{1}{\pi^2}.$$

So let $\Lambda \subset \mathbb{D}$ be a sequence in $\mathbb{D}$. When does

$$\bigvee \{k_\lambda^\Theta : \lambda \in \Lambda\} = (\Theta H^2)^\perp?$$

An easy exercise using the uniqueness theorem for analytic functions will show that if Λ has accumulation points in either the open unit disk or $\mathbb{T} \setminus \sigma_b(\Theta)$, then the set of kernel functions does indeed span. Hint: Use annihilators and the Hahn-Banach separation theorem. For other cases, things get a bit more complicated. What we want here is a orthonormal basis for $(\Theta H^2)^\perp$ consisting of (normalized) kernel functions!

Ah ha! But when U_α has discrete spectrum then indeed U_α has an orthoormal basis of eigenvectors

$$\left\{ \frac{k_\zeta^\Theta}{\sqrt{|\Theta'(\zeta)|}} : |\Theta'(\zeta)| < \infty, \Theta(\zeta) = \alpha \right\}$$

which can be used formulate Paley-Wiener type approximation theorems for model spaces $(\Theta H^2)^\perp$.

12. A connection to composition operators

For a self-map φ of $\mathbb{D}$ and an analytic function f on $\mathbb{D}$ define the composition operator

$$(C_\varphi f)(z) := f(\varphi(z)).$$

By an application of the Littlewood subordination principle [4], it is known that C_φ is a bounded operator form H^2 to itself. Two great places to learn about the basics of composition operators are [3, 10]. It also turns out that there are connections to AC measures.

The first one to mention involves the Aleksandrov operator. For a self map φ define, for a continuous function f on $\mathbb{T}$,

$$(A_\varphi f)(\alpha) = \int_{\mathbb{T}} f(\zeta) d\mu_\alpha(\zeta), \quad \alpha \in \mathbb{T}.$$

One can show that since f is continuous on $\mathbb{T}$ then $A_\varphi f$ is also continuous on $\mathbb{T}$. The above operator is called the *Aleksandrov operator.*

Theorem 12.1 (Aleksandrov (1987))**.** *If $\varphi(0) = 0$, the operator A_φ extends to be a bounded operator on H^2.*

Assuming that $\varphi(0) = 0$ one can work the identity

$$\int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda} d\mu_\alpha(\zeta) = \frac{\alpha + \varphi(\lambda)}{\alpha - \varphi(\lambda)}, \quad \lambda \in \mathbb{D}, \alpha \in \mathbb{T},$$

to get the formula

$$\int_{\mathbb{T}} \frac{1}{1-\overline{\lambda}\zeta} d\mu_\alpha(\zeta) = \frac{1}{1-\overline{\varphi(\lambda)}\alpha}.$$

Indeed,

$$\begin{aligned}\int \frac{1}{1-\overline{\zeta}\lambda} d\mu_\alpha(\zeta) &= \frac{1}{2}\int \left(\frac{\zeta+\lambda}{\zeta-\lambda}+1\right) d\mu_\alpha(\zeta)\\ &= \frac{1}{2}\left(\frac{\alpha+\varphi(\lambda)}{\alpha-\varphi(\lambda)}+1\right)\\ &= \frac{\alpha}{\alpha-\varphi(\lambda)}\\ &= \frac{1}{1-\overline{\alpha}\varphi(\lambda)}.\end{aligned}$$

Now take complex conjugates to get the desired identity. If we write

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$$

for the reproducing kernel for H^2, i.e.,

$$\langle f, k_\lambda\rangle = f(\lambda), \quad f \in H^2, \lambda \in \mathbb{D},$$

we can write the previous identity as

$$(A_\varphi k_\lambda)(\alpha) = \frac{1}{1-\overline{\varphi(\lambda)}\alpha}.$$

We can relate this to the composition operator. Indeed

$$\begin{aligned}(C_\varphi^* k_\lambda)(z) &= \langle C_\varphi^* k_\lambda, k_z\rangle\\ &= \langle k_\lambda, C_\varphi k_z\rangle\\ &= \langle k_\lambda, k_z(\varphi(\cdot))\rangle\\ &= \overline{\langle k_z(\varphi(\cdot)), k_\lambda\rangle}\\ &= \overline{k_z(\varphi(\lambda))}\\ &= \frac{1}{1-\overline{\varphi(\lambda)}z}.\end{aligned}$$

Thus,

$$(A_\varphi k_\lambda)(\alpha) = (C_\varphi^* k_\lambda)(\alpha), \quad \lambda \in \mathbb{D}, \alpha \in \mathbb{T}.$$

We are certainly taking a few liberties in the above formula since $\alpha \in \mathbb{T}$ and there is the issue of boundary limits at a particular point. However, all of this can be made precise by using radial limits m-almost everywhere and in fact we have.

Theorem 12.2. *If $\varphi(0) = 0$ then $C_\varphi^* = A_\varphi$.*

There are many other connections between composition operators and AC measures. Let us mention one more. The essential norm of C_φ, denoted by $\|C_\varphi\|_e$, is the distance from C_φ to the compact operators $\mathcal{K}$ on H^2, i.e.,

$$\|C_\varphi\|_e := \inf\{\|C_\varphi - K\| : K \in \mathcal{K}\}.$$

The essential norm of C_φ can be computed in terms of something called the Nevanlinna counting function [3, 10] but Cima and Matheson [2] computed it in terms of σ_α, where $\mu_\alpha = h_\alpha dm + \sigma_\alpha$ is the Lebesgue decomposition of $\mu_\alpha \in \mathcal{A}_\varphi$.

Theorem 12.3 (Cima-Matheson)**.** *For a self map φ,*

$$\|C_\varphi\|_e = \sqrt{\sup_{\alpha\in\mathbb{T}} \sigma_\alpha(\mathbb{T})}.$$

I am certainly not doing this subject justice here and there are many other connections between composition operators and AC measures. A more complete account is found in [9].

13. Higher dimensional analogs

Let Θ be a contractive $M_{n\times n}(\mathbb{C})$-valued analytic function on $\mathbb{D}$. By this we mean that $\|\Theta(z)\|$, the matrix norm of $\Theta(z)$, is bounded by one for all $z \in \mathbb{D}$. In the scalar case we know that if Θ is non-constant, then $|\Theta(z)| < 1$ for all $z \in \mathbb{D}$. This is no longer the case in the matrix case. What is true is the following:

Proposition 13.1. *Every contractive $M_{n\times n}(\mathbb{C})$-valued function Θ on $\mathbb{D}$ can be written in block form as $\Theta = \Theta_0 \oplus \Theta_1$, where Θ_0 is a constant unitary matrix and Θ_1 is purely contractive, i.e., $\|\Theta_1(z)\| < 1$ for all $z \in \mathbb{D}$.*

For a purely contractive Θ and a constant unitary matrix $A \in M_{n\times n}$ define the function

$$B_A(z) := (I + \Theta(z)A^*)(I - \Theta(z)A^*)^{-1}, \quad z \in \mathbb{D},$$

and note that this function is a purely contractive analytic function on $\mathbb{D}$. Furthermore, assuming that $\Theta(0) = 0$ (the zero matrix!), we have

$$\Re B_A(z) := \frac{1}{2}(B_A(z) + B_A(z)^*)$$

is a positive definite and, by a matrix version of the Herglotz formula, we can write

$$B_A(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \mu_A(d\zeta),$$

where μ_A is a positive matrix-valued measure on $\mathbb{T}$. That is to say, for every Borel set $E \subset \mathbb{T}$, $\mu_A(E)$ is a positive definite matrix. This yields the family of matrix-valued measures on $\mathbb{T}$, the Aleksandrov-Clark measures

$$\{\mu_A : A \in \mathcal{U}(n)\},$$

where $\mathcal{U}(n)$ is the unitary group. As before, these measures are associated with a higher dimensional analog of the Clark unitary operators discussed earlier (see [7] for a reference). We will focus on the following analog of the disintegration theorem: Let m_n denote $M_{n\times n}$-valued measure mI (n copies of normalized Lebesgue measure m along the diagonal) on $\mathbb{T}$. Also let dH_n denote Haar measure on $\mathcal{U}(n)$.

Theorem 13.2 (Martin (2012)). *For any continuous* $f : \mathbb{T} \to \mathbb{C}^n$*,*

$$\int_{\mathcal{U}(n)} \left(\int_{\mathbb{T}} \mu_A(d\zeta) f(\zeta) \right) dH(U) = \int_{\mathbb{T}} m_n(d\zeta) f(\zeta).$$

We will end here with a remark that much of the theory mentioned earlier for the scalar case (non-tangential limits, Ahern-Clark results, carriers, etc.) have analogs in this matrix-valued case. Details and references are in [7].

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