Lectures

Review .............................................................................................................................1-1
Nuclear spin ...............................................................................................................1-1
Nuclear spin and magnetic moment...........................................................................1-2
Nuclear spins in a uniform magnetic field .................................................................1-2
Spin Hamiltonian, operators and eigenfunctions .........................................................2-1
Energy levels and transitions ........................................................................................2-1
Means of generating transitions ....................................................................................2-3
continuous sweep .......................................................................................................2-3
pulsed NMR ...............................................................................................................2-3
Fourier Transform ............................................................................................................2-4
Time dependent Schrödinger equation ......................................................................3-1
Fourier transformed pulses.........................................................................................3-1
Models for looking at what is going on .........................................................................3-2
Quantum mechanical picture : Density operator approach........................................3-2
Classical picture : The vector model..........................................................................3-2
Pulses and observation .....................................................................................................3-3
generating transitions .................................................................................................3-3
Rotating frame of reference .............................................................................................4-1
Pulses .........................................................................................................................4-2
pulses and coherences .................................................................................................4-3
Observation ................................................................................................................4-3
Phase sensitive detection and CYCLOPS.....................................................................5-1
Relaxation effects.............................................................................................................5-1
T\textsubscript{1} relaxation, spin lattice relaxation...........................................................5-2
T\textsubscript{2} relaxation, spin-spin relaxation ...................................................................5-2
Experimental considerations ..........................................................................................6-1
data acquisition ..........................................................................................................6-1
multiple scans..............................................................................................................6-2
Data Manipulation, windowing functions......................................................................7-1
increasing signal/noise ...............................................................................................7-1
line sharpening ............................................................................................................7-2
trigonometric transforms............................................................................................7-3
The 1 pulse experiment .................................................................................................7-2
Multiple pulse 1D experiments .................................................................7-2
Vector model versus Product Operator Formalism........................................7-3
T₁ Determination : T1IR ..............................................................................7-3
T₂ Determination .......................................................................................8-1
Coupling .....................................................................................................8-2
  Spin echo and homo/hetero nuclear coupling .........................................9-1
Attached Proton Test (APT) .......................................................................9-2
  The APT sequence and the vector model ..............................................10-1
Selective Polarization Transfer .................................................................10-3
  Energies and populations .......................................................................11-1
  Population inversion .................................................................................12-1
INEPT ........................................................................................................12-2
Refocused INEPT ......................................................................................13-2
DEPT ..........................................................................................................14-1

2D NMR Spectroscopy ..............................................................................15-1
Introduction .............................................................................................15-1
  2D convolution ......................................................................................15-2
  2D line shapes ......................................................................................15-3
  Two dimensions of complex data ............................................................15-3
Homo nuclear shift correlation spectroscopy (COSY) ..................................16-1
  Hetero nuclear coupling in a homo nuclear COSY .................................17-1
Hetero nuclear shift correlation spectroscopy (HETCOR) ...............................18-1
Practical Aspects of 2D NMR ..................................................................19-1
  T₁ relaxation during t₁ ............................................................................19-1
  Quadrature detection in t₁ .................................................................19-1
    RUSH ..................................................................................................19-2
    TPPI : time proportional phase increment ........................................19-2
  Digital resolution and experiment time ...............................................19-3
Additional Tricks .....................................................................................19-4
  MC calculation .....................................................................................19-4
  SY symmetrization ..............................................................................19-5
# Overheads

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INEPT</td>
<td>13-1</td>
</tr>
<tr>
<td>DEPT</td>
<td>14-1</td>
</tr>
</tbody>
</table>

# Maple Worksheets

<table>
<thead>
<tr>
<th>Worksheet</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>QM_spins</td>
<td>1-2</td>
</tr>
<tr>
<td>Fourier</td>
<td>2-4</td>
</tr>
<tr>
<td>pulse FT</td>
<td>3-1</td>
</tr>
<tr>
<td>FID</td>
<td>5-3</td>
</tr>
<tr>
<td>DAQ</td>
<td>6-2</td>
</tr>
<tr>
<td>FFT</td>
<td>6-2</td>
</tr>
<tr>
<td>Gaussian</td>
<td>7-1</td>
</tr>
<tr>
<td>T1IR</td>
<td>7-4</td>
</tr>
<tr>
<td>spinecho</td>
<td>9-1</td>
</tr>
<tr>
<td>APT</td>
<td>10-2</td>
</tr>
<tr>
<td>Populat(ion)</td>
<td>10-3</td>
</tr>
<tr>
<td>INEPT</td>
<td>12-3</td>
</tr>
<tr>
<td>INEPT (refocused INEPT)</td>
<td>13-2</td>
</tr>
<tr>
<td>DEPT</td>
<td>14-1</td>
</tr>
<tr>
<td>2DFT</td>
<td>15-2</td>
</tr>
</tbody>
</table>

---

*The numbers refer to the page in the lectures that contains the overhead

‡ The numbers refer to the lecture page that refers to the particular work sheet
Problem Sets‡‡

Fourier Transformation ................................................................. PS-1
  pure trigonometric functions..................................................... 1
  exponentially decaying trigonometric functions ....................... 2
Convolution ..................................................................................... 4

Vector and Product operator formalism transformation rules .......... PS-2
  transformations rules...................................................................... 1
Spin Hamiltonian .............................................................................. 1
  rotations......................................................................................... 2

Vector model transformations
  pulses along Cartesian axes.......................................................... 2
  chemical shift evolution ................................................................. 3
  coupling evolution.......................................................................... 3
  observation.................................................................................... 4

Product Operator Transformation rules
  time dependence of the density operator ....................................... 5
  commutivity of spin operators ......................................................... 5
  orthogonal spin operator basis sets ............................................... 6
  transformation rules ....................................................................... 7
  Cartesian subspace rotations ......................................................... 8

DEPT of menthol ............................................................................... PS-3

Quantum coherences and 2D sequences ....................................... PS-4
  step operators and quantum coherence orders .............................. 1
  quantum filtering (double) ............................................................ 2

2D Spectra ....................................................................................... PS-5

‡‡ Several of the problem sets introduce very important new material
Lecture Notes

Chemistry 558

Organic Spectroscopy

Part II: FT-NMR

by

René P. F. Kanters

1994
**Review**

The basis for the possibility to measure nuclear magnetic resonance is the presence of a nuclear magnetic moment. The magnetic moment arises from the presence of nuclear spin, I. NMR is a spectroscopic technique, which means that we will look at the excitations/emission possible due to splitting of energy levels arising from the interaction of magnetic moments in an external magnetic field.

### Nuclear spin

The presence of the nuclear spin (or electron spin) is completely quantum mechanical, so we do not have a gut feeling of what it means. Spin is an intrinsic property of the nucleus and is related to the number of protons and total number of nucleons. A few cases are

<table>
<thead>
<tr>
<th>spin I</th>
<th>#n + #p</th>
<th>#p examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>n+1/2</td>
<td>odd</td>
<td>even/odd</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=1/2 1H, 3H, 13C, 15N, 19F, 29Si, 31P, 195Pt</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=3/2 11B, 23Na</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=5/2 17O, 27Al</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=7/2 59Co</td>
</tr>
<tr>
<td>n</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=1 2H, 14N</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=3 10B</td>
</tr>
<tr>
<td>0</td>
<td>even</td>
<td>even</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I=0 12C, 16O</td>
</tr>
</tbody>
</table>

We will focus mainly on systems that have nuclei with spin I=1/2 (1H, 13C, 19F).

A nucleus with spin number I can be considered to have a spin angular momentum (i.e. a vector), \( \vec{I} \), whose size is \( \sqrt{\vec{I} \cdot \vec{I}} = \sqrt{I(I+1)}\hbar \), where \( \hbar \) is Planck’s constant (h) divided by \( 2\pi \). Combined with this we have a spin operator, \( \hat{I} \), which is dimensionless. From quantum mechanics we know that this vector can not have just any direction. The projection of \( \vec{I} \) on any axis, e.g. z axis, is equal to \( m_I \hbar \), where \( m_I = -I, -I+1, \ldots, I-1, I \), thus

\[
\begin{align*}
\vec{I} &= \sqrt{I(I+1)}\hbar \\
I_z &= m_I \hbar \\
m_I &= -I, -I+1, \ldots, I-1, I
\end{align*}
\]

For a spin I=1/2 we only have two states, \( m_I = +1/2 (\alpha) \) and -1/2 (\( \beta \)).
We can measure the component along the z-axis, but due to the Heisenberg uncertainty principle, we can not at the same time get the components along the x and y axis accurately, so we only know that the spin vector lies in a cone that has an angle of $\arccos(I_Z/|I|)$ with the z-axis.

\textbf{(Maple : QM SPINS)}

\textbf{Nuclear spin and magnetic moment}

Associated with the spin momentum is a magnetic moment, $\vec{\mu}$, that is proportional to the spin momentum by the gyro-magnetic ratio, $\gamma$. The gyro-magnetic ratio is specific for a particular particle. Thus in equations: \textbf{(put this next to the spin equations)}

\begin{align*}
\vec{\mu} &= \gamma I \\
|\vec{\mu}| &= \gamma \sqrt{I(I+1)} \hbar \\
\mu_z &= \gamma m_1 \hbar
\end{align*}

\textbf{Nuclear spins in a uniform magnetic field}

The interaction of the spin momentum with a magnetic field is such that the energy of the system is lowered by the amount equal the product of the size of the projection of the magnetic moment vector and the magnetic field. Thus we can write for the energy, $E$, for a single spin

\[ E = -\vec{\mu} \cdot \vec{B}_0 = -\gamma \hat{l} \cdot \vec{B}_0 = -\gamma m_1 B_0 \]

We should keep in mind, though, that the $B_0$ stands for the magnetic field felt by the nucleus. This means that this does not have to be equal to the external field, so we should allow for that by introducing a chemical shielding term, $\sigma$, thus we really have:

\[ E = -\gamma m_1 (1 - \sigma) B_0 \]

Let’s look at an example of two $I=1/2$ spins with different shielding terms, but the same gyro-magnetic ratio. We see that at the same external field strength, they will have different resonance frequencies. The other way we can look at this is that they come into resonance with a fixed energy source at different field strengths.
The shielding for spin 1 is larger than that for spin 2, so that the energy difference between the spin \( \alpha \) and spin \( \beta \) states for spin 1 are smaller than those for spin 2. From the picture we see that there are two ways to get resonance. Either by having a fixed frequency and sweeping the field, or having a fixed field and sweeping the frequency. From looking at the graph we can see what relationships hold between shielding, effective field strength and resonance frequency: if \( \sigma_1 > \sigma_2 \) then \( B_1 > B_2 \) and/or \( \nu_1 < \nu_2 \). Thus spin 1 is strongly shielded, lies at high field, and at low frequency.

End of Lecture
Spin Hamiltonian, operators and eigenfunctions

It is very convenient to use the spin Hamiltonian in combination with a set of orthonormal spin functions, that are eigenfunctions of the Hamiltonian, to calculate the energies of the system. The spin Hamiltonian consists, so far, only of the Zeeman term for the spin:

\[ H = -\gamma (1-\sigma) \mathbf{B} \cdot \mathbf{I} = -\gamma (1-\sigma) B_0 I_z \]

Thus we can write for the Hamiltonian operator:

\[ \hat{H} = -\gamma (1-\sigma) B_0 \hat{I}_z = -\Omega \hat{I}_z \]

where \( \Omega = \gamma (1-\sigma) B_0 \hbar \). The \( \hbar \) gets in there because the spin angular momentum is in units of \( \hbar \), and spin operators are dimensionless.

Thus the eigenfunctions for this operator are the same as the eigenfunction of the \( \hat{I}_z \) operator. These are given by

\[ \hat{I}_z |\phi_s\rangle = m_I |\phi_s\rangle \]

where \( \phi_s \) is a spin function. The spin functions used for \( I = 1/2 \) are \( \alpha \) (\( m_I = 1/2 \)) and \( \beta \) (\( m_I = -1/2 \)).

Energy levels and transitions

To calculate the energy we can determine the expectation value of the Hamiltonian (the operator that gives us the energy), as follows:

\[ E_\alpha = \langle \alpha | H | \alpha \rangle = \langle \alpha | -\Omega \hat{I}_z | \alpha \rangle = -\Omega \langle \alpha | \hat{I}_z | \alpha \rangle = -\Omega \langle \alpha | \frac{1}{2} | \alpha \rangle = -\frac{\Omega}{2} \langle \alpha | \alpha \rangle = -\frac{\Omega}{2} \]

\[ E_\beta = \langle \beta | H | \beta \rangle = \langle \beta | -\Omega \hat{I}_z | \beta \rangle = -\Omega \langle \beta | \hat{I}_z | \beta \rangle = -\Omega \langle \beta | -\frac{1}{2} | \beta \rangle = -\frac{\Omega}{2} \langle \beta | \beta \rangle = \frac{\Omega}{2} \]

we can do this because we know that the two spinfunctions are eigenfunctions of this Hamiltonian and we have for the off-diagonal elements \( \langle \alpha | H | \beta \rangle = \langle \beta | H | \alpha \rangle = 0 \). If we have additional operators in the Hamiltonian we may introduce off-diagonal elements, thus making the eigenfunctions of the new Hamiltonian will be linear combinations of our initial spin-functions.

So we see that with only the Zeeman term in the Hamiltonian, the energies are \(-m_I \Omega\).
In general, for a single nucleus with spin $I$, we have $2I+1$ states, the number of $m_I$'s possible, whose energies are proportional to the magnetic field strength. The selection rules for transitions is that $\Delta m_I = \pm 1$, thus there are $2I$ excitations possible.

Since any transition has $\Delta m_I = \pm 1$ we have for the energy difference of the states involved

$$\Delta E = \pm \Delta m_I \gamma \hbar (1 - \sigma)B_0$$

where the negative sign signifies emission. This energy difference can be expressed into excitation/emission frequencies by

$$\Delta E = h\nu = \frac{\gamma \hbar (1 - \sigma)B_0}{2\pi}$$

$$\nu = \frac{\gamma (1 - \sigma)B_0}{2\pi}$$

Example: we have a single $^1$H atom in a field of 7 Tesla. What is the frequency of e.m. radiation required to generate transitions?

solution: First, the fact that we have a single atom suggests that the shielding must be zero. We need to know $\gamma$ for $^1$H, which is $26.7519 \times 10^7$ rad T$^{-1}$ s$^{-1}$. Thus we write

$$\nu = \frac{\gamma (1 - \sigma)B_0}{2\pi} = \frac{26.7519 \times 10^7 \text{ rad T}^{-1} \text{ s}^{-1} \times (1 - 0) \times 7 \text{ T}}{2\pi} = 298 \times 10^6 \text{ s}^{-1}$$

Thus the excitation frequency is 298 MHz. This is in the radio-frequency region, so the transitions can be made by irradiating the sample with radio-frequent electro magnetic radiation (radio waves: antennas for transmitting and receiving : copper coils). Other nuclei will have other gyro-magnetic constants and thus will resonate at other frequencies.

There are several different ways of writing the spin-Hamiltonian (the Hamiltonian for spin functions). The difference lies in the units that one wants the eigenvalues in. So far we had our Hamiltonian returns the eigenvalues in Joules. In NMR we often want our results in Hz, so we can divide the Hamiltonian by $\hbar$, or if one is interested in rotational frequencies (for modulations, as we will see later) one divides the Hamiltonian by $\hbar$. The operators will, of course, stay the same but the coefficients they have change.

**Means of generating transitions**

The reason to generate transitions is that this allows us to determine the intensity of the absorption or emission of radiation as a function of energy. There are basically two methods to perform transitions in nuclear magnetic resonance spectroscopy: continuous wave, and pulsed (FT).
continuous wave (sweep method)

Determine the amount of a weak continuous, single frequency, r.f. signal absorbed as the field is being swept. Sweeping of the field is done since that is easier. By keeping the r.f. signal the same frequency all the time, the transition will come in resonance as $B_0$ is varied.

Disadvantage: slow, since if we want a resolution of 1 Hz (on the energy scale) the Heisenberg uncertainty principle for $\Delta E$, $\Delta t$, tells us that

$$\Delta E \Delta t \geq \frac{\hbar}{2} = \frac{\hbar}{4\pi}$$

$$\hbar \Delta v \Delta t \geq \frac{\hbar}{4\pi}$$

$$\Delta t \geq \frac{1}{4\pi \Delta v}$$

which will give $\Delta t \geq 0.08$ s if $\Delta v = 1$ Hz. Thus on a 300MHz machine with a scan range or 10 ppm we need a $10 \times 10^{-6} \times 300 \times 10^6$ Hz = 3000 Hz range. In total that should takes us more than $0.08 \times 3000 = 240$ sec = 4 minutes. In addition to that low concentration would result in a low signal to noise ratio and multiple scans would be required (each of 4 minutes).

pulsed NMR (Fourier Transform)

Make use of the fact that a short, strong r.f. signal can be considered to be built up of a range of frequencies, thus exciting the whole range that we are interested in. This thus allows us to determine the response of all resonances at the same time. Detection times are usually in the range of 1s (in some cases, when the system returns to equilibrium really fast the detection time can be made shorter) and after that time period a new pulse can be given and a second spectrum taken. The experimental time is greatly reduced, this is called the Felgett advantage. We will see in a bit how the detection of the response of the system is performed, but first we need to look at how it is possible with a e.m. pulse to get a range of frequencies.

Fourier Transform

Fourier transform is a mathematical method to unravel a signal as a function of time into it’s frequency constituents (Mathematically it really re-expresses a function into a new basis in the reciprocal space, not just in time->freq, also distance->reciprocal space, e.g., in X-ray.) It is specially useful for modulating functions. It works because it does:
FT\left(f(t)\right) = F(\omega) = \int_{t=-\infty}^{\infty} f(t) e^{-i\omega t} dt

The expression $e^{-i\omega t}$ is the mathematical notation for a precessing signal of which we know the direction of precession, i.e., $e^{i\alpha} = \cos \alpha + i\sin \alpha$ (which can be shown by series expansions). From this equation it is easily shown how a cosine and sine function can be expressed in the exponential forms:

$$\cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$\sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

(Maple : Fourier)

So, now let’s look what is going on in the Fourier transform for a cosine modulated signal

$$F(\omega) = \int_{t=-\infty}^{\infty} \cos(\omega_0 t) e^{-i\omega t} dt = \int_{t=-\infty}^{\infty} \left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) e^{-i\omega t} dt$$

$$= \int_{t=-\infty}^{\infty} \frac{1}{2} e^{i(\omega_0 - \omega)t} dt + \int_{t=-\infty}^{\infty} \frac{1}{2} e^{i(-\omega_0 - \omega)t} dt$$

$$= \delta_{\omega_0} + \delta_{-\omega_0}$$

Thus we see that if $\omega = \omega_0$ the first term gives $\infty$, if $\omega = -\omega_0$, the second term, and we get the so-called Dirac functions. Notice that the ‘intensity’ of these functions is half of that of a complex exponential only.

End of Lecture
Time dependent Schrödinger equation

NMR spectroscopists usually want to have the Hamiltonian so that it gives the energies in angular frequencies. The reason for this lies in the quantum mechanical picture of what is happening to a single system when a certain Hamiltonian is present. To figure out what is going on, one looks at the time dependent Schrödinger equation. This is given as:

\[ \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H \psi \]

When the Hamiltonian is time independent, we can solve the time dependent wave functions as follows:

\[ \frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H \psi \]
\[ \frac{\partial \psi}{\psi} = -\frac{i}{\hbar} H \Delta t \]
\[ \ln \psi - \ln \psi_0 = -\frac{i}{\hbar} H \Delta t + 0 \]
\[ \psi = \psi_0 e^{-\frac{i}{\hbar} H \Delta t} \]

This means that the wave function will undergo complex rotations with the angular frequency of \( H / \hbar \).

Note that this only holds for a single system. In order to use the time dependent Schrödinger equation for a large number of spin systems, which is the case when we measure something with NMR (sample in orders of \( \mu \)moles: huge number of molecules) you need to introduce the statistical distribution that is present, i.e., statistical Quantum Mechanics. We will see how that works later.

Fourier transformed pulses

Now back to where we were: the use of FT to figure out what frequencies can be considered to be present when we have a pulse of a certain length in time:

We can use Maple to see what the Fourier transform of a pulse in time is to figure out which frequencies it will generate.

\((\text{Maple : pulse FT})\)
The intensities of the frequency distribution comprised in a pulse of length \( \tau \) has nodes when \( \nu = \frac{n}{\tau} \). This means that a of pulses of 5 - 10 \( \mu \)s has ranges from 200 kHz – 100 kHz. This is more than large enough for the 3 kHz range in which our protons resonate. We do want the pulse to be short, so that the intensity of all frequencies in the range we are interested in is approximately the same (so that the intensities in the measured NMR signal does not depend on the frequency, or how far away the frequency is from the carrier frequency).

Another way to see why a frequency range is obtained by a short pulse is that the uncertainty in energy is real big (Heisenberg again: \( \Delta E \Delta t \geq \hbar/4\pi \), so \( \Delta E \) inversely proportional to \( \Delta t \)).

**Models for looking at what is going on**

**Quantum mechanical picture : Density operator approach**

In order to look at systems that are not in equilibrium, we need to determine what the time dependence of statistical average of the observables will be. The method that does that is the density operator approach, which uses the time independent solutions to the Schrödinger equation (determined by the Hamiltonian) for the initial state of the system, and then let's the time dependent solution to the Schrödinger equation (relates the time derivative of the wavefunction to the hamiltonian) govern how the time dependence of the wave function evolves, by using a Taylor series expansion.

A simplified model that can be derived from this model is the product operator formalism, which has some additional assumptions and generates pretty easy to use ‘rules’ for determining what happens during an NMR experiment. We will see that in our second problem set.

**Classical Picture : The vector model**

This is the model that that the book uses. It is pretty simple, and pictorial, so we do not have to write down mathematical expressions for what is going on. There is one major short coming: it does not describe things properly. It works for most 1D experiments, but we will see cases where this model completely breaks down.

The classical motion of a magnetic moment, \( \mu \), in a uniform magnetic field at constant energy says that a torque will be exerted on the magnetic moments to align them with the field. The torque only works on the component perpendicular to the field and the result is a precession around the field. (just like a gyroscope). The induced angular velocity of the precession is given by

\[
\mathbf{\omega} = -\gamma \mathbf{B}_0
\]
The minus sign tells us that with a positive $\gamma$ the direction of rotation is given by the right-hand rule such that the thumb opposes (-) the direction of the field. In other words, the precession occurs in a clockwise fashion looking down on the magnetic field vector. Thus the frequency of the precession is equal to

$$\nu = \frac{\omega}{2\pi} = \frac{\gamma B_0}{2\pi}$$

This frequency $\nu$ is also called the **Larmor frequency** and gives the precession frequency of the spin in a magnetic field of strength $B_0$. In NMR we often express the field strength in the Larmor frequency that it has for protons, e.g., the NMR we have is a 300MHz machine (thus the field it operates with is approximately 7T).

Notice that this precession frequency is equal to the quantum mechanically determined excitation frequency, so these two models agree with each other on this part.

The way that we will often look at the state of our spin system is by looking at the **total magnetization vector**, $\tilde{M}$. *(draw spins with different phases around a field vector)* This is just the sum of all microscopic moments added the way vectors have to be added. Since in the equilibrium state of a sample there is no preferred phase of the precessions of the (classical) microscopic moments around the external field, the component in the xy planes add up to zero and $\tilde{M}$ will be aligned with the field vector and will be constant (thus equilibrium).

**Pulses and observation**

In general we will want to perturb the static field with a magnetic field perpendicular to it, so that we can have the magnetization precess around the perturbing field and tip it into the xy plane so that due to the precession around $B_0$, time dependent magnetization in the xy plane will be generated that can be observed.

**Generating transitions**

Transitions are caused by having a perturbing e.m. field that has the rotates with the same frequency as $\tilde{M}$ and in the same direction. From the point of view of the spinning $\tilde{M}$ this r.f. field seems to stand still and $\tilde{M}$ will start to precess around this perturbing field. As the angle of $\tilde{M}$ with the external field changes, the total energy of the system changes, so from a QM picture we generate transitions, and as we saw the frequency needed to do that is equal to the Larmor frequency.
But how do we achieve rotating magnetization with a specific frequency needed for the pulse?

An alternating current in a coil will generate a linear polarized oscillating magnetic field, $\tilde{B}_1$. The field is described by $B_1 \cos(\omega_{\text{rf}} t)$. We know that an oscillating field can be considered to be built up of two opposite turning rotating fields, i.e.,

$$B_1 \cos(\omega_{\text{rf}} t) = B_1 \left( \frac{e^{i\omega_{\text{rf}} t} + e^{-i\omega_{\text{rf}} t}}{2} \right) = \frac{B_1}{2} \left( e^{i\omega_{\text{rf}} t} + e^{-i\omega_{\text{rf}} t} \right)$$

One of these component will rotate in the same direction as the magnetization vector. If $\omega_{\text{rf}}$ is equal to $\omega_{\text{Larmor}}$ this one component will seem to be stationary, and the magnetization will start to rotate towards the $xy$ plane. The other part, rotating in the opposite direction, will be (classically) rotating too quickly for the total magnetization to be able to precess around it, and is ignored.

End of Lecture
Rotating frame of reference

Another, easier, way to describe what is going on is achieved by rotating around the frame of reference that our spins are in with the Larmor frequency. This is equivalent to rotating the coordinate system that has the magnetization vector in it in the opposite direction, $-\omega_r$, so in our ‘laboratory’ frame of reference the magnetization vector will stand still, and effectively we have taken the effect of the external magnetic field into account by using this rotating frame of reference. Thus the external field strength in this rotating frame of reference is reduced, since the precession for spins that precess with the rotating frame of reference’s angular frequency $\omega_r$ becomes zero.

The mathematical way to do this transformation to the rotating frame of reference is a multiplication with this rotation function, $e^{-i\omega_r t}$. What will happen to the spinning magnetization, is the following

$$\vec{M}'(t) = e^{-i\omega_r t}\vec{M}(t) = e^{-i\omega_r t}Me^{i\omega t} = Me^{i(\omega-\omega_r)t}$$

In other words, the angular velocity of the precessing magnetization is reduced by the angular velocity of the rotating frame, which is exactly what we set out to do to begin with.

In a pictorial way we can draw this as:

How large is now the external field in this rotating frame of reference? We saw that the effective field in this reference system is reduced since the angular frequencies are smaller also, so we have

$$B_r = -\frac{\omega_r}{\gamma}$$

$$B' = B + B_r = B - \frac{\omega_r}{\gamma}$$

This means that only for 1 angular frequency the effective external field becomes equal to 0, i.e., for the angular frequency equal to the rotating frame of reference’s angular frequency. Usually one
uses for the rotating frame of reference’s angular frequency the frequency of the pulse, since this is the center of the frequencies present in the excitation signal.

In general we will look at things in the rotating frame of reference, since all rotations become low frequent. In a way it is equivalent to shifting our frequency axis down with the frequency of the rotating frame, so that the range of 299,998,500 Hz – 300,001,500 Hz frequencies become the range of -1,500 to 1,500 Hz.

Pulses

Now that we can work in the rotating frame, we can determine what happens when a perturbing e.m. field is present that rotates along with the rotating frame. In the rotating frame of reference this is described as a stationary magnetic field with a certain phase angle with the x-axis. The net magnetization will start to precess around that and the tip-angle will depend on the larmor frequency of the spins in a field of strength $B_1/2$, and the time that the $B_1$ field is on.

Notice that the direction that the spins end up is dependent on the frequency of the spins, due to their angular velocities around both the remnant of the external field (affecting $\varphi$) and the perturbing field (affecting $\theta$) and also because of the intensity distribution of the excitation frequencies present in the pulse. This problem is partly reduced by using short pulses. The spins that have an $\omega \neq \omega_{rf}$ will during the pulse time also precess around the z-axis, since the effective external field isn’t completely canceled out by the rotating frame of reference. As a result of that, they will also have some magnetization along the y-axis, i.e. a different phase than the spins that are on resonance. This part of the problem is dealt with by doing a phase correction in the spectrum, that is $\omega$ dependent.

In general we will assume that all spins after a $\pi/2$ pulse will end up with the same phase, thus a $\pi/2$ pulse along x will rotate all spins (that the pulse operates on) along the y-axis.

Question:
What is the effective field strength that a protons need in order to rotate 90°, in 10µs?

Solution:
We know that the magnetization will precess with the precession angular frequency determined by the effective field of $B_1$, thus we can write, assuming no shielding:
\[ \omega t = \frac{\gamma B_1}{2\pi} t = \frac{2}{\pi} \longrightarrow B_1 = \frac{\pi^2}{\gamma t} \]

filling out \( \gamma = 26.7519 \times 10^7 \text{ rad/(Ts)} \) and \( t = 10 \times 10^{-6} \text{ s} \), we get 0.0005871718744 T. The real amplitude of the pulse should be twice as high since only half of the signal can be considered to rotate in the right direction thus \( B_1 \) should be approximately 1 mT.

We notice that the effective field is very small, specially compared to the 7T that the external field has in a 300MHz machine. We did say earlier that the FT technique used short, intense pulses. What’s going on here? Well recall that the FT for a really short pulse has a broad frequency range, but at the same time has a low intensities for these frequencies…

What would happen to the magnetization, in terms of the phase angle, \( \varphi \), for a spin that has a resonance frequency that is 1500 Hz higher than the pulse frequency?

A resonance frequency that is 1500 Hz above the pulse frequency means that the magnetization will still precess around the z-axis with a rate of 1500 x 360 °/s. Thus in 10 µs that results in an angle of 1500 x 360 x 10^{-5} ° = 5.4°. This means that we will need to do a phase correction whose magnitude is proportional to the frequency difference with the pulse frequency, in order to make it look like every magnetization vector ended up in the same direction after the \( \pi/2 \) pulse (along x).

**Pulses and Coherences**

In the quantum mechanical picture of large number of spins we think of a pulse as a method to create excitations from the ground state level to the excited state. If the phases of the spins remain random one would end up with the ‘sum vector’ zero, when the populations of the states are equal, thus at the same time a phase ordering has to develop that makes that the resultant magnetization lies in the xy plane. This is called the creation of **coherences**.

**Observation**

The precessing magnetization can be observed by placing a coil in the xy plane in which a current will be induced with the same frequency as the Larmor frequency of the spin. Thus a signal will be detected with that is modulated in time with the specific resonance frequency for that spin. The signal that one observes is called the **Free Induction Decay**, (FID). Usually the carrier frequency of the pulse is mixed in to this signal so that we really observe the difference frequency (frequency modulation gives sum and difference of frequencies) and we detect a low frequent signal. This mixing in of the pulse frequency has the effect of changing our laboratory coordinate system into
the rotating frame system during observation, thus we can get (and should observe) both positive and negative frequencies relative to the carrier frequency (center of the spectrum).

The way we can picture the observation of the precessing magnetization is that we determine the size of the projection of the magnetization on a certain axis. This turns a rotation expression into a modulation expression. We all know that for modulating signals we use sine and cosine functions. But the problem is that we can not determine both the intensity and the sign of the frequency at the same time, e.g.

\[ A \cos(\omega t) = A \cos(-\omega t) \]
\[ A \sin(\omega t) = -A \sin(-\omega t) \]

Thus for a sine or cosine modulated signal we can’t distinguish between a positive or negative frequency.

By choosing an orthogonal direction and assigning that to the imaginary part of a complex number we obtain a complex expression for the rotation. This allows us to distinguish between counter clock wise (+\( \omega \)) and clock wise (-\( \omega \)) rotations. If we assign the cosine modulated signal to the real part and the sine modulated one to the imaginary part we get a complex modulation that upon Fourier transformation will give us the signs of the rotations also:

\[ \cos(\omega_0 t) + i\sin(\omega_0 t) = e^{i\omega_0 t} \quad \text{FT} \rightarrow \quad 2\pi \text{Dirac}(\omega_0) \]
\[ \cos(\omega_0 t) - i\sin(\omega_0 t) = e^{i\omega_0 t} \quad \text{FT} \rightarrow \quad 2\pi \text{Dirac}(-\omega_0) \]

Thus we can distinguish positive and negative frequencies (with respect to a carrier frequency).

Observation can be considered to be performed by two coils, in which an alternating signal is induced proportional to the magnitude of the precessing magnetization vectors. One of the coils is positioned along the x- and one along the y-axis. In reality, though, a single coil is used and the signal is split in 2. Both of these signals are mixed with the carrier frequency, but with phases that are \( \pi/2 \) apart, thus through FM modulation we get two low frequent signals that have a 90 degree phase shift with each other (like cosine and sine do), and one is assigned to an real buffer, the other to an imaginary buffer.
This method of detection is called **quadrature detection**. It allows us to set the pulse frequency to the center of the range of frequencies we are interested in since it allows us to distinguish between positive and negative directions of rotation.

| End of Lecture |   |
Phase sensitive detection and CYCLOPS

By having our signal split in two parts, after which each of them will be modulated with the carrier frequency and subsequently digitized, we may introduce errors. The split signals may have different intensities, either from non equal split or from different sensitivities of the analog to digital converters used to turn the signal into data that the computer can work with. In order to get rid of this problem we can make sure that in one spectrum the magnetization ends up along y, and the next time along -x, -y and x (changing the phases of the observation). Doing this makes that we will use a specific phase cycle. This particular one is called CYCLOPS.

<table>
<thead>
<tr>
<th>pulse #</th>
<th>direction</th>
<th>( \vec{M} )</th>
<th>x proj</th>
<th>y proj</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>y</td>
<td>sin</td>
<td>cos</td>
</tr>
<tr>
<td>2</td>
<td>y</td>
<td>-x</td>
<td>-cos</td>
<td>sin</td>
</tr>
<tr>
<td>3</td>
<td>-x</td>
<td>-y</td>
<td>-sin</td>
<td>-cos</td>
</tr>
<tr>
<td>4</td>
<td>-y</td>
<td>x</td>
<td>cos</td>
<td>-sin</td>
</tr>
</tbody>
</table>

In order to get the final magnetization with real cosine modulation and complex sine modulation, we will add for the real part \( y(1)-x(2)-y(3)+x(4) \), and for the imaginary part \( x(1)+y(2)-x(3)-y(4) \). So we see that for both the real and the imaginary part we have the difference of two signals coming from the same ADC. If we didn’t use CYCLOPS and we had offset errors in our AD converters we would get a part of the signal that would be constant in time, so that Fourier transformation would turn that into peaks at 0Hz (at the carrier frequency).

Other problems that can arise in quadrature detection are related to phase errors of the two signals used to obtain low frequent cosine and sine modulated signals. CYCLOPS will reduce those problems too. (Part of these problems are incorrect phases of the peak, but also occurrence of peaks at negative frequencies with respect to the carrier, thus still in our observed spectrum!. For more information see “Effects of CYCLOPS” worksheet in Maple).

Relaxation effects

In real life signal intensities decay with time, i.e. the system relaxes back to the equilibrium state. This must be the case, since when we observe the signal we measure the emitted energy, thus the energy of the system has to decrease. The name Free Induction Decay, already suggested that the observed signal decay at time goes on.

We distinguish two types of relaxation. These are spin-lattice relaxation, or T1-relaxation, and spin-spin relaxation, or T2 relaxation.
**T1-relaxation, spin-lattice relaxation**

T1 relaxation is the loss of magnetization in the xy-plane due to the build-up of magnetization along the z-axis, i.e., return to equilibrium. This process is 1st order and has a rate constant $R_1 = 1/T_1$. This loss of excitation energy is caused by interactions of the magnetic moments with oscillating fields present in the surroundings (or lattice), e.g. due to molecular motion. It is also called longitudinal relaxation, since it builds up magnetization along the static field vector (compare with longitudinal waves).

\[
M(t) = M_0 (1 - e^{-t/T_1})
\]

\[
M_0 e^{-t/T_1}
\]

**T2-relaxation, spin-spin relaxation**

T2 relaxation, also called transversal relaxation, is caused by spin-spin interactions and can be considered as a ‘fanning’ out of the magnetization in the xy plane. This is another way of saying that there is *loss of phase coherence* (that was built up during the pulse). There is no loss of energy, thus this process can occur faster than T1 relaxation, thus $R_2 \geq R_1$ or $T_2 \leq T_1$. At a certain point in time, the phases become totally random and there is no net magnetization left in the xy-plane, so no signal is detected anymore. The dephasing of the magnetization is in part inherent to the spins, but it is also caused by field inhomogeneity, thus we speak of a $T_2^*$ (the observed $T_2$) = $T_2 + \text{inhomogeneity effects}$. This relaxation is first order with a rate constant $R_2 = 1/T_2$.

What will a FID look like if we have different spins present, each with their own chemical shift, and with their own relaxation times? Well, the total magnetization is just the sum of the individual magnetization vectors, so we will have:

\[
I(t) = \sum_k \left[ \cos((\omega_k - \omega_r)t) + i\sin((\omega_k - \omega_r)t) \right] e^{-t/T_{2,k}}
\]
Where $T_k$ is the effective relaxation rate for spin $k$. In general $T_1 >> T_2^*$, the effective relaxation rate ($1/T_k$) is determined by the fastest one, thus one usually considers only the $T_2^*$ relaxation to affect the decay and thus the line shapes.

**Maple : FID**

The Fourier transform of a function of the type $\, \text{e}^{i\omega t} \text{e}^{-t/T_2^*} \,$ will not produce a real Dirac function, but a Lorentzian absorption peak in the real domain and in the imaginary domain a Lorentzian dispersion signal. The line shape functions for these signals are:

\[
\begin{align*}
\text{Ab}(\nu) &= \frac{2T_2}{1 + \left[2\pi T_2(\nu_0 - \nu)\right]^2} \\
\text{Di}(\nu) &= \frac{4\pi T_2^2(\nu_0 - \nu)}{1 + \left[2\pi T_2(\nu_0 - \nu)\right]^2}
\end{align*}
\]

From this we can derive the height of the absorption signal is half of that of it’s peak, the linewidth, is equal $\Delta \nu_{1/2} = 1/(\pi T_2)$, or more precisely $1/(\pi T_2^*)$.

End of Lecture
Experimental considerations

Now that we know what conceptually is going on, we should focus for a moment on how we physically really work with all this. In general the FID is not observed as a continuous function and we do not know the frequencies that are comprised in it.

data acquisition

As we remarked earlier, when we perform the detection of the signal we mix in the carrier frequency of the observed spins, in order to obtain a low frequent signal. In quadrature detection this frequency lies in the middle of the range that we want to observe, so that we will have positive and negative frequencies and we will need a complex Fourier transform to unravel these. The advantage, and need, of making the signal low frequent is that we have relatively more time to convert the analog signal as a function of time into a digital one. The analog-digital conversion takes a bit of time, and if in the mean time the signal has progressed 1 revolution, one would not get much of that signal. The digital acquisition makes that we have a temporal resolution of the signal, that is reflected into a frequency resolution. How does this work?

Let’s assume that we have N datapoints measured in a time span of t seconds. The time interval between datapoints, Δt, is therefor equal to

\[ \Delta t = \frac{t}{N} \]

It is often easier to think of this as a sample rate : \( R = \frac{1}{\Delta t} = \frac{N}{t} \)

The Nyquist theorem says that you need at least two datapoints per cycle in order to determine whether that frequency is present, using Fourier transformation. This gives us an upper limit of the frequency that can be determined based on our sample rate. The highest frequency, \( v_{Nyq} \) is thus half of the sample rate.

\[ v_{Nyq} = \frac{R}{2} = \frac{1}{2\Delta t} \]

With quadrature detection we will measure from \(-v_{Nyq}\) to \(+v_{Nyq}\), thus the total width of frequencies observed, sweep width, is equal to the sample rate:

\[ SW = 2v_{Nyq} = \frac{1}{\Delta t} \]
If we didn’t have quadrature detection, we would only have the range from 0 to +\(\nu_{Nyq}\), and the sweep width would be equal to the Nyquist frequency itself or half the sample rate. There are going to be problems if the frequency of the signal lies outside the sweep width range. Since we will sample less than 2 points per cycle we will effectively see a lower frequent signal

**Maple: DAQ and FFT**

We saw that if our signal lies outside our sweep range, it will roll back in from the other side of the spectrum. Thus it is important to make sure that we initially set our sweep width larger than needed, to avoid this. After that we can focus on the range in which our peaks lie. If we want to have a small range and we have peaks that lie outside of that range, we need a high frequency filter to cut of the intensity of those signals (or low-band pass filter) and the cut off frequency should be related to our step time (\(\Delta t\), related to SW).

The resolution in the frequency domain, the \(\Delta \nu\) between two successive points, can be calculated easily by noticing that with quadrature detection, we still have N frequencies over the whole sweep width, so we will have a resolution of

\[
\text{resolution} = \frac{SW}{N} = \frac{1/\Delta t}{N} = \frac{1}{N\Delta t} = \frac{1}{t}
\]

So ideally we want to measure as long as possible to get the highest resolution.

**Multiple scans**

The signal to noise S/N ratio will increase if we add several, identically obtained, spectra together. Since the noise is random it will not constructively add. The S/N improvement is proportional to \(\sqrt{\text{#spectra}}\).

This does poses a new problem. We assume that the state of the sample is going to be the same before every single pulse experiment in our series. This means that we should have returned to the equilibrium state. The rate with which this occurs is determined by T1. Since our observed magnetization has died out before that (\(T2 < T1\), \(R2 > R1\)) it is usually not needed to measure all that time (one will only measure the noise in the system by the end of the FID). The amount of magnetization that is excited is related to the tip angle, \(\theta\), (amount of energy we put in the system) and it is generally suggested that to be on the safe side we should meet the conditions that

\[
\cos \theta \leq e^{-\frac{t}{T1}}
\]
where \( t \) is the total time of observation plus a pre pulse delay time (post acquisition delay). It is easy to see that that means that the ratio of \( t/T_1 \) should be:

\[
\ln(\cos \theta) \leq -\frac{t}{T_1}
\]

\[
\frac{t}{T_1} \geq -\ln(\cos \theta) = \ln \left( \frac{1}{\cos \theta} \right)
\]

Thus \( \theta \) approaches 90°, \( t/T_1 \) should be larger than \( \ln(\infty) = \infty \). This is a mathematical fluke, people in general use for using pulses of 90° a experiment interval time of \( 5T_1 \), which in the case of \(^{13}\text{C} \) and \(^1\text{H} \) can be pretty long (\( T_1 \)'s in the range of seconds long…).

Thus in general for a single pulse experiment, one uses pulse angles of 30° or so.

Problem : what should the pre-pulse delay time be if our longest \( T_1 \) is 8 seconds, and we sweep for 4500 Hz having quadrature detection of 4096 points. Our pulse angle is 30°.

Answer : Let’s rewrite our condition:

\[
\frac{t}{T_1} \geq -\ln(\cos \theta) = \ln \left( \frac{1}{\cos \theta} \right)
\]

\[
t = t_{\text{obs}} + t_{\text{delay}} \geq T_1 \ln \left( \frac{1}{\cos \theta} \right)
\]

\[
t_{\text{delay}} \geq T_1 \ln \left( \frac{1}{\cos \theta} \right) - t_{\text{obs}}
\]

We know that \( t_{\text{obs}} \) is determined by \( N \Delta t \) is \( N/SW \), so we have:

\[
t_{\text{delay}} \geq T_1 \ln \left( \frac{1}{\cos \theta} \right) - \frac{N}{SW}
\]

Filling in the numbers shows that \( t_{\text{obs}} = 0.91 \) s, and the first term is 1.15 s so that \( t_{\text{delay}} \geq 0.24 \) s.

End of Lecture
Data manipulation, windowing functions

With the data that are obtained we can do some tricks before we do a Fourier transform. These are usually done, in order to either increase the signal to noise ratio, or sharpen the lines.

increasing signal/noise

Since the magnetization in the xy-plane will decay with an approximate first order rate for transversal magnetization $R_2 = 1/T_2$, at a certain point in time the magnetization becomes almost negligible. The detector at that point will mainly detect noise in the electronic system. Thus if we multiply our FID with an $e^{-at}$ function (with $a > 0$) we will force the intensity of the high t datapoints to zero, thus reducing the noise. At the same time, this will increase the effective line width, since

$$f(t)e^{-R_2t} \times e^{-at} = f(t)e^{-(R_2+a)t}$$

so that the perceived $R_2^*$ is really $R_2 + a$, and a higher relaxation rate means a larger linewidth, since $\Delta v_1^2 = \frac{1}{\pi T_2} = \frac{1}{\pi} R_2^* = \frac{1}{\pi} (R_2 + a)$. This is the reason that this kind of manipulation is also called line-broadening.

increasing resolution

Since the resolution is given by $1/t$, we can pretend that we measured for a longer time by padding our N data points with a bunch of zero’s. This will only work properly if by the end of the acquired FID the magnetization has gone completely to 0. If it hasn’t we can use a line-broadening function to force this to happen. This technique of interpolation using Fourier transform, is in NMR terms also called zero-filling.

line-sharpening

As we know signal with an envelope function of $e^{-tR_2}$ yields a Lorentzian line shape after Fourier transform. The Gaussian lineshape is narrower at the baseline, and stems from an envelope function of $e^{-at^2}$. To make our lines narrower we can multiply our FID with $e^{+itR_2}$ first, to undo the $T_2$ relaxation, followed by a multiplication with the $e^{-at^2}$ function to yield the Gaussian line. Since that has $at^2$, the a value can be kept pretty small.

(Maple: Gaussian.ms)
trigonometric transforms

These functions are in general used for 2D spectra. If you know that your modulation should be purely sine, you can multiply the FID in that time domain with a sine function and making that the node in the sine ends up at the last data point. This will dampen the short time information and increase the later times. The effect of that in a FT is that the lines will become sharper, but also that one can get ‘ringing’ around the peak(s).

The 1 pulse experiment

So far we have been talking about an experiment in which we give one pulse of angle \( \theta \) after which we detect the modulation of the magnetization along two directions in the xy plane (quadrature detection). We saw that in order to repeat this experiment we had to be careful about the time we give the system to reach equilibrium again. In practice this equilibrium will not be completely reached, but a sort of steady state will be obtained. As a result of the first few pulses do not start with the steady state magnetization, and often it is suggested to not detect the magnetization for the first few pulses (dummy scans). We also saw that we need to use a phase cycle in order to balance out differences between the two receiver coils.

The way this pulse sequence is written is, if one uses a 30° pulse along the x-axis: \( \frac{\pi}{6} \) or pictorially:

\[
\text{PD} \quad \frac{\pi}{6} \quad \text{FID}
\]

Multiple pulse 1D experiments

From now on we are going to look at new forms of NMR spectroscopy that have become available with the use of pulsed NMR. These techniques require more than one pulse, a pulse sequence, so they can never be achieved by CW NMR. Depending on the particular pulse sequence used we will observe a particular spectrum, and they will be named according to what its function is.

The 1D pulse sequences we will look at are:

- T1IR – \( T_1 \) Inversion Recovery
- APT – Attached Proton Test
- DEPT – Distortionless Enhancement by Polarization Transfer
- INEPT – Insensitive Nuclei Enhanced by Polarization Transfer
Vector model versus Product Operator Formalism

In order to understand what happens during a pulse sequence we will use both the vector model and the product operator formalism. They both have their specific advantages and disadvantages. The book will only use the vector model, but since we can use Maple and a set of procedures that implements the algebra involved in the POF model has been developed, we will also use the POF model. The POF model works with so-called transformation rules that are a recipe for how terms in the spin state will change when pulses are applied, chemical shift evolves, coupling evolves, and the magnetization is observed. In the second problem set you will derive a set of algebraic rules based on the vector model. You will see that they are wrong in only one term, and as a result of that the vector model will not work properly in all cases. In particular, the vector model will go wrong when so-called multiple quantum coherences play a role. We will not go into what these are at this point in time, but it will suffice to say that these will give observable magnetization only if we have more the one 90° pulse on a particular set of spins.

T$_1$ Determination: T1IR

It is important to know what the longest T$_1$ is since that one will determine the time needed to re-establish the equilibrium magnetization, so that in a sense the sample does not know anymore what had happened to it before. Often people do not determine the T$_1$ and as a result of that they use recycle delays that are longer than necessary and waste time, or the recycle delay is to short and peak intensities are not right (recall in $^{13}$C NMR the quaternary carbons usually are very weak since they have very long T$_1$’s). If you run only a 1-pulse NMR spectrum that is usually OK, but if you want to really investigate a molecule by NMR, e.g. structure elucidation by NMR, and you need to run lots of spectra of your compound, you had better be sure that you know what your T$_1$’s are so that you can be as efficient as possible. We already saw before that the T$_1$ ≥ T$_2$ so that you can not figure out what the T$_1$ is from looking at your FID and seeing that it is practically zero at the end or from the line shape, both of which depend on T$_2^*$. How do we figure out what the T$_1$’s of your peaks are? T1IR: T$_1$ determination through Inverse Recovery. The name suggests that we look at the recovery of magnetization after it has been inverted. So we give a π pulse to get it along -z, what a bit to give it some time, τ, to recover magnetization along +z (and loose it along -z) and then flip that back in the xy plane, using a π/2 read pulse, so we can detect how much was recovered depending on τ. Thus in a pulse sequence:

\[ \pi_x \quad \tau \quad \frac{\pi}{2}_x \quad \text{FID} \]
The effect of this on a single spin can be seen as follows:

\[ \pi_x \rightarrow \\tau \text{ short} \rightarrow \left[ \frac{\pi}{2} x \rightarrow \text{negative peak} \right] \]

\[ \tau \text{ long} \rightarrow \left[ \frac{\pi}{2} x \rightarrow \text{positive peak} \right] \]

Thus the intensity of the signal depends on the time, \( \tau \), between the pulses.

In order to determine the intensity of the signal as a function of \( \tau \), we need to solve the Bloch equation for longitudinal relaxation. This equations says that the magnetization along the z-axis will be built in a rate proportional to how far away from equilibrium the system is, with a first order rate constant of \( 1/T_1 \). Thus we can write down the differential equation:

\[
\frac{dM_z(t)}{dt} = \frac{(M_0 - M_z(t))}{T_1}
\]

<Maple : T1IR.ms>

Maple showed that we get for the z magnetization before the read pulse

\[
M_z(t) = M_0 \left( 1 - 2e^{-\frac{t}{T_1}} \right)
\]

After our read pulse all this magnetization is tipped in the xy plane and thus its \( \tau \) dependence describes the peak intensity. To get \( T_1 \) from all this we can rewrite our equation as:

\[
\ln \left( \frac{M_0 - M(\tau)}{2M_0} \right) = -\frac{\tau}{T_1}
\]

and we see that plotting this ln versus \( \tau \) will give a slope of \(-1/T_1\).

We can also adjust \( \tau \) so that the slowest relaxing peak is exactly zero. In that case, we have that

\[
T_1 = \frac{\tau}{\ln(2)}
\]
**T$_2$ Determination**

So much for the T$_1$ determination, how about the T$_2$?

Well we know that the linewidths in the spectrum are determined by the T$_2$ in combination with the effect of field inhomogeneities. We need to be able to get rid the effect of those inhomogeneities in order to determine the real T$_2$.

Let’s consider what happens to the magnetization of a single site as a result of the field inhomogeneities. Some of the molecules will be at a position in the field such that they will feel a slightly stronger field than the others. As a result of that they will precess faster than the other ones. If we were able to reverse the direction of precession, we would be able to get them to refocus. This is the idea of the spin-echo experiment and was initially developed by Hahn in 1950.

The spin echo pulse sequence is:

\[ \frac{\pi}{2} \_ x \_ \tau \_ \pi \_ x \_ \tau \_ FID \]

Let’s look at the vector model to see what is going to happen during this sequence. We will focus on two peaks $\omega_1$, and $\omega_2$. We will assume that there is some field inhomogeneity present, so we will have a fast and slow precessing spins present for each of the two spins.

We hadn’t mentioned this before, but in a single pulse experiment it is not possible to immediately start the observation of the magnetization after the pulse. The pulse has to ‘die out’ before we can open the receiver and as a result of that the phases of the magnetization that is being observed will not be 0, but depend on the chemical shift and the time between the pulse and start of the observation. This is the reason that we need a frequency dependent phase correction in our 1D spectra. We looked at the results of non-zero phases in the first problem set and saw that they will give rise to peaks that consist of the sum of absorption and dispersion contributions, so they won’t look too nice. In order to show that in our analysis for this sequence we will split our $\tau$ evolution into two parts, one DE long, and the other $\tau$-DE long:
Thus we see that the problems of the phases, due to DE, is fixed.

The fanning out, due to field inhomogeneity, during the first \( \tau \) is refocused during in second \( \tau \), i.e. after the \( \pi \) pulse. Because of the refocusing effect, the intensity of the signals will be reduced due to the pure \( T_2 \) and is now dependent on the time elapsed between the end of the \( \pi/2 \) pulse and the start of the observation domain, so \( M(\tau) = M_0 e^{-\frac{2\pi}{T_2}} \), thus a plot of the natural log of the intensities versus \( \tau \) will give a straight line with slope of \(-\frac{2}{T_2}\). (This only works if no coupling is present...)

**Coupling**

So far we have only focused on what happens with singlets, or in other words we only looked at the chemical shift evolution during the wait times and during observation of the magnetization. This has been an oversimplification since NMR spectra are not only determined by the chemical shifts of the spins but also by coupling between the spins. The Hamiltonian part that has the coupling term in it, is of the form (proportional to the dot product of the spin angular momenta)

\[
H_{k,l} = J_{k,l} \mathbf{I}_k \cdot \mathbf{I}_l
\]

We already had the Zeeman term as the dot product between the spin and the magnetic field. The eigenfunctions for the Zeeman term are our spin functions \( \alpha \) and \( \beta \), with \( m_1 = 1/2 \) and -1/2,
respectively. The dot product of the two spin angular momenta will introduce $x$ and $y$ spin operators in the Hamiltonian:

$$\hat{H} = -\sum_k v_k \hat{I}_{kz} + \sum_{k<l} J_{k,l} \left( \hat{I}_{kx} \hat{I}_{lx} + \hat{I}_{ky} \hat{I}_{ly} + \hat{I}_{kz} \hat{I}_{lz} \right)$$

where the Hamiltonian is set up to give energies in Hz.

This causes a problem, because we know that the $I_x$ and $I_y$ do not commute with $I_z$, and our initial eigenfunctions of the Zeeman term are not eigenfunctions of new spin Hamiltonian anymore. Therefore we will use perturbation theory and assume that the $I_x$ and $I_y$ terms can be separated out. Thus a first order approximation, leading to first order spectra, is obtained using the following Hamiltonian (where we now drop the hats on the operators):

$$H = -\sum_k v_k I_{kz} + \sum_{k<l} J_{k,l} I_{kz} I_{lz}$$

Now we can use spin functions that are products of $\alpha$ and $\beta$’s for the spins that are present, e.g., for a two spin system we have $\alpha\alpha$, $\alpha\beta$, $\beta\alpha$, $\beta\beta$, and they will still be eigenfunctions of our first order Hamiltonian. The result is that the Hamiltonian matrix will be diagonal, since there are no operators that will change these spin functions, i.e. $H|\imath\rangle = E_\imath |\imath\rangle$

$$H_{ii} = \langle \imath | H | \imath \rangle = E_\imath \langle \imath | \imath \rangle = E_\imath$$

$$H_{ij} = \langle \imath | H | j \rangle = E_{ij} \langle \imath | j \rangle = 0$$

and the energies of the system can be determined directly from the diagonal elements of the Hamiltonian. Thus we can calculate the energies for a two spin system, spin 1, and 2, as follows

$$E_\imath = \langle \imath | v_1 I_{1z} | \imath \rangle + \langle \imath | v_2 I_{2z} | \imath \rangle + \langle \imath | J_{1,2} I_{1z} I_{2z} | \imath \rangle$$

$$= -v_1 \langle \imath | I_{1z} | \imath \rangle - v_2 \langle \imath | I_{2z} | \imath \rangle + J_{1,2} \langle \imath | I_{1z} I_{2z} | \imath \rangle$$

$$= -v_1 m_{1,1} \langle \imath | \imath \rangle - v_2 m_{1,2} \langle \imath | \imath \rangle + J_{1,2} m_{1,1} m_{1,2} \langle \imath | \imath \rangle$$

$$= -v_1 m_{1,1} - v_2 m_{1,2} + J_{1,2} m_{1,1} m_{1,2}$$

Now it is simple to calculate the energies of the four basis functions:
The energy splitting of this system can be drawn as:

\[
\begin{align*}
\alpha\alpha: & -v_1\left(\frac{1}{2}\right) - v_2\left(\frac{1}{2}\right) + J_{1,2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{v_1}{2} - \frac{v_2}{2} + \frac{J_{1,2}}{4} \\
\alpha\beta: & -v_1\left(-\frac{1}{2}\right) - v_2\left(-\frac{1}{2}\right) + J_{1,2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{v_1}{2} + \frac{v_2}{2} - \frac{J_{1,2}}{4} \\
\beta\alpha: & v_1\left(-\frac{1}{2}\right) - v_2\left(\frac{1}{2}\right) + J_{1,2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = +\frac{v_1}{2} - \frac{v_2}{2} - \frac{J_{1,2}}{4} \\
\beta\beta: & v_1\left(-\frac{1}{2}\right) - v_2\left(-\frac{1}{2}\right) + J_{1,2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = +\frac{v_1}{2} + \frac{v_2}{2} + \frac{J_{1,2}}{4}
\end{align*}
\]

And we see that we get a splitting of the signal for spin 1 in two signals with a separation of $J_{1,2}$ Hz between them, centered at $v_1$. Notice that the excitation for which the other spin is $\beta$ will be at higher energy when the coupling value is positive.

End of Lecture
Spin echo and homo/hetero nuclear coupling

The way the vector model introduces the idea of coupling is by treating each of the peaks in a multiplet as a singlet that also carries around some information about the spin state of the other nucleus. We know that the spectrum for spin A consists of a doublet, with resonances as $\nu_A \pm J/2$ due to the two spin states, $\beta$ (+J/2) and $\alpha$ (-J/2) that spin X can be in. The angular frequencies that go with these are $2\pi$ times as large, i.e., $\omega_A \pm \pi J$. Thus for a hetero nuclear coupling a spin echo experiment will result in (starting after the $\pi/2$ pulse)

![Diagram of spin echo experiment](Maple : spinecho.ms)

i.e., the chemical shift and hetero nuclear coupling will refocus.

In the case of a homo nuclear coupling, the $\pi$ pulse will also flip the $\alpha$ and $\beta$:

![Diagram of homo nuclear coupling](Maple : spinecho.ms)

and we see that homo nuclear coupling is not refocused in a spin-echo experiment.

What does that mean? Well, it means that the intensities of the signals observed will depend on $\tau$ for the homo nuclear spins, thus the intensities of signals that have homo nuclear couplings will modulate.

But wait a minute, we can control whether or not a coupling will evolve in a time domain. We can ‘decouple’ our spectrum by irradiating the sample with a broad band e.m. signal that will cause the spins to flip so fast between their spin states, that the other spins will only effectively see a single state. As a result of that there will be no coupling, but the peaks can shift a little bit (Bloch-
Siegert shifts). There are other effects caused by broad band decoupling, but we will not go into those. The notation for decoupling spins A, i.e. irradiating spins A, is \{A\}.

**Attached Proton Test (APT)**

If we use broad band decoupling during one $\tau$ period in our spin echo experiment and not the other, we prevent the hetero nuclear coupling from refocusing. If all of our hetero nuclear couplings are strong and of about the same strength we can play with the $\tau$ time so that the vectors will cancel out in specific cases. This is the basis for the APT, attached proton test. This technique allows us to distinguish spins (e.g., $^{13}$C) that are coupled to an even number of hetero nuclei (e.g. $^1$H) from those that are coupled to an odd number of them. Notice that you could do this for other nuclei also.

The pulse sequence for the APT experiment is:

```
13C \[\pi_x/2\] \[\tau\] \[\pi_x\] \[\tau\] FID

1H \[BB\] \[BB\]
```

First we should figure out how far a signal that is $\Delta v$ away from the precession frequency of the rotating frame precesses in a time $t$. That is of course given by $\Delta \phi = t \times \Delta v$. Let’s look at the splitting pattern that arises from 0, 1, 2, and 3 nuclei of spin 1/2 coupled with another spin 1/2 in a 1-pulse experiment:

```
\[\begin{array}{c}
n=0 \\
n=1 \\
n=2 \\
n=3 \\
\end{array}\]
```

This picture reeks of FT of observed signals described by $e^{2\pi i vt} \cos^n(2\pi \frac{J}{2} t)$.

End of Lecture
The APT sequence and the vector model

Let’s restate the APT pulse sequence:

\[ \begin{array}{c}
\text{FID} \\
\pi_x \\
\tau \\
\pi_x \\
\end{array} \]

\[ \begin{array}{c}
\text{BB} \\
\text{BB} \\
\end{array} \]

\[ \begin{array}{c}
\text{13C} \\
\text{1H} \\
\end{array} \]

Independent of the number of \(^1\text{H}\)”s the carbon is coupled to, the state of the system is the same after the \(\pi\) pulse (coupling with the hydrogen spins will not evolve during the proton decoupled \(\tau\) interval). For convenience sake we assume that the rotating frame frequency is equal to the \(^{13}\text{C}\) frequency. We can draw the following pictograms to get to the start of the second \(\tau\) evolution:

Now, depending on the length of \(\tau\) and the kind of carbon we are looking at, things will develop differently. We will look at two cases \(\tau\) times: \(\frac{1}{2J}\), and \(\frac{1}{J}\) for C, CH, CH\(_2\), and CH\(_3\) respectively.

With protons present, the amount of revolutions, \(\Delta\phi\), that each of the vectors due to the presence of \(J_{\text{CH}}\) will rotate during a time \(\tau\) is given by \(\Delta\phi = \Delta\nu \tau\), where \(\Delta\nu\) is \(\pm nJ/2\). For instance a signal at \(+J/2\), with respect to the chemical shift, will during a time \(1/(2J)\) evolve for \(\Delta\phi = J/2 \times 1/(2J) = 1/4\) revolution. (in vector model positive rotation is clock wise, opposite of mathematical positive)
Thus we get positive peaks for odd numbers of protons, and negative peaks for even numbers when $\tau = 1/J$. When $\tau = 1/(2J)$ we only get quaternary C peaks which are negative. Keep in mind that it is possible to get exactly the reverse by choosing a $\pi$ phase shift for detection (UR: APT spectra have odd H down, even H up, actually W&M now too, have an additional $\pi$ pulse at end of the sequence).

(Maple : APT.ms)
Selective Polarization Transfer

(Maple: POPULAT.ms)

So far we haven’t really looked into the peak intensities of our signals due to the sensitivity for particular nuclei. In other words, we have not looked at population differences in the spin states and the effect that pulses will have on them. From the Boltzman distribution we would expect the ratio of the two populations to be proportional to $\exp(-\Delta E/kT)$. When $\Delta E$ is small compared to $kT$ we can expand this series into a MacLaurin series truncated after the second term, i.e.,

$$\frac{N_\beta}{N_\alpha} = e^{-\frac{\Delta E}{kT}} = e^{-\frac{\gamma h B_0}{kT}} \equiv 1 - \frac{\gamma h B_0}{kT}$$

since the MacLaurin series of an exponential function is given by

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

Thus the excess of systems in the lower energy state, $\alpha$, and thus the excitation probability is proportional to $\gamma$:

$$N_\alpha - N_\beta = N_\alpha - N_\alpha \left(1 - \frac{\gamma h B_0}{kT}\right) = N_\alpha \frac{\gamma h B_0}{kT}$$

The sensitivity for a particular nucleus in NMR is also dependent of the effectiveness of inducing a current in the detector (proportional $\gamma$), and the size of the magnetization vector, also proportional to $\gamma$. Thus the sensitivity is proportional to $\gamma^3$. In addition to that, we need to account for the natural abundance, $N$, of NMR sensitive nuclei. Thus for the relative sensitivity to $^1H$ versus $^{13}C$, we can write, since $\gamma$ of $^1H$ is 4 times $\gamma$ of $^{13}C$:

$$\frac{S(^1H)}{S(^{13}C)} = \frac{\gamma_{^1H}^3 / N(^1H)}{\gamma_{^{13}C}^3 / N(^{13}C)} = \frac{4^3}{1^3} \cdot \frac{100\%}{1.1\%} = 5818$$

Thus means that it any NMR is about 6000 times less sensitive to $^{13}C$ than it is to $^1H$. That is terrible. There are some tricks in which, through what we call population inversion, some of the sensitivity for protons is transferred to the carbons.

End of Lecture
Energies and populations

We know that the high-resolution, solution, spin Hamiltonian consists of a Zeeman term and a coupling term as follows, (Hamiltonian to give energy in Hz, for rad/s multiply with $2\pi$):

$$\hat{H} = -\nu_k \hat{I}_{kz} + \sum_{k<l} J_{k,l} \hat{I}_{kz} \hat{I}_{lz}$$

The expectation value of the energy of the system is given by the expectation value of this Hamiltonian. We already noticed that the products of the nuclear spin functions, which we use as the spin functions for the collection of nuclei, are still eigenfunctions of this Hamiltonian. As a result of that we can write:

$$\langle E_{ii} \rangle = \langle i|\hat{H}|i \rangle = \langle i|\sum_k -\nu_k \hat{I}_{kz} + \sum_{k<l} J_{k,l} \hat{I}_{kz} \hat{I}_{lz}|i \rangle = \sum_k -\nu_k m_{I,k} + \sum_{k<l} J_{k,l} m_{I,k} m_{I,l}$$

Or more specific for a two spin case of a $^{13}$C and a $^1$H spin, which are coupled to each other:

$$E = -m_{I,H}\nu_H + -m_{I,C}\nu_C + m_{I,H}m_{I,C}J$$

The relative populations of the spin states are proportional to $-E$ (lower $E$, higher population). Since the $J$ value is very small compared to the $\nu$’s we will ignore them in the population analysis, and we can write down for the relative populations:

$$P = m_{I,H}P_H + m_{I,C}P_C$$

Now let’s put all this in a picture:
The energy of the transitions are $E_2 - E_1$ and the intensity is proportional to how many more spins are in the ground state than in the excited state, i.e., $P_1 - P_2$:

<table>
<thead>
<tr>
<th>$^1$H transitions</th>
<th>$\Delta E$</th>
<th>Int</th>
<th>$^{13}$C transitions</th>
<th>$\Delta E$</th>
<th>Int</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \alpha \rightarrow \beta \alpha$</td>
<td>$v_H - \frac{J}{2}$</td>
<td>$P_H$</td>
<td>$\alpha \alpha \rightarrow \alpha \beta$</td>
<td>$v_C - \frac{J}{2}$</td>
<td>$P_C$</td>
</tr>
<tr>
<td>$\alpha \beta \rightarrow \beta \beta$</td>
<td>$v_H + \frac{J}{2}$</td>
<td>$P_H$</td>
<td>$\beta \alpha \rightarrow \beta \beta$</td>
<td>$v_C + \frac{J}{2}$</td>
<td>$P_C$</td>
</tr>
</tbody>
</table>

So we see that we get two lines with equal intensities in both the $^1$H and $^{13}$C spectra at the chemical shift $\pm J/2$, where the higher frequency one is caused by the $\beta$ spin of the other nucleus:

This is of course exactly what we expected.

End of Lecture
Population inversion

Population inversion is achieved by giving a $\pi$ pulse on only one of the resonances in a multiplet, i.e. flipping the net magnetization from the $+z$ axis to the $-z$ axis. Consider again our CH system where we give a $\pi$ pulse on the peak $\nu_H+J/2$ (i.e., $\alpha\beta\rightarrow\beta\beta$) (right hand side of the picture) or on the other $^{13}\text{C}$ satellite ($\alpha$, for the left hand side). The populations of the spin states will be affected as follows

\[
\begin{align*}
\text{Energy} & \quad \text{intensity $\beta$ inversion} & \quad \text{intensity $\alpha$ inversion} \\
\alpha\alpha\rightarrow\alpha\beta & \quad \nu_C - \frac{J}{2} \quad - \frac{P_H}{2} - \frac{P_C}{2} \quad - \frac{P_H}{2} + \frac{P_C}{2} \\
\beta\alpha\rightarrow\beta\beta & \quad \nu_C + \frac{J}{2} \quad - \frac{P_H}{2} + \frac{P_C}{2} \quad + \frac{P_H}{2} - \frac{P_C}{2}
\end{align*}
\]

The transitions intensities of the $^{13}\text{C}$ transitions are thus given by, since $P_H = 4 \ P_C$ (based on $\gamma$ ratios),

<table>
<thead>
<tr>
<th>$^{13}\text{C}$ transitions</th>
<th>Energy</th>
<th>$\beta$ inversion</th>
<th>$\alpha$ inversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha\alpha\rightarrow\alpha\beta$</td>
<td>$\nu_C - \frac{J}{2}$</td>
<td>$-\frac{P_H}{2} - \frac{P_C}{2}$</td>
<td>$+P_H + P_C \approx 5P_C$</td>
</tr>
<tr>
<td>$\beta\alpha\rightarrow\beta\beta$</td>
<td>$\nu_C + \frac{J}{2}$</td>
<td>$-\frac{P_H}{2} + \frac{P_C}{2}$</td>
<td>$-P_H + P_C \approx -3P_C$</td>
</tr>
</tbody>
</table>

Thus due to $\beta$ inversion, the low energy transition in the $^{13}\text{C}$ spectrum has an intensity increased by a factor 5, and the high energy transition now has a negative intensity 3 times as strong. This is caused by the transfer of the population difference of the $^1\text{H}$’s to the $^{13}\text{C}$’s. By taking the difference between the two transfers we see that we can get an effective transfer of sensitivity of $\frac{\gamma_H}{\gamma_C} \approx 4$.

The problem is that we can only enhance 1 C at a time, since if we give a $\pi$ pulse on the whole $^1\text{H}$ range we would not get the right population levels. The solution to these problems are the pulse sequences INEPT and DEPT.
INEPT

In order to get the inversion of either $\alpha$ or $\beta$ of the $^{13}$C satellites of the protons, we can make the coupling between $^1$H and $^{13}$C to seem homo nuclear (by giving $\pi$ pulses on both spins) and make that the spin echo total time is $1/(2J)$ so that the doublet evolves into two anti parallel vectors along $x$ (see also APT for CH spin), which after a $\pi/2$ pulse along $y$ will turn it into the picture in which we have the $\alpha$ along $z$ and $\beta$ along -$z$ (or vice versa with a pulse along -$y$), i.e., selective inversion of either the $\alpha$ or $\beta$ $^{13}$C satellite in the proton spectrum!

The pulse sequence for Inensitive Nuclei Enhanced by Polarization Transfer is:

$$
\begin{align*}
^1H & \quad \frac{\pi}{2}x \quad \frac{1}{4J} \quad \pi_x \quad \frac{1}{4J} \quad \frac{\pi}{2} \pm y \\
^{13}C & \quad \pi_x \quad \frac{\pi}{2}x \quad \text{FID}
\end{align*}
$$

In order to determine the effect of the pulse sequence step by step, we will first focus on the $^1$H spins. Notice with a coupling of $J$ Hz, an evolution of $1/(4J)$ will result in $\pm J/2 1/(4J) = \pm 1/8$ cycle for the coupling. With a $J$ of 150 Hz, this translates to intervals of $1/(4J) = 1.67$ msec.

Thus after the $\pi/2$ y-pulse on the protons we effectively have the high frequency signal of the doublet inverted. The effectiveness of the inversion is strongly dependent on the correspondence between the evolution times $1/4J$ with the real $J$ value.
Looking at the $^{13}\text{C}$ after the inversion has been established. Still left with a $\pi/2$ x pulse on $^{13}\text{C}$ and the observation:

\[
\begin{align*}
\text{satellite} & \quad \text{inversion} \\
\alpha & \quad \beta \\
& \quad \text{pulse} \quad \text{on} \quad 13\text{C} \\
& \quad \text{at} \quad \text{all} \quad \text{inversion} \quad \pi \times \text{pulse} \quad \text{on} \quad \text{13C} \\
& \quad \text{and} \quad \text{we} \quad \text{end} \quad \text{up} \quad \text{with} \quad \text{an} \quad \text{anti-phase} \quad \text{doublet}, \quad \text{where} \quad \text{the} \quad \text{high} \quad \text{frequency} \quad \text{part} \quad \text{is} \quad \text{along} \quad \text{the} \quad \text{negative} \quad y \quad \text{axis}, \quad \text{giving} \quad \text{rise} \quad \text{to} \quad \text{a} \quad \text{negative} \quad \text{peak} \quad \text{at} \quad +J/2 (\beta \text{ spin}) \quad \text{and} \quad \text{a} \quad \text{positive} \quad \text{peak} \quad \text{at} \quad -J/2 (\alpha \text{ spin}). \quad \text{The} \quad \text{intensity} \quad \text{enhancement} \quad \text{are} \quad \text{a} \quad \text{factor} \quad 5 \quad \text{and} \quad 3 \quad \text{respectively}
\end{align*}
\]

By alternating the $\pi/2$ y pulse on the protons with a $\pi/2$ along -y and subtracting these spectra we get a polarization transfer of $(+J/2 : -J/2) = (-3 : +5) - (+5 : -3) = (-8 : +8)$ or $-1 : +1$. The intensity is enhanced by the factor $\gamma_{1H}/\gamma_{13C} \approx 4$ on average for each of the two spectra.

The intensities of the signals for doublet, triplet and quartet will be as follows (increasing $\nu$ to left):

- **doublet**: $-1 : 1$
- **triplet**: $-1 : 0 : 1$
- **quartet**: $-1 : -1 : 1 : 1$

One immediately sees that if one were to use proton decoupling during the observation time domain, the peaks would collapse on top of each other and the net intensity left would be 0! But it would have been nice if we could get a proton decoupled carbon spectrum...

**MAPLE : INEPT.ms**

Notice that in Maple I have to use an observation phase of $\pi/2$ ($\pi$ out of phase with APT). This is because there was a $\pi$ pulse on $^{13}\text{C}$, which I ignored in the vector model.

---

End of Lecture
INEPT for a CH$_2$ spin system using the Product Operator Formalism. Protons: A and B, carbon: C.

<table>
<thead>
<tr>
<th>$^1$H</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{1}{4J}$</th>
<th>$\pi$</th>
<th>$\frac{1}{4J}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\pi_x$</th>
<th>$\frac{1}{2}$</th>
<th>$\pi_x$</th>
<th>$\frac{1}{4J}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{13}$C</td>
<td>$\pi_x$</td>
<td>$\pi_x$</td>
<td>$\frac{\pi}{2}$</td>
<td>FID</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Shortcut for spin echos:
$I_{x,y} \xrightarrow{\omega I_z} \pi_x \xrightarrow{\omega I_z} I_{x,y}$

Don’t need to evolve chemical shift.

Reminder for rotations:

<table>
<thead>
<tr>
<th>$z$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
</tr>
</thead>
</table>

| $y$ |

System CH$_2$:

- $\pi$ $^2$ Ax,Bx
- $\pi$ $^2$ 2I$_{Az}^1$I$_{Cz}$
- $\pi$ $^2$ 2I$_{Bz}^1$I$_{Cz}$
- (system) Ax,Bx,Cx

Combination:

- $\frac{1}{2}$ $^2$ I$_{Ay}$ $- \frac{1}{2}$ 2I$_{Ax}^1$I$_{Cz}$ $- \frac{1}{2}$ 2I$_{Ax}^1$I$_{Cz}$ $- \frac{1}{2}$ I$_{Ay}$
- $\frac{1}{2}$ $^2$ I$_{By}$ $- \frac{1}{2}$ 2I$_{Bx}^1$I$_{Cz}$ $- \frac{1}{2}$ 2I$_{Bx}^1$I$_{Cz}$ $- \frac{1}{2}$ I$_{By}$

Enhanced modulations for C, stem from H spinoperators

Which means:

- $I_{Cx} (\pm 2 \cos \omega C t \cos \pi J t \sin \pi J t - \sin \omega C t \cos^2 \pi J t) + I_{Cy} (\pm 2 \sin \omega C t \cos \pi J t \sin \pi J t + \cos \omega C t \cos^2 \pi J t)$

Observed signal:

- $2e^{i\omega c^t} \sin 2\pi J t$

Notice that normal C spectrum (last terms between brackets) drops out and only the enhanced modulation is observed.
Refocused INEPT

(MAPLE : INEPT.ms continuation)

There is a modification to the INEPT sequence so that one can observe proton decoupled spectra. This is the so-called **Refocused-INTEPT** which has the following pulse sequence:

\[
\begin{align*}
1^H &: & \frac{\pi}{2} & \frac{1}{4} & \Delta & \frac{1}{4} & \Delta & \frac{\pi}{2} & \Delta & \frac{\pi}{2} & \Delta & \text{BB} \\
13^C &: & \Delta & \frac{\pi}{2} & \Delta & \frac{\pi}{2} & \Delta & \frac{\pi}{2} & \Delta & \text{FID}
\end{align*}
\]

where the $\Delta - \pi - \Delta$ echo sequence allows for the anti-phase patterns to evolve with a refocusing of the chemical shifts, analogous to the APT spectrum, into in-phase patterns, so that at that point in time the decoupler can be turned on. Suitable sizes for the $\Delta$ intervals are $1/4J$ for $\text{CH}$, $1/8J$ for $\text{CH}_2$, and $1/8J$ for $\text{CH}_3$ where for the latter the anti-phase pattern does not evolve into a pattern in which all peaks are in phase but will have the largest intensity. In order to have all kinds of C’s one sets the $\Delta$ value to $1/7J$ as a compromise. See figure 3.41. The phase distortions are more severe in the refocused INEPT than in the normal INEPT.

End of Lecture
DEPT

Distortionless Enhancement by Polarization Transfer allows one to observe proton decoupled enhanced $^{13}$C spectra where the phase distortion, due to incorrect pulse intervals (related to differences in $J_{C,H}$ values) is less severe. The pulse sequence is as follows:

This sequence works based multiple quantum coherence developments, which are impossible to obtain with the vector model. We will therefore use the Product Operator Formalism procedures in Maple to figure out what the effect of this pulse sequence is.

MAPLE : DEPT.ms

The absorption signals observed for different C’s are proportional to:

\[
\begin{array}{cccc}
\theta = \theta & \theta = 45^\circ & \theta = 90^\circ & \theta = 135^\circ \\
\text{CH} & \sin \theta & \sqrt{2}/2 & 1 & \sqrt{2}/2 \\
\text{CH}_2 & 2\sin \theta \cos \theta = \sin 2\theta & 1 & 0 & -1 \\
\text{CH}_3 & 3\cos^2 \theta \sin \theta & 3\sqrt{2}/4 & 0 & 3\sqrt{2}/4 \\
\end{array}
\]

In general for a CH$_n$ spin system and a pulse angle of $\theta$ we have for the peak intensity:

\[ n \cos^n \theta \sin \theta \]

overhead worked out example DEPT CH$_2$

End of Lecture
DEPT for a CH$_2$ spin system using the Product Operator Formalism. Protons: A and B, carbon: C.

From here on we only have shift evolutions, so no new single transversal operators can be produced (only by coupling). Observables are thus:

\[
\begin{align*}
\omega_{A,B} & \quad 1/I_{Az,Bz} \\
\omega_{C} & \quad I_{Cy} \\
\end{align*}
\]

which means:

\[
I_{Cx} (2 \cos \omega_C t \cos \theta \sin \theta - \sin \omega_C t \cos^2 \theta) + I_{Cy} (2 \sin \omega_C t \cos \theta \sin \theta + \cos \omega_C t \cos^2 \theta)
\]

Enhanced, absorption

\[
\begin{align*}
\cos \theta & \quad \text{enhanced} \\
\sin \theta & \quad \text{not enhanced}
\end{align*}
\]

Observed signal:

\[
(\cos \omega_C t + i \sin \omega_C t) \sin 2\theta + i (\cos \omega_C t + i \sin \omega_C t) \cos^2 \theta = e^{i \omega_C t} \sin 2\theta + i e^{i \omega_C t} \cos^2 \theta
\]

Enhanced, absorption

\[
\begin{align*}
\cos \theta & \quad \text{not enhanced, dispersion}
\end{align*}
\]
2D NMR spectroscopy

Introduction

A general pulse sequence can be considered to consist of three stages:

\[ \text{preparation} - \text{evolution} - \text{detection}(t) : I(t) \]

In 1D experiments only the detection stage introduces a time dependence of the observables (transversal magnetization). The modulations of the detected signal in the observation time domain is converted into a frequency spectrum by performing a Fourier Transform.

NMR spectroscopy becomes really fun when you also allow the evolution time domain to be a function of time:

\[ \text{preparation} - \text{evolution}(t_1) - \text{detection}(t_2) : I(t_1,t_2) \]

This means that we will have a series of spectra (observed in \( t_2 \)) with specific lengths of \( t_1 \) times. A Fourier transform of the \( t_1 \) time domain will give the frequencies present in the intensity modulation of the peaks in the \( \nu_2 \) spectrum. The \( t_1 \) time variable is just a time between pulses that is going to be varied from spectrum to spectrum. We will have to perform Fourier transforms for every independent variable time domain present in our pulse sequence.

The Fourier transform of a signal that is a function of more than one independent time variables can be considered to be the sequence of separate Fourier transforms for each variable. This is easily understood by keeping in mind that when one Fourier transforms the \( t_2 \) domain, the part of the functions that depends on anything but \( t_2 \), like \( t_1 \), can be considered constant and be taken out of the integration itself. Thus if we can write our time function \( I(t_1,t_2) \) as \( I_1(t_1)I_2(t_2) \), we can do the two Fourier transforms as follows:

\[
F(\omega_1,\omega_2) = FT_1\left( FT_2\left( I(t_1,t_2) \right) \right) = FT_1\left( FT_2\left( I_1(t_1)I_2(t_2) \right) \right) = FT_1\left( I_1(t_1)FT_2\left( I_2(t_2) \right) \right)
\]

\[
= FT_1\left( I_1(t_1) \int I_2(t_2)e^{-i\omega_2 t_2} dt_2 \right) = FT_1\left( I_1(t_1)F(\omega_2) \right)
\]

\[
= \int I_1(t_1)F(\omega_2)e^{-i\omega_1 t_1} dt_1 = F(\omega_2)\int I_1(t_1)e^{-i\omega_1 t_1} dt_1 = F(\omega_2)F(\omega_1)
\]

This, of course, holds for any number of time variables, not just 2. In other words, mathematically we can consider of FT’s as the product of the Fourier transforms of the separated expressions. We also see that The Fourier transforms can be be performed one by one, and in any order.
A physically observed, time-discreet signal already has the time variables separated out, since each spectrum has a constant \( t_1 \) value and will only have modulations depending on \( t_2 \). Thus one can Fourier Transform the observation time domain \( t_2 \) first, to obtain a series of spectra that are functions of \( \nu_2 \). Every point in those spectra has an intensity that is a function of \( t_1 \), and performing the Fourier transform for the \( t_1 \) series (one for each \( \nu_2 \) point) will give use the frequencies present in the intensity modulation. This may be more easily understood by looking at it pictorially:

\[
\begin{align*}
\text{FT}_2 &: t_2 \rightarrow \nu_2 \\
\text{FT}_1 &: t_1 \rightarrow \nu_1
\end{align*}
\]

(Maple: 2DFT.ms)

2D convolution

The 2D Fourier transform is the product of the result of two 1D Fourier transforms. We know how to do convolution to get our spectrum in 1D so we can use the same approach to get the spectrum of a 2D signal. Let’s look at the same modulation as we had in the Maple example. We will do two 1D convolutions, one for the \( t_1 \) dependent part, the other for the \( t_2 \) part. Focusing only on the positive frequencies:

\[
\begin{align*}
&\cos \pi t_1 \sin \omega_A t_1 \sin \pi t_2 \cos \omega_B t_2 \\
&\sin \omega_A t_1 &\cos \pi t_2 \cos \omega_B t_2 \\
&\cos \pi t_1 &\sin \pi t_2 \\
&(1/2)(1/2)(Ab+iDi) &\phi = \frac{\pi}{2} \rightarrow Ab+iDi/4 \\
&(1/2)(1/2)(Ab+iDi) &\phi = \frac{\pi}{2} \rightarrow Ab+iDi/4
\end{align*}
\]

In most 2D techniques, not all of them, we look at the spectrum the real part of \( \nu_2 \) and the real part of \( \nu_1 \). To construct the 2D spectrum, we multiply the real parts of the \( \nu_2 \) spectrum at the bottom with that of the \( \nu_1 \) spectrum at the right. Note the direction of the axes.
2D line shapes

In schematic drawings for 2D spectra it is customary to use circles for the location of the peaks and a shading depicts where the intensity is positive or negative (just as you did with atomic and molecular orbitals).

The peak combinations for two dimensional peaks can be, either correctly phased (equal phase in both frequency domains) pure absorption (AbAb) or pure dispersion (DiDi)

\[ +\nu_1 -\nu_2 \]

\[ +\nu_1 +\nu_2 \]

incorrectly phased (90° out of phase in the two frequency domains). The latter of the two can be avoided by using the proper phase correction.

Or mixed-phase, or phase-twisted lines. These are usually the result from the combination of the real-real and imaginary-imaginary parts of the spectrum, or in other words the real part of the product of the real and imaginary parts, i.e., \( \Re((\text{Ab} + i\text{Di})(\text{Ab} + i\text{Di})) = \text{AbAb} - \text{DiDi} \).

\[ \text{AbAb} - \text{DiDi} = \text{AbAb-DiDi} \]
Two Dimensions of Complex Data

What part of the spectrum will we really be looking at when we draw our 2D spectrum? Well, a 2D spectrum really consists of 4 data buffers: a real and an imaginary part for each frequency domain. If we have quadrature detection we can set the carrier in the center of the range and look at positive and negative frequencies. If we do not have quadrature detection we can only look at the part of the spectrum that has frequencies higher than the carrier frequency. Let’s put this in our 2D representation of our Fourier transformed data:

A no quadrature detection in $t_1$ or $t_2$. $\mathbb{R}\mathbb{R}$ is $1/16$th of the spectrum. Combination of $\mathbb{R}\mathbb{R}$ and $\mathbb{I}\mathbb{I}$ gives $1/8$th of the spectrum. For $B$, where we have quadrature detection only in $t_2$ that gives for $\mathbb{R}\mathbb{R}$ 1/8th and the combination of $\mathbb{R}\mathbb{R}$ and $\mathbb{I}\mathbb{I}$ gives 1/4th of the spectrum. Finally, in $C$ with quadrature in both $t_1$ and $t_2$ we get 1/4th for $\mathbb{R}\mathbb{R}$ and 1/2 for $\mathbb{R}\mathbb{R}$ and $\mathbb{I}\mathbb{I}$ as the fraction of the signal we really look at.

End of Lecture
**Homo nuclear shift COrrelation SpectroscopY (COSY)**

We saw in the INEPT, refocused INEPT, and DEPT experiments that it is possible to transfer polarization from one spin to another. This was possible if the two spins are coupled to each other ($J \neq 0$). This means that we can, after a $\pi/2$ x pulse, let the coupling and shift evolve for a variable length $t_1$ (not fixed as in the 1D versions) and read out the created polarization transfer (and untransferred ones too) by giving a $\pi/2$ x pulse after which we observe the signal. This leads to the basic sequence for a 2D version of polarization transfer as was suggested by Jeener. This sequence is called COSY:

$$\frac{\pi}{2} x \quad t_1 \quad \frac{\pi}{2} x \quad \text{FID, } t_2$$

*(Maple COSY.ms)*

We can determine the pathway of how the cross and diagonal peak signals come into being by calculating our way through the sequence for only the branch that will give us the signal we are interested in. In order to do that we have to be smart in choosing which of the two branches we need to follow when we do the shift and the coupling evolution.

For the cross peak it is clear that during $t_1$ we need to create a spin operator that will have an $I_{Ay}$ and $I_{Bz}$, so that the x pulse turn that term into a $I_{Az}$ ($z$ for every one) and $I_{By}$ (the only one that we do want the chemical shift modulation for in $t_2$). Notice that if the spin operator after the arrow is different than the one before we have a sine modulation, otherwise a cosine.

$$I_{Az} \xrightarrow{\frac{\pi}{2} x} -I_{Ay} \xrightarrow{\pi t_1 2I_{Az}I_{Bz}} 2I_{Ax}I_{Bz} \xrightarrow{\omega_A t_1 I_{Az}} 2I_{Ay}I_{Bz} \xrightarrow{\frac{\pi}{2} x} -2I_{Az}I_{By} \xrightarrow{\pi t_2 2I_{Az}I_{Bz}} I_{Bx} \xrightarrow{\omega_B t_2 I_{Bz}} I_{By}$$

So we can write for the complex observed signal:

$$e^{i\omega_B t_2} \sin \pi t_2 \sin \omega_A t_1 \sin \pi t_1$$

Note that in the observation time domain in order to obtain observable magnetization (only a single transversal operator), the sine part of the coupling of B with any spin that has a longitudinal spin operator will get rid of that longitudinal operator. Other spins that the transversal spin may be coupled with, but whose longitudinal spin operators are not present, will follow the cosine branch of the coupling so as not to introduce the longitudinal spin operators.
For the diagonal peak we want to have only $I_{Ax}$ at the end of $t_1$ so that after the x pulse we start out with only $I_{Ax}$ so we get chemical shift information of A in $t_2$ too:

$$I_{Az} \xrightarrow{\pi/2_x} I_{Ay} \xrightarrow{\pi J_1 2I_{Ax} I_{Bz}} I_{Ay} \xrightarrow{\omega A_1 I_{Ax}} I_{Az} \xrightarrow{\pi/2_x} I_{Ax}$$

$$I_{Az} \xrightarrow{\pi J_2 2I_{Ax} I_{Bz}} I_{Ax} \xrightarrow{\omega A_2 I_{Bz}} I_{Az} \xrightarrow{\pi/2_x} I_{Ax}$$

and the signal is therefor:

$$e^{i\omega A_1 t_2} \cos \pi J t_1 \sin \omega A t_1 \cos \pi J t_1$$

Thus we immediately see that the phase correction needed to get the diagonal peak in pure absorption will be $(\pi/2,0)$ since we have 1 sine in $t_1$ only. The phase correction needed to get the diagonal peak in pure absorption is: $(0,\pi/2)$ since we have two sines in $t_1$ and one in $t_2$. This means that we will have AbAb lines for one and DiDi lines for the other one, as we saw in Maple. Let’s do the 2D convolution for the cross peak where we use the $(\pi/2,0)$ phase correction to get the diagonal peaks in pure absorption:

$$\Re \left( i \frac{A_i B + i D_i}{2iZ} \right)_1 \Re \left( \frac{A_i B + i D_i}{2i} \right)_2$$

$$= \frac{A_i A_2}{8}$$

$$\Re \left( i \frac{A_i B + i D_i}{2iZ} \right)_1 \Re \left( \frac{A_i B + i D_i}{2i} \right)_2$$

$$= \frac{D_i D_2}{8}$$

We see that we get an anti-phase pattern of pure dispersion lines for our cross peak with intensities of 1/8 if we have the diagonal in pure absorption. If we make the cross peak pure absorption (still anti-phase pattern) the diagonal peak will be pure dispersion (in-phase pattern).

End of Lecture
Hetero nuclear coupling in a homo nuclear COSY

Let’s look at the peak pattern for a cross peak in a ABF COSY where spins A and B are of the same kind, and spin F is a hetero nucleus (e.g. A and B are $^1$H and F = $^{19}$F, e.g. CHF=CHCl). We are only interested in the cross peak between A and B, so we will start with the magnetization for spin A only. Just before the observation starts, we can select only the terms with a transversal spin operator for spin B and no, or only longitudinal spin operators for the other spins that B is coupled to. Then after the coupling evolutions we will only observe the magnetization with only a transversal term for spin B. Let’s set up the tree:
Intermezzo : chemical shift evolution and observation trick

Consider the case of an observation time domain where the couplings have already evolved. We are left with the chemical shift evolutions of only single transversal spin operators, followed by turning the spin state into the observed complex signal. We can write in general

\[ I_x \cos \omega t_{\text{obs}} \rightarrow \cos \omega t_{\text{obs}} + \sin \omega t_{\text{obs}} = e^{i \omega t_{\text{obs}}} \]

\[ I_y \cos \omega t_{\text{obs}} \rightarrow \cos \omega t_{\text{obs}} - \sin \omega t_{\text{obs}} = i e^{i \omega t_{\text{obs}}} \]

Thus we can do the chemical shift evolution and turning our spin state into a signal in one swoop.

Thus the observed signal for the cross peak between A and B is given by

\[ e^{i \omega_B t_2} \cos \pi J_{B,F} t_2 \sin \pi J_{A,F} t_2 \sin \omega_A t_1 \cos \pi J_{A,B} t_1 \sin \pi J_{A,B} t_1 \]

\[ + i e^{i \omega_B t_2} \sin \pi J_{B,F} t_2 \sin \pi J_{A,B} t_2 \cos \omega_A t_1 \sin \pi J_{A,F} t_1 \sin \pi J_{A,B} t_1 \]

Notice that in time domain 1 only the shift of A with the couplings that spin A has are present, and in time domain 2 only shift of B with the couplings that B has. The hetero nuclear coupling gives rise to the sum of two terms for the cross peak where one term has cosine modulation with the coupling while the other has sine modulation. If F was a homo nucleus, we would the coupling with F would have been a passive one and shown only cosine modulations. This again shows that hetero nuclear coupling does something different than homo nuclear coupling.

The full complex observed signal transformed, first the two 1D transforms for each term. Notice that we need a phase correction of \((0, \pi/2)\), but since we have two sines in \(t_1\) the intensity will be negative \((i^2 = -1)\).

\[ \mathcal{R}(i \frac{\Delta b + i \Delta d}{2i \Delta t}) = \frac{\Delta b}{4} \]

\[ \mathcal{R}(\frac{\Delta b + i \Delta d}{2i \Delta t}) = - \frac{\Delta b}{8} \]

\[ \mathcal{R}(i \frac{i(\Delta b + i \Delta d)}{2i \Delta t}) = \frac{\Delta b}{4} \]

\[ \mathcal{R}(\frac{\Delta b + i \Delta d}{2i \Delta t}) = - \frac{\Delta b}{8} \]
Now combining these into the cross peak pattern that each term would give rise to, and adding the two together:

This means that we can determine the relative sign of the coupling of spins A and B with the hetero nucleus F. If the line through the centers of the active coupling patterns lies in the same direction as the diagonal, the couplings have the same sign. If it lies in the opposite direction as the diagonal the couplings have opposite signs. (In our drawing we assumed that both the couplings were positive.)
**HETero nuclear CORrelation spectroscopy (HETCOR)**

We saw before that it is possible to transfer polarization from \(^1\)H (high \(\gamma\) nucleus) to \(^{13}\)C (low \(\gamma\)) to increase sensitivity. Thus polarization transfer makes it possible to have chemical shift evolution for protons in \(t_1\) that will give rise to carbon chemical shift modulation in \(t_2\). The spectrum that we thus observe is a \(^{13}\)C spectrum in \(v_2\) and a \(^1\)H spectrum in \(v_1\). We will get peaks for the carbons that have polarization transfer due to \(^1\)J\(_{\text{HC}}\) coupling (i.e. are connected to protons).

Let’s see how a pulse sequence that accomplishes this should work.

Remember the refocused INEPT sequence:

\[
\begin{array}{cccccccc}
1^H & \frac{\pi}{2} & \frac{1}{4}J & \pi_x & \frac{1}{4}J & \frac{\pi}{2} & \Delta & \delta & 1^H \rightarrow \delta \rightarrow \delta \rightarrow \delta \\
1^{13}C & \pi_x & \pi_x & \frac{\pi}{2} & \pi_x & \Delta & \Delta & BB & FID
\end{array}
\]

create transversal H magnetization

create anti-phase magnetization

turn anti-phase into in-phase to allow decoupling

With the \(\Delta\) value set to obtain enhancement for all kinds of carbons, i.e. \(\Delta = 1/(7J)\).

We still want the same thing, but now we also want our protons to evolve only their chemical shifts (and homo nuclear couplings) during a time period \(t_1\), but have the \(J_{\text{CH}}\) refocused so that we have in phase magnetization just like after the \(\pi/2\) x pulse in the INEPT sequence. To ‘decouple’ the H and C in \(t_1\) we need to give a \(\pi\) pulse on \(^{13}\)C. Thus the pulse sequence would look like:

\[
\begin{array}{cccccccc}
1^H & \frac{\pi}{2} & \frac{1}{2}J & \frac{1}{2}J & \pi_x & \frac{1}{4}J & \frac{\pi}{2} & \Delta & \delta & 1^H \rightarrow \delta \rightarrow \delta \rightarrow \delta \\
1^{13}C & \pi_x & \pi_x & \frac{\pi}{2} & \pi_x & \Delta & \Delta & BB & FID
\end{array}
\]

Let’s see how this newly introduced part of the sequence will affect the results modified ‘initial’ spin state for the refocused INEPT.

\[
I_{Hz} \xrightarrow{\frac{\pi}{2} x} I_{Hy} \xrightarrow{\frac{1}{2} \cdot \pi_{Cx} \cdot \frac{1}{2} x} ?
\]

During the \(t_1\) evolution we can consider the chemical shift to work for the whole period, since the pulse on the C will only affect the hetero nuclear coupling evolution:
so we end up for the transformation during $t_1$: 

$$-I_{Hy} \frac{\pi J_{t_1}}{2} 2I_{Hz} I_Cz \pi_{Cx} \pi J_{t_1} 2I_{Hz} I_Cz \rightarrow -I_{Hy} (\cos^2 \pi J_{t_1} + \sin^2 \pi J_{t_1}$$

$$+ 2I_{Hz} I_Cz \left( \cos \pi J_{t_1} \sin \pi J_{t_1} - \cos \pi J_{t_1} \sin \pi J_{t_1} \right) = -I_{Hy} \rightarrow \omega t_1 I_{Hz} \rightarrow -I_{Hy}$$

and we see the coupling between C and H in $t_1$ is refocused, and we only ‘label’ the spin state with the chemical shift of protons in that time domain. For the remainder of the sequence we get the same results as in the refocused INEPT, which gave the observed signal (Maple worksheet, starting after the $\pi/2$ x pulse on protons):

$$-I_{Hy} \rightarrow \text{refocused INEPT} \rightarrow 2e^{i\omega C t_2} \sin 2\pi J_{t1}$$

where the 2 before the exponential arises from the difference of the two spectra with the +y and -y pulse on H. The observed signal for our whole sequence will be:

$$2e^{i\omega C t_2} \sin 2\pi J_{t1} \cos \omega H t_1$$

Due to the fact that this observed signal started with proton spin operator that the observed carbon magnetization is coupled with, the peaks in the 2D spectrum are only from carbons that have H’s connected to them. Fourier transformation of the signal will give only ‘cross peaks’ between the carbon and the proton(s) that it is connected to. This is extremely useful if we have CH$_2$’s whose H’s are not equivalent.

Note, however, that we will still have the evolution of the homo nuclear H-H coupling in $t_1$, so our $\nu_1$ spectrum will be effectively a $^1H\{^{13}C\}$ spectrum, where we achieved decoupling using an inversion pulse and not a decoupler. Our $\nu_2$ spectrum will be a $^{13}C\{^1H\}$ spectrum.

End of Lecture
Practical Aspects of 2D NMR

T₁ relaxation during t₁

T₁ relaxation during the t₁ time domain will give rise to components of the signal that will have no modulation in t₁. Fourier transformation will turn them into peaks on the ν₁ = 0 axis, the so-called axial peaks. Let’s look at an example for a single spin system using the vector model:

This problem can be corrected with an appropriate phase cycle, in which the non t₁ modulated signal ends up along the -y axis in one spectrum, and +y in the other, by alternating the read out pulse between +x and -x:

Adding these two together shows that only the x term remains, and the non modulated signal (along y) will cancel out.

Quadrature detection in t₁

In order to obtain phase sensitive spectra also in t₁ we need real (cosine modulation of chemical shift) and imaginary (sine modulation of chemical shift) parts of the signal for t₁ too, just like in the observation time domain. There are basically two methods that achieve this. Both of them involve, of course, phase cycles.
RUSH

The RUSH method was devised by Ruben, States and Habekorn. The phase cycle is 16 long. It involves playing around with the phase of the read pulse and being smart about where you put the observed spectra. Keep in mind that we have real and imaginary buffers for both $t_1$ and $t_2$ where we store our observed spectrum...

We’ll look at two of the spectra that will be taken in the sequence of 16 spectra with a particular $t_1$ value. The trick is to increment the read out phase to either pick the sine or cosine modulated signal in $t_1$:

And we see that we now can get the sine and cosine modulated signals in $t_1$ separately and store them in the appropriate real and imaginary $t_1$ buffers: the spectrum obtained after the x pulse goes in the imaginary $t_1$ buffer and the one after the y pulse in the real $t_1$ buffer.

TPPI : Time Proportional Phase Increment

This method was figured out by Redfield. In this method the sample rate in $t_1$ is doubled and the observer phase is cycled by $90^\circ$ intervals. This introduces an observer rotation frequency that is half the Nyquist frequency ($2\,\nu_N$ [from doubling] x 1/4 (phase increment) = $\nu_N/2$) that will offset every frequency in the signal. This can also be understood using the following reasoning.

In order to get a spectrum that has the frequency range of $-\frac{\nu_N}{2} \ldots +\frac{\nu_N}{2}$, one needs to sample with the Nyquist frequency. The sample rate we choose now is $2\,\nu_N$ so that the time per point is $1/(2\,\nu_N)$. Let’s focus on the lowest frequency we are interested in, $-\nu_N/2$. This one will have in the next $t_1$ point an additional cycle of $\Delta\varphi = \frac{1}{2\,\nu_N} \left(-\frac{\nu_N}{2}\right) = -\frac{1}{4}$. Thus in subsequent spectra it will end up at $y$, $-x$, $-y$, $x$, etc. By following the observation with this $-\pi/4$ rotation the signal at $-\nu_N/2$ will seem to have a frequency that is 0. In effect the whole range of frequencies is shifted up by
+\nu_N/2. The TPPI cycling can be pictured as follows, where we focus on the signal with an angular frequency of -\nu_N/2:

Thus by changing the detector phase, the signal at -\nu_N/2 will seem to have a frequency of 0 Hz.

**Digital Resolution and experiment time**

Let’s consider the homo nuclear COSY for protons on a 300 MHz machine, where we want a range of 10 ppm in both phase sensitive frequency domains. The frequency range is 3000 Hz. Our narrowest line is about 0.5 - 1.5 Hz wide, so we want a resolution that is better than 0.5 Hz/pt. This means that we need 3000 Hz/(< 0.5 Hz/pt) = >6000 pt’s, or 8K (=8192) data points, in both dimensions. How long would it take to get that 2D spectrum (where we will even ignore the recycle delay needed to get the equilibrium state for the next spectrum).

Each 1D spectrum will take 8192 x 1/(3000 Hz) = 2.731 sec. We will need 8K of those spectra, and we have a phase cycle of 16, so a rough estimate of the total time is 16 x 8192 x 2.731 sec = 357 913.9413 sec = 99 hr 25 min. The more precise equation is:

\[ t_{exp} = nc \sum_{n_1=1}^{N_1} \left( \frac{n_1}{v_{N_1}} + \sum_{n_2=1}^{N_2} \frac{1}{v_{N_2}} \right) = nc \sum_{n_1=1}^{N_1} \left( \frac{n_1}{v_{N_1}} + \frac{N_2}{v_{N_2}} \right) = nc \left( \frac{N_1(N_1+1)}{2v_{N_1}} + \frac{N_1N_2}{v_{N_2}} \right) \]

\[ = nc \left( \frac{N_1+1}{2v_{N_1}} + \frac{N_2}{v_{N_2}} \right) \]
Where \( n_c \) is the length of the whole phase cycle and \( \nu_{N_1} \) and \( \nu_{N_2} \) are the Nyquist frequencies in time domains \( t_1 \) and \( t_2 \) respectively. The first term between the brackets was the part that we ignored in the rough estimate.

In our case this yields
\[
16 \times 8192 \left( \frac{8193}{6000} + \frac{8192}{3000} \right) = 536,892 \text{ sec} = 149 \text{ hr} \ 8 \text{ min} > 6 \text{ full days} !!!
\]

It is customary to use, for small molecules, data buffers that are 512 x 512 (resolution of 6Hz) and cut down on the phase cycle by doing only 1/4 of the cycle, our total time would become
\[
4 \times 512 \left( \frac{513}{6000} + \frac{512}{3000} \right) = 524.6 \text{ sec} < 9 \text{ min}.
\]
For larger molecules one often uses size like 1K x 2K (requires about 4 hr. with the full phase cycle).

Notice that the function is approximately proportional to \( 1/2 \times N_1^2 \) and \( N_1N_2 \) thus it is smart to have the lower resolution (lower N) in the first time domain (not the second)!

Also remember that when your peaks lie in a smaller region, the Nyquist frequency can be reduced. This would initially seem to give larger values for the duration of the experiment, but keep in mind that the resolution will be higher if you don’t also reduce the number of data points.

### Additional tricks

**MC calculation**

You will have to do phase correction in \( t_1 \). This can be very tedious. The phase correction can be avoided by calculating the magnitude spectrum (MC, also called power spectrum).

This in effect will determine the **length** of the complex vectors and you loose (i.e. do no need) the phase information. It is easily achieved by adding the squares of the real parts and imaginary parts of the complex numbers (buffers) and taking the square root of them:

\[
L = \sqrt{A^2 + B^2}
\]

The result of a magnitude spectrum is that the lines are **broader** and you **loose information about peaks** being up or down (i.e. phases). We saw this in the COSY Maple worksheet where we proposed this as a solution to the presence of pure absorption and pure dispersion line shapes for cross and diagonal peaks.
SY symmetrization

Since the coupling is symmetrical, \( J_{A,B} = J_{B,A} \) the COSY spectra are supposed to be symmetrical. This means that if there are additional problems that arise in the \( t_1 \) dimension, e.g. \( t_1 \) noise whose source is not completely understood, but the book ascribes these in part to non constant \( t_1 \) increment times (timing problems). You can get rid of these by doing a symmetrization. There are different kinds of symmetrization functions one can use, but the most used one is to take the lowest of the two absolute values of data points that are mirrored in the diagonal:

\[
(a, b) = \left(b, a \right) \leftarrow \text{symmetrization} \min \left( \text{abs}(a, b), \text{abs}(b, a) \right)
\]

The other advantage of doing symmetrization is that the resolution obtained in \( \nu_2 \) (usually higher than that in \( \nu_1 \)) is transferred to \( \nu_1 \), viz.:

To perform a symmetrization, equal sizes of the data buffers for both frequency domains is required. If your acquisition data buffers are not equal in size, the smallest of the two (usually \( t_1 \)) will have to be zero-filled before the Fourier transform.

End of Lecture
Maple Worksheets

Chemistry 558

Organic Spectroscopy

Part II: FT-NMR

by

René P. F. Kanters

1994
We are going to see what angles of the spin momentum vectors can have with the external field. We will use $S$ in stead of $I$, since Maple considers $I$ to be $\sqrt{-1}$.

We saw that the projection on the field has length of $m[S]$ and the length of the vector itself is $\sqrt{S(S+1)}$ both in multiples of $h/(2\pi)$.

\[
> \mu_z := m[S] \frac{h}{2\pi};
\]

\[
> \mu_t := \sqrt{S(S+1)} \frac{h}{2\pi};
\]

\[
> \cos(\theta_s) = \frac{\mu_z}{\mu_t};
\]

\[
> \text{solve(", theta[s]);}
\]

Thus the angle with the field vector is $\arccos(m[S]/\sqrt{S(S+1)})$

We want to have a little procedure that will calculate this value for us in degrees, so we write

\[
> \text{angle:=(m,S) \to evalf(arccos(m/(sqrt(S*(S+1))))*180/Pi)};
\]

We want to look at the cones that have these lengths for their projection on the z-axis. To do this we first define a procedure that will draw a cone that has the angle theta with the z-axis, using the parametric notation in cylindrical coordinates.
> cone:=proc(theta)
    cylinderplot([r*sin(theta),phi,r*cos(theta)], r=0..1, phi=0..2*Pi);
end:

The whole set can be generated by displaying the cones for all of the angles that I can have with the z-axis. We will also have an argument to this procedure that will set the title of the plot.

> cones:=proc(S,t) local m;
    display3d({seq(cone(arccos(m/sqrt(S*(S+1)))),m=-S..S)},axes=BOXED,
               orientation=[30,80],title=t);
end:

Now we can plot these cones, e.g. for a $I=5/2$ case

> cones(5/2,`I=5/2`);

$>5/2$

$>$ cones(3/2,`I=3/2`);
A worksheet to look at Fourier transformations.

We first need to read the library procedure that can do the Fourier transform algebraically.

> readlib(fourier):
> help(fourier);

FUNCTION: fourier - Fourier transform

CALLING SEQUENCE:
  fourier(expr, t, w)

PARAMETERS:
  expr - expression to be transformed
  t    - variable expr is transformed with respect to
  w    - parameter of transform

SYNOPSIS:
- The fourier function applies the Fourier transformation to expr with respect
to t, using the definition

\[
F(w) = \lim_{\omega \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]

- Expressions involving complex exponentials, polynomials, trigonometrics
  (sin, cos) with linear arguments, and a variety of functions and other
  integral transforms can be transformed. References to linearity are with
  respect to the integration variable t.

- The fourier function recognizes derivatives (diff or Diff) and integrals (int
  or Int).

- Users can define fourier transforms for their own functions. For example, if
  FunctionName(arg1,...,argn) is a function with n arguments, then defining
  'fourier/FunctionName' := proc(t,w,arg1,arg2,...,argn) ... end; will add
  FunctionName to the list of functions for which fourier will give an answer.

- fourier recognizes the Dirac-delta (or unit-impulse) function as Dirac(t) and
  Heaviside's unit step function as Heaviside(t).

- This function should be defined by the command readlib(fourier) before it is
  used.

Let's do a fourier transform of a complex modulation

> fourier(exp(I*omega[1]*t),t,omega);

\[
2\pi \text{Dirac} \left( \omega - \omega_1 \right)
\]

We see that we obtain the Dirac function. This function is 1 when the argument is 0.
Thus we have a spike at omega=omega1 in the real buffer.

This is equivalent to doing it for the following function (rewriting the exponential notation in the
\cos + I \sin form):

> expand(fourier(cos(omega[1]*t)+I*sin(omega[1]*t),t,omega));
Let's see what would have happened if we added a phase shift of \( \frac{\pi}{2} \):

\[
> \text{fourier}(\exp(i*(\omega[1]*t+\phi)),t,\omega);
\]

\[
2 e^{(I \phi) \pi \text{Dirac}(\omega - \omega_1)}
\]

Thus the constant phase shift itself can be taken out of the integration and ends up as a complex exponential in front of our previous result, we will evaluate it is a complex number:

\[
> \text{evalc}(*);
\]

\[
2 \cos(\phi) \pi \text{Dirac}(\omega - \omega_1) + 2 I \sin(\phi) \pi \text{Dirac}(\omega - \omega_1)
\]

We see that we really have peaks in both the real and/or the imaginary part of the buffer, depending on the initial phase of the complex modulation.

For \( \phi = \frac{\pi}{2} \), e.g., we have only a peak in the complex buffer at \( \omega_1 \).

By adjusting the phase of the modulation in the complex and real buffers, we can play around with getting the peak either in the real, or the imaginary part.

The fourier transform on non-complex modulation will give two peaks, either in the imaginary buffer (for the sin modulation) or the real buffer (for the cosine), but with peaks as positive and negative omega's

\[
> \text{expand(fourier(cos(omega[1]*t),t,omega))};
\]

\[
\pi \text{Dirac}(\omega - \omega_1) + \pi \text{Dirac}(\omega + \omega_1)
\]

\[
> \text{expand(fourier(sin(omega[1]*t),t,omega))};
\]

\[
-I \pi \text{Dirac}(\omega - \omega_1) + I \pi \text{Dirac}(\omega + \omega_1)
\]

we should have expected that since \( \sin(x) = -\sin(-x) \) thus a sine can be considered to be built up of two sines with opposite frequencies and intensities. Analogously \( \cos(x) = \cos(-x) \) so we can not determine whether the angle is positive or negative in either case.
We are going to do a fourier transform of a pulse, in order to determine the intensities of the frequencies that are comprised in one.

>`readlib(fourier)`:  

We first need to define the procedure that will give us a step function (a pulse with a carrier frequency of 0Hz). To do that we need to use the mathematical function that is called the Heaviside function is defined as follows:

>`help(Heaviside)`:  

FUNCTION: Dirac - the Dirac delta function  
FUNCTION: Heaviside - the Heaviside step function  

CALLING SEQUENCE:  
- Dirac(t)  
- Dirac(n, t)  
- Heaviside(t)  

PARAMETERS:  
- t - algebraic expression  
- n - nonnegative integer  

SYNOPSIS:  
- The Dirac(t) delta function is defined as zero everywhere except at t = 0 where it has a singularity. It has an additional property, specifically:  
  \[ \text{Int}(\text{Dirac}(t), t = -\infty..\infty) = 1 \]  
- Derivatives of the Dirac function are denoted by the two-argument Dirac function. The first argument denotes the order of the derivative. For example, \( \text{diff}(\text{Dirac}(t), t^n) \) will be automatically simplified to \( \text{Dirac}(n, t) \) for any integer \( n \).  
- The Heaviside(t) unit step function is defined as zero for \( t < 0 \), 1 for \( t \geq 0 \). It is related to the Dirac function by \( \text{diff}(\text{Heaviside}(t), t) = \text{Dirac}(t) \).  
- These functions are typically used in the context of integral transforms such as \text{laplace()}, \text{mellin()}, or \text{fourier()}.  

SEE ALSO: laplace, mellin, fourier  

>`step:=(t,l)->Heaviside(t+l/2)-Heaviside(t-l/2)`:  

Let's check whether we did this right  

>`plot(step(t,3),t=-2..2,title=`step(t,3)`);`  

![Plot of step function](image)  

Looks good: let's Fourier transfor this step function:
let's simplify this expression to get something nicer:

> F:=simplify(");

$$F := 2 \sin\left(\frac{1}{2} \omega \tau \right)$$

We see that the function will turn 0 whenever \(\omega \tau / 2 = n \cdot \pi\), or in other words, with \(\omega = 2 \cdot \pi \cdot \nu\), we have \(\pi \nu \tau = n \cdot \pi\), or \(\nu = n / \tau\):

> solve(subs(omega=2*Pi*nu,omega*tau/2)=n*Pi,{nu});

$$\{ \nu = \frac{n}{\tau} \}$$

We will plot the frequency distribution comprised in a pulse for two values of the pulse length (\(\tau\)) as a function of the radial frequency, \(\omega\):

> plot({subs(tau=3,F),subs(tau=6,F)},omega=-10..10,title=`FT(step(t,3)) and FT(step(t,6))`);
Now let's see what happens if we have a carrier with angular frequency \( w \). In order to do this, I will have to use a complex exponential so that we have the direction of rotation defined

\[
\text{simplify(fourier(step(t,\tau) \cdot \exp(i \cdot w \cdot t), t, \omega))};
\]

\[
\frac{\sin \left( \frac{1}{2} (-\omega + w) \tau \right)}{2 \cdot (-\omega + w)}
\]

This is the same function we had before but now with an offset of our carrier frequency. We can see this by using that \( \delta \omega = \omega - w \), or \( \omega = \delta \omega + w \) and simplifying this

\[
\text{simplify(subs(omega=w+delta[omega],"))};
\]

\[
\frac{\sin \left( \frac{1}{2} \delta \omega \tau \right)}{2 \cdot \delta \omega}
\]

Thus we have our initial frequency distribution but now around the carrier frequency.
> read `pof.m`;

The pof.m file has definitions for the use of the product operator formalism. It also contains the Lorentzian lineshape functions, so we can look at what these line shapes look like.

> ?pA;

FUNCTIONS: pA, pD - absorbtion and dispersion lineshape functions

CALLING SEQUENCE: pA(v0,v,T2); pD(v0,v,T2);

PARAMETERS: v0 - peak position in Hz 
v - frequency domain 
T2 - relaxation time in seconds

SYNOPSIS:
Returns the algebraic expression for an absorption (pA) and dispersion (pD) peak centered at position v0 Hz, in frequency domain v, with a relaxation time of T2 seconds.

EXAMPLES:
> evalf(pA(0,crit_A/T2,T2)/pA(0,0,T2));
0.2470452302
> evalf(pD(0,crit_D/T2,T2)/pD(0,1/(2*Pi*T2),T2));
0.03182292777
> map(expand,{solve(diff(pD(v0,v,T2),v)=0,v)}); # dispersion extremes
{v0 - 1, v0 + 1}

2 Pi T2   2 Pi T2
> plot({pA(v,0,2.0),pD(v,0,2.0)},v=-1..1);

SEE ALSO: crit_A, crit_D

Let's plot both the absorption and dispersion lineshape of a spins that has a T2 of 1s:
> plot({pA(0,nu,1),pD(0,nu,1)},nu=-5..5,title=`Lorentzians T2=1s`);
Let's check what the linewidth of an absorption signal is, by solving for the frequency where the intensity of the signal is half of that at the center of the peak (0), and taking the difference between the two solutions:

\[
> \text{hh:=}\{\text{solve}(\text{pA}(v0,v,T2) = \text{pA}(v0,v0,T2)/2,v)\};
\]

\[
\text{hh} := \left\{ \frac{1}{8} \frac{8 \pi^2 T2^2 v0 - 4 \pi T2}{\pi^2 T2^2}, \frac{1}{8} \frac{8 \pi^2 T2^2 v0 + 4 \pi T2}{\pi^2 T2^2} \right\}
\]

\[
> \text{op}(2,\text{hh})-\text{op}(1,\text{hh});
\]

\[
\frac{1}{8} \frac{8 \pi^2 T2^2 v0 + 4 \pi T2}{\pi^2 T2^2} - \frac{1}{8} \frac{8 \pi^2 T2^2 v0 - 4 \pi T2}{\pi^2 T2^2}
\]

\[
> \text{simplify}(\text{"});
\]

\[
\frac{1}{\pi T2}
\]

We can also use the pof procedures to calculate what happens if we have a two spin system, and observe the signal after we give a 90 degrees pulse:

In the procedures define in pof.m we also have the coupling between spins.

We define our spinsystem to consist of two spins:

\[
> s:=\text{spinsystem([A,B])};
\]

\[
s := I_{z A} + I_{z B}
\]

At this point in time we have not talked about coupling, so we should set the J[A,B] to 0:

\[
> J[A,B]:=0;
\]

we are going to give a π/2 pulse along the x (notice that the rotation directions in pof are defined to be just the other way around (negative gyro-magnetic ratio, or positive angular velocities)).

\[
> \text{xpulse}(s,\{A,B\},\text{Pi}/2);
\]

\[
-Iy_A - Iy_B
\]

\[
> \text{obs:=observe(\text{"},\{\},t,0)};
\]

\[
os := -I \cos\left( 2 \pi W_A t \right) + \sin\left( 2 \pi W_A t \right) - I \cos\left( 2 \pi W_B t \right) + \sin\left( 2 \pi W_B t \right)
\]

Let's try to plot the real part of the FID that we will observe, if both spins have a T2 of 1/2 s

\[
> \text{FID:=subs(}\{\text{Sin=\text{sin},W[A]=8, W[B]=21}\}, \text{evalc(Re(\text{obs}))})*\exp(-t/(1/2));
\]

\[
FID := (\sin(16 \pi t) + \sin(42 \pi t)) e^{(-2 t)}
\]

\[
> \text{plot}(\text{FID},t=0..3,\text{numpoints}=300);
\]
We see that we have observed sin - I cos signals, so we should do a phase correction (we didn't look along the right direction), before we do the Fourier transform.

> spc:=FT(obs,Pi/2,t,nu);

we have absorption signal in the real part and dispersion signal in the imaginary part

> real:=evalc(Re(spc)); imag:=evalc(Im(spc));

> T2:=table([[(A)=1,(B)=1]]); Wv:=table([[(A)=3,(B)=-5]]);
> nmrplot({real,imag},nu=-10..10);
let's make the relaxation time of spin B really fast

> T2[B]:=0.1;
> nmrplot({real,imag},nu=-10..10);

We see that the signal for B (at -5 Hz) becomes very broad and is not very intense anymore.
We will look at digitally acquired data from modulation signals (time discrete functions)

> with(plots):

Let's define a procedure that will sample a function at specific intervals, given by dt, n times. It will assign the plot structure for the sampled points to s, and the function itself to l.

First we write a little function that will give the three sets of x,y values needed for the linedrawing of a spike, so we can use that in our sample function.

> spike:=(a,b) -> (a,0,a,b,a,0):
> sample:=proc(f,dt,n,s,l) local i;
> s:=plot([seq(spike(i*dt,evalhf(f(i*dt))),i=0..(n-1))],x=0..(n-1)*dt);
> l:=plot(f(x),x=0..(n-1)*dt);
> end:

Define a function that we are going to play with: a cosine with a frequence of 30 Hz:

> f:=(t) -> cos(30*2*Pi*t):

We'll sample that with intervals of 1/90 sec, for a total of 16 points. We should have 3 data points before a cycle is complete.

> sample(f,1/90,16,s,l): display({s,l});

Lo and behold we see that in each cycle we get 3 data points.

Now, let's see what happens if we sample it at a rate such that we have less than two points per cycle. We would have exactly 2 points per cycle if our increment time were 1/60 sec, so we should make the time interval longer than that, e.g. 1/32 sec.

> sample(f,1/32,20,s,l): display(s);
this doesn't look like a 30 Hz cosine at all. It looks more like a cosine with a cycle completed in
0.5 sec, i.e. 2 Hz.

From this spectrum we can't see whether it will seem to be +2 or -2 Hz, so we need to look at
what the sine would have given us:

> sample((t)->sin(30*2*Pi*t),1/32,20,s,l): display(s);

This looks like a - sin of 2Hz.

Thus if we had a signal cos(30Hz) + I sin(30Hz) sampled at a rate of 32 Hz, we would see
the same spectrum as if we had sample the function cos(2Hz) - I sin(2Hz). The Nyquist frequency
for 32Hz sample rate is half of that rate, i.e. 16 Hz, and the range of frequencies we can measure
goes from -16Hz to 16Hz. So it seems as if the 30Hz rotation shifts back in from the negative
side.
The result of Fourier transforming a time discrete signal (with or without decay of intensity)

> readlib(FFT);

FFT will result in an array where for the first n/2 data points the positive frequencies are, and the the last n/2 points have negative frequencies (both running up in value). Thus we need to turn those into some kind of a spectrum where the data run from -f .. f and where we also know the frequencies at each point.

> spec:=proc(x,dx,n) local n2,i;
    n2:=n/2;
    [seq(((i-1)*dx,x[i]),i=1..n2),seq(((i-1-n)*dx,x[i]),i=n2+1..n)];
end:

We will define a procedure that will sample both an fx(t) and fy(t) function for 2^N points with an acquisition time increment of dt. The returned value is the list of both the real and imaginary spectra respectively.

> sampleFT:=proc(fx,fy,dt,N) local i,dnu,NN,re,im;
    NN:=2^N;
    re:=array(1..NN); im:=array(1..NN);
    for i from 1 to NN do re[i]:=fx(dt*(i-1)); im[i]:=fy(dt*(i-1)); od:
    evalhf(FFT(N,var(re),var(im)));
    dnu:=1/((NN-1)*dt);
    [spec(re,dnu,NN),spec(im,dnu,NN)];
end:

Well work with a single frequency of +30 Hz:

> fx:=(t)->evalhf(cos(30*2*Pi*t)): fy:=(t)->evalhf(sin(30*2*Pi*t)):
> s1:=sampleFT(fx,fy,1/80,8):
> plot(s1[1]);

So we see that we get a peak at 30 Hz.
Let's see what an exponential decay function (no modulations) will give when we Fourier transform that.

```plaintext
> s2:=sampleFT((t)->evalhf(exp(-t)),(t)->0,1/5,8):
> plot(s2[1],title=`Re part of decaying signal`);
```

Let's look at the imaginary part of the spectrum:

```plaintext
> plot(s2[2],title=`Im part of decaying signal`);
```
Now let's do the example we looked at before: a 30 Hz signal sampled with a sample rate of 32 Hz, giving a Nyquist frequency of 16 Hz. First let's assume that we do not have quadrature detection, so or imaginary values are 0:

\[ s_3 := \text{sampleFT}(f_x, (t) \rightarrow 0, 1/32, 6) : \text{plot}(s_3[1]); \]

As we had noticed before, we can not tell whether the peak should be at + or - 2Hz. The quadrature detection allows us to figure that out, since for that we also have the sine modulation in the real part. Let's first look at the latter only:

\[ s_4 := \text{sampleFT}((t) \rightarrow 0, f_y, 1/32, 6) : \text{plot}(s_4[1]); \]
So if we have both of them together, the peak at +2 Hz would exactly cancel out:

> plot({s3[1],s4[1]});

Let's do the quadrature detected signal:

> s5:=sampleFT(fx,fy,1/32,6):plot(s5[1]);

Indeed what we expected.
We will look into the line sharpening using Gaussian functions.

We first need to read the library functions to be able to do fast Fourier transforms

```maple
> readlib(FFT);
```

Define a procedure that will make the lists returned by FFT go up in frequencies from -Nyquist to + Nyquist.

```maple
> spec:=proc(x,dx,n) local n2,i;
>   n2:=n/2;
>   [seq([[i-1-n]*dx,x[i]],i=n2+1..n),seq([[i-1]*dx,x[i]],i=1..n2)];
> end:
```

Define a procedure `sampleFT`, that allows us to sample a function \( f(x) \) at time intervals \( dt \) for \( 2^n \) datapoints. The imaginary part is considered to remain 0 (i.e. observe a signal that is in resonance with the rotating frame of reference)

```maple
> sampleFT:=proc(f,dt,N) local i,dnu,NN,im,re;
>   NN:=2^N;
>   re:=array(1..NN): im := array(1..NN):
>   for i from 1 to NN do re[i]:=f(dt*(i-1)); im[i]:=0; od;
>   evalhf(FFT(N,var(re),var(im)));
>   dnu:=1/((NN-1)*dt);
>   [spec(re,dnu,NN),spec(im,dnu,NN)];
> end:
```

First we will define our exponential decaying functions both with a constant of 1 before \( t \)’s

```maple
> f:=(t) -> evalhf(exp(-t)):g:=(t)->evalhf(exp(-t^2)):
```

Let’s see what these two functions look like

```maple
> plot({'f(t)','g(t)'},t=0..5,title=`Lorentzian and Gaussian Decays`);
```

It is obvious that with the same coefficient the \( \exp(-at^2) \) decays more rapidly than the \( \exp(-at) \).

Now let’s do the sampling and FFT’s of both decaying functions:

```maple
> ff:=sampleFT(f,1/5,6):gg:=sampleFT(g,1/5,6):
```

```maple
> with(plots):```
Let's plot the Lorentzian first, followed by the Gaussian:

```plaintext
> plot(ff[1], x = -2..2, y=0..6, title=`Lorentzian absorption`);
```

![Lorentzian absorption graph]

```plaintext
> plot(gg[1], x = -2..2, y=0..6, title=`Gaussian absorption`);
```

![Gaussian absorption graph]
Let's put them together so we can compare them better

```plaintext
> plot({ff[1],gg[1]},x=-2..2,title=`Lorentzian and Gaussian absorptions`);
```

The linewidth of the gaussian is larger, but the width at the base is smaller. You can now 'set' the linewidth to what you want it to be, since you put the gaussian line broadening in after you 'undo' the T2 relaxation itself. This means that you should set the factor $a$ in the Gaussian decay to a number smaller than $1/T2^*$. The imaginary parts of the buffers look like this:

```plaintext
> plot({ff[2],gg[2]},x=-2..2, title=`Lorentzian and Gaussian dispersions`);
```

```plaintext
```

```plaintext
```

```plaintext
```
Let's solve the Bloch equation for reestablishing the longitudinal magnetization. This differential equation is given by:

\[
deq := \frac{\partial}{\partial t} M_z(t) = \frac{M_0 - M_z(t)}{T1}
\]

Since the Pi pulse has inverted the magnetization, we know that \(M(z)(0) = -M[0]\), so use that as a boundary condition in solving the differential equation:

\[
dsolve\{deq,M[z](0)=-M[0]\},M[z](t)
\]

To assign the result to an expression we do

\[
M(t) := \text{collect(op(2,"),M[0])}
\]

in order to plot this, we need to substitute the values for T1, and M0

\[
\text{plot}\{\text{subs}\{\{T1=1,M[0]=1/2\},M[z](t)\},\text{subs}\{\{T1=3,M[0]=1\},M[z](t)\},t=0..10\};
\]

Now, let's see where the general function goes through zero:

\[
T1 = \text{solve}(M(t)=0,T1)
\]

\[
T1 = \frac{t}{\ln(2)}
\]
In this worksheet we will look at the effect of a spin echo sequence on homo and hetero nuclear couplings.

We first read the product operator transformation rules:

```plaintext
> read `pof.m`;
```

Now we are going to perform the pulse program (Pi/2)x - tau - (Pi)x - tau and see what we end up with at that point in time.

Before we go on, let's define a procedure that will generate the spinecho itself, by giving the correct pulses on the set of spins in P:

```plaintext
> spinecho:=proc(s,P,tau)
  evolve(s,{},tau): xpulse("P,Pi):evolve("{},tau);
end:
```

Let's do the heteronuclear case of two spins, A and X. Since X is a heteronucleus, all the pulses will only work on spin A. Start with our spinsystem definition

```plaintext
> spinsystem([A,X]);
```

```
Iz A Iz X
```

give our first x-pulse on A:

```plaintext
> xpulse("{A},Pi/2);
```

```
-Iy A + Iz X
```

do the echo part only for spin A only:

```plaintext
> spinecho("{A},tau);
```

```
Iy A + Iz X
```

A first observation is that the magnetization is not along the -y axis, as it was in the vector model. This is because in the product operator formalism the positive rotations are counter clockwise, not clock wise. But as long as we are consistent, it should work out OK. What is more important, though, is to note that there is no dependency on either chemical shift, nor any coupling constant.

Thus both the chemical shift and heteronuclear coupling refocus, in a spin-echo experiment.

Now let's do the homonuclear case:

```plaintext
> spinsystem([A,B]): xpulse("{A,B},Pi/2);
```

```
-Iy A - Iy B
```

now the spin echo pulse works on both spins, A and B:

```plaintext
> spinecho("{A,B},tau);
```

```
Iy A Cos(1); 2 Ix A Iz B Sin(1) + Iy B Cos(1) - 2 Ix B Iz A Sin(1)
%1 := 2 \pi J A, B \tau
```
We see that the result is more complex, but it is clear that the magnetization is dependent on the size of the $J_{A,B}$ coupling: the homonuclear coupling is not refocused. There is no dependency on chemical shifts, so the chemical shift still refocusses.
APT spectra
Copyright (1994) Rene P.F. Kanters
KANTERS@urvax.urich.edu

> read `pof.m`:

The pulse sequence for an APT spectrum is:

13C channel :  (Pi/2)x - tau {H} - (Pi)x - tau - observe {H}

where {H} stands for broad ban decoupling on proton during that particular time domain.

We will define the procedure that will do an APT experiment for us. Note that in order for this to work we need to make sure that all the couplings between the C and H spins are equal, thus this simulation will only work for one CHn set of spins.

> APT:=proc(ALL,H,C,tau,t) local i,j,s; global J;
  s:=spinsystem(ALL);
  for i in H do for j in C do J[i,j]:=J; od; od;
  s:=xpulse(s,C,Pi/2);
  decouple(H): s:=shift(s,{},tau); decouple({}):
  s:=xpulse(s,C,Pi);
  s:=evolve(s,{},tau);
  decouple(H); s:=observe(s,{},t,-Pi/2); decouple({}):
  s;
end:

Also we will define the procedure used for obtaining the real part of the Fourier transformed spectrum

> spectrum:=(s) -> evalc(Re(FT(s,0,t,nu))):

Let's calculate some APT spectra so that we can see how the appearance of the spectrum depends on relationships between timing in the pulse sequence and physical constants of our spin system (tau and J). We will do this for a quaternary (CH0), tertiary (CH1, methine), secondary (CH2, methylene), and a primary (CH3, methyl) system:

> CH0:=APT([C0],{H},{C0},tau,t);

\[
CH0 := -\cos\left(2 \pi W_{C0} t\right) - I \sin\left(2 \pi W_{C0} t\right)
\]

> CH1:=APT([H,C1],{H},{C1},tau,t);

\[
CH1 := -\cos(\pi J \tau) \cos\left(2 \pi W_{C1} t\right) - I \cos(\pi J \tau) \sin\left(2 \pi W_{C1} t\right)
\]

> CH2:=APT([[H,Hb],C2],{H},{C2},tau,t);

\[
CH2 := -\frac{1}{2} \cos\left(2 \pi W_{C2} t\right) - \frac{1}{2} I \sin\left(2 \pi W_{C2} t\right) - \frac{1}{2} \cos(2 \pi J \tau) \cos\left(2 \pi W_{C2} t\right)
- \frac{1}{2} I \cos(2 \pi J \tau) \sin\left(2 \pi W_{C2} t\right)
\]

> CH3:=APT([[H,Hb,Hz],C3],{H},{C3},tau,t);
\[ \text{CH3} := -\frac{3}{4} \cos(\pi J \tau) \cos(2 \pi W_{C3} t) - \frac{3}{4} i \cos(\pi J \tau) \sin(2 \pi W_{C3} t) \]
\[ -\frac{1}{4} \cos(3 \pi J \tau) \cos(2 \pi W_{C3} t) - \frac{1}{4} i \cos(3 \pi J \tau) \sin(2 \pi W_{C3} t) \]

Notice that quaternary C’s have no signal dependency on J, but CH and CH3 have dependencies of \(\cos((2n+1) \pi J \tau)\), and CH2 has dependency of \(\cos(2 \pi J \tau)\). Thus for \(\tau = 1/(2J)\) all these \(\cos\) will evaluate to \(\cos((2n+1)\pi/2) = 0\), so only a signal for quaternary C’s is observed. When \(\tau = 1/J\), the CH and CH3 will have \(\cos((2n+1) \pi) = -1\) as multipliers thus give negative signals. The CH2 in that case will have \(\cos(2\pi) = 1\) as multiplier and has positive signals (just as CH0).

We will look at that by doing the the Fourier transforms and performing the substitutions for the particular tau lengths we’re interested in

\[ > \text{print(seq(spectrum(nmrsubs(tau=1/(2*J),CH.i)),i=0..3))}; \]
\[ -\text{Ab}(W_{C0}^i)0,0,0 \]
\[ > \text{print(seq(spectrum(nmrsubs(tau=1/J,CH.i)),i=0..3))}; \]
\[ -\text{Ab}(W_{C0}^i)\text{Ab}(W_{C1}^i)\text{Ab}(W_{C2}^i)\text{Ab}(W_{C3}^i) \]

So we see that indeed in an APT spectrum where \(\tau\) is set to \(1/J\) we get positive peaks for carbons that have an even number of protons attached, and negative peaks for carbons with an odd number of protons.

Let’s see the general expressions for the FT’d spectra we would expect:

\[ > \text{spectra}:=\{\text{seq(spectrum(CH.i),i=0..3)}\}; \]
\[ \text{spectra} := \left[ -\text{Ab}(W_{C0}^i), -\cos(\pi J \tau) \text{Ab}(W_{C1}^i), \text{Ab}(W_{C2}^i) - \frac{1}{2} \text{Ab}(W_{C2}^i), \text{Ab}(W_{C3}^i) \right] \]

We see from the modulations how the signals will have intensities that depend on \(J^*\tau\).

Let’s make that the absorptions signal is just going to give us an intensity of 1

\[ > \text{Ab}:=(\cdot) \rightarrow 1; \]

Our carbons will have the following intensities depending on \(J^*\tau\)

\[ > \text{spectra}; \]
\[ \left[ -1, -\cos(\pi J \tau), -\frac{1}{2} - \frac{1}{2} \cos(2 \pi J \tau), -\frac{3}{4} \cos(\pi J \tau) - \frac{1}{4} \cos(3 \pi J \tau) \right] \]

Let’s plot the modulation intensities of the observed signals depending on \(\tau^*J\), i.e. \(\tau = JT/J\) (or in other words JT is # 1/J’s that tau is long), which we will turn into one variable: JT.

\[ > \text{plot(convert(subs(Cos=cos,tau=JT/J,spectra),set),JT=0..4)}; \]
The color orders are black, yellow, red, green for (C,CH2,CH3).

We see that with \( \tau = 0.5/J \) (\( JT = 0.5 \)) we have only 1 negative signal, the other three are zero, and with \( \tau = 1/J \) (\( JT = 1 \)) we get negative peaks for C,CH, and positive ones for CH, and CH3.
We will calculate the population difference between the two states present when we have \(N\) hydrogen atoms (\(I=1/2\)) at 300K in a field of 7Tesla (proton frequency is 298 MHz).

We have two equations that need to be solved simultaneously:

1) Boltzman equation for the ratio of the populations
2) the sum of the populations is \(N\)

\[
\begin{align*}
eq 1: & \quad \frac{N_b}{N_a} = e^{-\frac{h \nu}{kT}} \\
eq 2: & \quad N = N_a + N_b
\end{align*}
\]

We can assign the results to the variables

\[
\begin{align*}
N_a &= \frac{N}{1 + e^{-\frac{h \nu}{kT}}} \\
N_b &= \frac{e^{-\frac{h \nu}{kT}} N}{1 + e^{-\frac{h \nu}{kT}}}
\end{align*}
\]

and calculate the fraction of the total number of atoms that the groundstate has more than the excited state has:

\[
\begin{align*}
dN &= \text{simplify}(N_a - N_b) \\
&= -N \frac{1 + e^{-\frac{h \nu}{kT}}}{1 + e^{-\frac{h \nu}{kT}}}
\end{align*}
\]

by filling out the numbers for the variables we get for \(dN\) in a 298 MHz machine:

\[
\begin{align*}
\text{subs}\left(\{h=6.626*10^{-34}*J*s, k=1.381*10^{-23}*J/K,T=300*K, \nu=298*10^6/s\}, dN\right);
\end{align*}
\]

This number is how many protons we have more in the ground state relative to the excited state.

\[
\begin{align*}
\text{evalf}(*)
\end{align*}
\]

we see that this is a very small number (less than 24 ppm).
which means that in 1 million atoms only 24 more are in the ground state than the excited state. Thus when we make the populations equal, so that the net magnetization lies in the xy-plane, we really excite about 12 in 1,000,000 atoms.....

This is a result of the small value of the energy difference between the ground state and the excited state (h \(\nu\)) compared to the thermal energy present (kT). Let's look at the ratio of the excess of spins in the excited state relative to the total number of spins present:

\[
\text{ratio} := \frac{-1 + e^{-\frac{h \nu}{kT}}}{1 + e^{-\frac{h \nu}{kT}}}
\]

Now let's see how this ratio is a function of the excitation frequency, \(\nu\), by expanding this as a series

\[
\text{series(ratio,nu)};
\]

This expression can be approximated by the first term if the \(h \nu/(kT)\) is very small. Let's see if it is for a 300MHz field and 300K:

\[
x := 6.626 \times 10^{-34} \text{ J s} \times 300 \times 10^6 \text{ s}/(1.381 \times 10^{-23} \text{ J K}^{-1} \times 300 \text{ K})
\]

Less than 0.005%. This means that for the next term in the series, the third power, we only have a value of:

\[
x^3;
\]

This is a very small fraction of our first term.

Thus we can approximate the fraction of atoms in the that we have more in the ground state than in the excited state with a linear function that is proportional to the energy difference.
INEPT and Refocused INEPT spectra
Copyright (1994) Rene P.F. Kanters
KANTERS@urvax.urich.edu

> read `pof.m`

We will define the procedure that will do an INEPT experiment for us. Note that in order for this to work we need to make sure that all the couplings between the C and H spins are equal, thus this simulation will only work for one CHn set of spins.

Note also that in order to label a particular spin operator in the equilibrium spin state, a multiplication with it’s gamma is performed. This way we can see which gyro-magnetic ratio affects the transition intensity.

We will subtract the spectrum with the -y pulse from that with the y pulse on H to get rid of natural 13C signals and average out the population inversions.

> INEPT:=proc(ALL,Hs,Cs) local i,j,s,tmp,delta; global J;
spinsystem(ALL);
spinlabel("\gamma[H],Hs");
s:=spinlabel("\gamma[C],Cs");
for i in Hs do for j in Cs do J[i,j]:=J; od; od;
delta:=1/(4*J);
xpulse(s,Hs,Pi/2);
evolve("\{},delta"); xpulse("Hs union Cs,Pi"); tmp:=evolve("\{},delta");
pulse(tmp,Hs,Pi/2,Pi/2); xpulse("Cs,Pi/2"); s:=observe("\{},t,Pi/2");
s-observe("\{},t,Pi/2"); end:

Also we will define the procedure used for obtaining the real part of the Fourier transformed spectrum

> spectrum:=(s) ->evalc(Re(FT(s,0,t,nu))):

Let’s do the INEPT spectrum for a CHn system:

> CH:=INEPT([H,C],{H},{C});

\[
CH := -2 I \gamma_H \cos(2 \pi W_C t) \sin(\pi J t) + 2 \gamma_H \sin(2 \pi W_C t) \sin(\pi J t)
\]

> spectrum(CH);

\[
-\gamma_H \text{Ab}\left(W_C + \frac{1}{2} J, \nu\right) + \gamma_H \text{Ab}\left(W_C - \frac{1}{2} J, \nu\right)
\]

Thus the intensities are -1 : +1 for +J/2 : -J/2.

We also see that polarization transfer has occurred since the peak intensities are proportional to the gamma of H, and not that of C.

Now for a methylene carbon:

> spectrum(INEPT([[H,Hb],C],{H},{C}));

\[
-\gamma_H \text{Ab}\left(W_C + J, \nu\right) + \gamma_H \text{Ab}\left(W_C - J, \nu\right)
\]
The intensities are -1: 0: +1 for +J: 0: -J. The central peak of the triplet is cancelled out.

Finally for a methyl carbon:

\[ \begin{align*}
&\frac{3}{4} \gamma_H A \left( W_C + \frac{3}{2} J, v \right) + \frac{3}{4} \gamma_H A \left( W_C - \frac{3}{2} J, v \right) - \frac{3}{4} \gamma_H A \left( W_C + \frac{1}{2} J, v \right) \\
&+ \frac{3}{4} \gamma_H A \left( W_C - \frac{1}{2} J, v \right)
\end{align*} \]

The relative intensities are -1: -1: 1: 1 for +3J/2 : +J/2 : -J/2 : -3J/2.

Now let's do a refocused INEPT experiment:

We first need to define our pulse procedures:

\[
RINEPT := \text{proc}(\text{ALL}, \text{Hs}, \text{Cs}, \delta) \text{ local } i, j, s, \text{tmp}, \text{obs1}, \text{del}; \text{ global } J; \\
\text{spinsystem}(\text{ALL}); \\
\text{spinlabel}(\text{"}, \gamma[H], \text{Hs}); \\
s := \text{spinlabel}(\text{"}, \gamma[C], \text{Cs}); \\
\text{for } i \text{ in } \text{Hs} \text{ do for } j \text{ in } \text{Cs} \text{ do } J[i,j] := \text{J}; \text{ od}; \text{ od}; \\
\text{del} := 1/(4 \ast \text{J}); \\
\text{xpulse}(s, \text{Hs}, \text{Pi}/2); \\
\text{evolve}(\text{"}, \text{del}); \text{xpulse}(\text{"}, \text{Hs union Cs, Pi}); \text{tmp} := \text{evolve}(\text{"}, \text{del}); \\
\text{pulse}((\text{tmp, Hs, Pi}/2, \text{Pi}/2)); \text{xpulse}(\text{"}, \text{Cs, Pi}/2); \\
\text{evolve}(\text{"}, \text{del}); \text{xpulse}(\text{"}, \text{Hs union Cs, Pi}); s := \text{evolve}(\text{"}, \text{del}); \\
\text{decouple}(\text{Hs}); \text{obs1} := \text{observe}(s, \text{del}, 0); \text{ decouple}(\text{"}); \\
\text{pulse}((\text{tmp, Hs, -Pi}/2, \text{Pi}/2)); \text{xpulse}(\text{"}, \text{Cs, Pi}/2); \\
\text{evolve}(\text{"}, \text{del}); \text{xpulse}(\text{"}, \text{Hs union Cs, Pi}); s := \text{evolve}(\text{"}, \text{del}); \\
\text{decouple}(\text{Hs}); s := \text{observe}(s, \text{del}, 0); \text{ decouple}(\text{"}); \\
\text{obs1-s}; \text{ end;}
\]

\[
RCH1 := \text{spectrum}(RINEPT([H, C1], \{H\}, \{C1\}, \delta));
\]

\[
RCH1 := 2 \gamma_H \sin(2 \pi J \delta) A \left( W_{C1}, v \right)
\]

\[
RCH2 := \text{spectrum}(RINEPT([H, Hb, C2], \{H\}, \{C2\}, \delta));
\]

\[
RCH2 := 2 \gamma_H \sin(4 \pi J \delta) A \left( W_{C2}, v \right)
\]

\[
RCH3 := \frac{3}{2} \gamma_H \sin(2 \pi J \delta) A \left( W_{C3}, v \right) + \frac{3}{2} \gamma_H \sin(6 \pi J \delta) A \left( W_{C3}, v \right)
\]

Let's now change these expressions so that we can plot them with nmrplot2d. To do that we have to assign a value to gamma[H], change the Sin into sin, and make that J delta is rewritten as a single variable Jd (which is equal to the length of the delta time interval expressed in 1/J's):

\[
CH1 := \text{subs}(\text{Sin=sin}, \delta = \text{Jd}/J, \gamma[H] = 1, \text{RCH1});
\]

\[
CH1 := 2 \sin(2 \pi Jd) A \left( W_{C1}, v \right)
\]
In order to plot the intensities of the three kinds of carbons we can also do the following:

```maple
> with(plots):
> display(seq(nmrplot(subs(W[C.i]=0,nu=0,CH.i),Jd=0..1),i=1..3));
```

![Graph showing the intensities of three kinds of carbons](image-url)
DEPT spectra
Copyright (1994) Rene P.F. Kanters
KANTERS@urvax.urich.edu

> read `pof.m`;

We will define the procedure that will do a DEPT experiment for us. Note that in order for this to work we need to make sure that all the couplings between the C and H spins are equal, thus this simulation will only work for one CHn set of spins. We will also label the initial spin operators with the gamma of that particular nucleus:

> DEPT:=proc(ALL,Hs,Cs,theta) local i,j,s; global J;
  spinsystem(ALL);
  spinlabel(",gamma[H],Hs);
  s:=spinlabel(",gamma[C],Cs);
  for i in Hs do for j in Cs do J[i,j]:=J; od; od;
  xpulse(s,Hs,Pi/2);
  evolve("{},1/(2*J));
  xpulse("Hs,Pl); xpulse("Cs,Pi/2);
  evolve("{},1/(2*J));
  ypulse("Hs,theta); xpulse("Cs,Pi);
  s:=evolve("{},1/(2*J));
  decouple(Hs): s:=observe(s,"t,0); decouple({});
  s;
end:

Also we will define the procedure used for obtaining the real part of the Fourier transformed spectrum

> spectrum:=(s) ->evalc(Re(FT(s,0,"t,nu))):

Let's do a DEPT spectrum, with theta = 3*Pi/4, for a CH spin system.

> obs:=DEPT([H,C],{H},{C},3*Pi/4);

\[
\begin{align*}
obs &= \frac{1}{2} \gamma_H \sqrt{2} \cos\left(2 \pi W_C t\right) + \frac{1}{2} \gamma_H \sqrt{2} \sin\left(2 \pi W_C t\right) - \frac{1}{2} \gamma_C \sqrt{2} \cos\left(2 \pi W_C t\right) \\
&\quad + \frac{1}{2} \gamma_C \sqrt{2} \sin\left(2 \pi W_C t\right)
\end{align*}
\]

If we FT this with a phase correction of zero we get:

> spectrum(obs);

\[
\begin{align*}
&\quad \frac{1}{2} \gamma_H \sqrt{2} \text{Ab}\left(W_C v\right) + \frac{1}{2} \gamma_C \sqrt{2} \text{Di}\left(W_C v\right)
\end{align*}
\]

So we see that we have a peak whose absorption part is enhanced due to the polarization transfer. There is also a dispersion contribution of the non enhanced carbon signal.

Let's take some DEPT spectra where we keep the ypulse of angle theta on the protons variable, so that we can see what happens with specific values of theta. We will calculate the spectra for the spinsystems CH, CH2, and CH3.

> for i from 1 to 3 do
  CH.i:=spectrum(DEPT([[seq(H,j=j..i)],C.i],[H1],[C.i],theta)) od;
Let's set the disgusting dispersion parts to zero:

\[
CH1 := \gamma_H \sin(\theta) \text{Ab}\left( W_{Cl}, v \right) - \gamma_C \cos(\theta) \text{Di}\left( W_{C1}, v \right)
\]

\[
CH2 := -\gamma_C \cos(\theta)^2 \text{Di}\left( W_{C2}, v \right) + 2 \gamma_H \sin(\theta) \cos(\theta) \text{Ab}\left( W_{C2}, v \right)
\]

\[
CH3 := 3 \gamma_H \cos(\theta)^2 \sin(\theta) \text{Ab}\left( W_{C3}, v \right) - \gamma_C \cos(\theta)^3 \text{Di}\left( W_{C3}, v \right)
\]

The values for theta most often used is Pi/4, Pi/2 and 3Pi/4, so we will substitute these in our expressions to see what the spectrum would have looked like

\[
\text{Let's look at those spectra. We'll put the CH carbon at 10Hz, CH2 at 20, and CH3 at 30 Hz:}
\]
> Wv:=table([(C1)=10,(C2)=20,(C3)=30]): T2:=table([(C1)=1,(C2)=1,(C3)=1]):
> nmrplot(subs(gamma[H]=1,addspectra(1,0,-1)/2),nu=0..40);
The spectra we would obtain if we used the three standard angles of 45, 90 and 135 degrees:

\[
> \text{nrmplot(}\text{eval(}\text{subs(}\theta=\pi/4, \gamma[H]=1, \{\text{CH1, CH2, CH3}\})\text{), nu=0..40, title='theta=45 degrees'}}); \\
\theta=45 \text{ degrees}
\]

\[
> \text{nrmplot(}\text{eval(}\text{subs(}\theta=2\pi/4, \gamma[H]=1, \{\text{CH1, CH2, CH3}\})\text{), nu=0..40, title='theta=90 degrees'}}); \\
\theta=90 \text{ degrees}
\]
\begin{verbatim}
> nmrplot(eval(subs(theta=3*Pi/4, gamma[H]=1, \{CH1, CH2, CH3\})), nu=0..40, title=\textquote{\textit{theta=135 degrees}});
\end{verbatim}

\textit{theta=135 degrees}
We are going to see what happens when we have a signal that modulates with two different (independent) time variables, e.g.,

\[
FID1 := \cos(\pi J_{12} t1) \sin(2 \pi W_1 t1);
\]

\[
FID2 := \sin(\pi J_{12} t2) \cos(2 \pi W_2 t2);
\]

\[
FID := FID1 \times FID2;
\]

Let's assign some values to the coupling, shift and relaxation times

\[
J_{v12} := 4; W_{v1} := 5; W_{v2} := 10; T2_{1} := 1; T2_{2} := 1;
\]

Let's Fourier transform the t2 time domain (our observation). Since I do not have quadrature in t2, I will only do a CosFT or SinFT. The term has an odd numbers of sines for t2 so we will do a real FT (CosFT) with a Pi/2 phase correction to get the lines in absorption mode.

Note that the procedures FT, SinFT, and CosFT, are implemented under the assumption of quadrature detection, so that they will only give positive chemical shift informations

\[
spec2FID1 := \frac{1}{4} \cos(\pi J_{12} t1) \sin(2 \pi W_1 t1) \left[ \cos \left( W_2 + \frac{1}{2} J_{12}, v2 \right) - \frac{1}{4} \cos(\pi J_{12} t1) \sin(2 \pi W_1 t1) \left[ \cos \left( W_2 - \frac{1}{2} J_{12}, v2 \right) \right] \right]
\]
To get the frequency information in t1 we will use the CosFT with phase correction of Pi/2.

\[ spec12 := \frac{1}{16} Ab \left\{ W_2 + \frac{1}{2} J_{1,2}, v_2 \right\} Ab \left\{ W_1 + \frac{1}{2} J_{1,2}, v_1 \right\} \\
+ \frac{1}{16} Ab \left\{ W_2 + \frac{1}{2} J_{1,2}, v_2 \right\} Ab \left\{ W_1 - \frac{1}{2} J_{1,2}, v_1 \right\} \\
- \frac{1}{16} Ab \left\{ W_2 - \frac{1}{2} J_{1,2}, v_2 \right\} Ab \left\{ W_1 + \frac{1}{2} J_{1,2}, v_1 \right\} \\
- \frac{1}{16} Ab \left\{ W_2 - \frac{1}{2} J_{1,2}, v_2 \right\} Ab \left\{ W_1 - \frac{1}{2} J_{1,2}, v_1 \right\} \]
We also could have determined this with convolution for each dimension and then forming the 2D version of the pattern (each point in the 2D grid is the product of the intensities of the 1d spectra with that particular frequency in $v_1$ and $v_2$).

Let's look at some of the lineshapes possible for a 2D peak, at 0,0

```maple
> T2[0]:=1;

The peak can be $Ab(1) Ab(2)$ (pure absorption), which will look like:

```maple
> nmrplot2d(Ab(0,v1)*Ab(0,v2),v2=-2..2,v1=-2..2,numpoints=30*30,
axes=FRAME,orientation=[120,60]);
```

or pure dispersion, i.e. dispersion in both frequency domains

```maple
> nmrplot2d(Di(0,v1)*Di(0,v2),v2=-2..2,v1=-2..2,numpoints=30*30,
axes=FRAME,orientation=[120,60]);
```
The line can be incorrectly phased, i.e., absorption in one dimension and dispersion in other. This line shape can be avoided (and therefore does not have a name)

```maple
> nmrplot2d(Ab(0,v1)*Di(0,v2),v2=-2..2,v1=-2..2,numpoints=30*30,axes=FRAME,orientation=[120,60]);
```

Finally, one can have a so-called phase twisted, or mixed phase, line shape. These result if one combines the real real and the imaginary imaginary parts of the spectrum and the line shape is the real part of \((Ab+iDi)(Ab+iDi) = AbAb-DiDi\). This will look like:

```maple
> nmrplot2d(Ab(0,v1)*Ab(0,v2)-Di(0,v1)*Di(0,v2),v2=-2..2,v1=-2..2,numpoints=30*30,axes=FRAME,orientation=[120,60]);
```
We are going to look at some homonuclear shift correlation (COSY) spectra:
The COSY pulse sequence is: \((\pi/2) x -- t1 -- (\pi/2)x --\) FID

> read `pof.m`:

Let's define a procedure that will do a COSY spectrum where we focus in the spins in the set d1
during time domain t1, and the ones in set d2 during the observation time domain:

> COSY:=proc(s,d1,d2)
spinsystem(s):
  xpulse("",{},Pi/2): evolve("",d1,t1): xpulse("",{},Pi/2):
  observe("",d2,t2,0);
end:

> COSY([H],{},{});

\[
\begin{align*}
\text{Sin}\left(2 \pi W_H t1\right) \cos\left(2 \pi W_H t2\right) + i \text{Sin}\left(2 \pi W_H t1\right) \text{Sin}\left(2 \pi W_H t2\right)
\end{align*}
\]

We see that we have a signal that is modulated with the chemical shift of H in both time
domains. Thus this should give a peak on the diagonal. We have complex modulation for t2
(quadrature detection) so we can do an FT for that time domain. For the t1 domain we only have
the sine part so we need to do a CosFT with a Pi/2 phase shift, and get the real part of the
spetrum:

> evalc(Re(CosFT(FT("",0,t2,v2),Pi/2,t1,v1)));

\[
\frac{1}{2} \text{Ab}\left(W_H v1\right) \text{Ab}\left(W_H v2\right)
\]

Voila, our real part of the spectrum will show a pure absorption peak on the diagonal.

Now let's define the required fourier transforms. We will allow ourselves to set the phase
corrections for the transforms.

> spectrum:=(s,p1,p2)->evalc(Re(CosFT(FT(s,p2,t2,v2),p1,t1,v1))):

But what if we have a slightly more complicated case of two protons, A and B, that couple
eachother. Let's look at the real part of the Fourier transformed spectrum of that. We will use the
same phase corrections as in the single spin case, to get the diagonal peak(s) in pure absorption:

> spc:=spectrum(COSY([A,B],{},{}),Pi/2,0);
We see that we have pure absorption signals in both v1 and v2 directions for peaks that lie on the diagonal (same chemical shift for in both time domains -> same frequencies in v1 and v2).

All these peaks have a positive intensity.

We also have peaks that have a different chemical shift modulation in one frequency domain than in the other. These are called cross-peaks. With the used phase correction (to get the diagonal peaks in pure absorption mode) they will be pure dispersion peaks.

Let’s plot this spectrum, with a very low resolution. We will give A a shift of 5 Hz, B one of 15 Hz and a coupling between the two of 2Hz. We will also make the lines broad, so we can see them with a low resolution graph

```maple
spc := 1/8 Ab(%,v2) Ab(%,v1) + 1/8 Di(%,v2) Di(%,v1) + 1/8 Ab(%,v2) Ab(%,v1)
+ 1/8 Di(%,v1) Di(%,v1) + 1/8 Di(%,v2) Di(%,v1)
+ 1/8 Ab(%,v2) Ab(%,v1) + 1/8 Ab(%,v2) Ab(%,v1)
+ 1/8 Di(%,v2) Di(%,v1) + 1/8 Ab(%,v1) Ab(%,v1)
+ 1/8 Ab(%,v2) Ab(%,v1) - 1/8 Di(%,v2) Di(%,v1)
- 1/8 Di(%,v2) Di(%,v1) - 1/8 Di(%,v1) Di(%,v1)
+ 1/8 Ab(%,v2) Ab(%,v1) + 1/8 Ab(%,v2) Ab(%,v1)
- 1/8 Di(%,v2) Di(%,v1)

%1 := W_B + 1/2 J_A,B
%2 := W_A - 1/2 J_A,B
%3 := W_A + 1/2 J_A,B
%4 := W_B - 1/2 J_A,B
```

We see that we have pure absorption signals in both v1 and v2 directions for peaks that lie on the diagonal (same chemical shift for in both time domains -> same frequencies in v1 and v2).

All these peaks have a positive intensity.

We also have peaks that have a different chemical shift modulation in one frequency domain than in the other. These are called cross-peaks. With the used phase correction (to get the diagonal peaks in pure absorption mode) they will be pure dispersion peaks.

Let’s plot this spectrum, with a very low resolution. We will give A a shift of 5 Hz, B one of 15 Hz and a coupling between the two of 2Hz. We will also make the lines broad, so we can see them with a low resolution graph

```maple
> T2:=table([A=-1,B=-1]):Wv:=table([A=5,B=15]):
> Jv:=table(symmetric,[(A,B)=2]):
> nmrplot2d(spc,v2=0..20,v1=0..20,orientation=[120,60],axes=FRAME, title=`homo nuclear COSY AB`);```
That looks disgusting. What we were able to see from this that we have peaks on the diagonal, and crosspeaks. We also see that the spectrum is symmetrical around the body diagonal. That is because coupling is symmetrical, i.e., $J[A,B] = J[B,A]$.

Let's focus on the diagonal and cross peaks separately:

The diagonal peaks for A:

> obs_diag := COSY([A,B],{A},{A});

$$
\begin{aligned}
\text{obs} \text{diag} &:= \sin \left( 2 \pi \frac{W_A}{t_1} \right) \cos \left( \frac{\pi J_{A,B}}{t_1} \right) \cos \left( 2 \pi \frac{W_A}{t_2} \right) \cos \left( \frac{\pi J_{A,B}}{t_2} \right) \\
&+ I \sin \left( 2 \pi \frac{W_A}{t_1} \right) \cos \left( \frac{\pi J_{A,B}}{t_1} \right) \sin \left( 2 \pi \frac{W_A}{t_2} \right) \cos \left( \frac{\pi J_{A,B}}{t_2} \right)
\end{aligned}
$$

Notice that we have the same observed signal as in the case of a single A spin, with the difference that we now have a coupling modulation with a cosine present. This means that if the coupling is 0 Hz we get exactly the same spectrum.

> spc_diag := spectrum(obs_diag,Pi/2,0);

$$
\begin{aligned}
\text{spc} \text{diag} :&= \frac{1}{8} \text{Ab}(\%2, \nu_2) \text{Ab}(\%2, \nu I) + \frac{1}{8} \text{Ab}(\%2, \nu 2) \text{Ab}(\%1, \nu I) \\
&+ \frac{1}{8} \text{Ab}(\%1, \nu 2) \text{Ab}(\%2, \nu I) + \frac{1}{8} \text{Ab}(\%1, \nu 2) \text{Ab}(\%1, \nu I)
\end{aligned}
$$

We see that this consists of 4 positive absorption peaks, $(A+J/2,A+J/2)$, $(A+J/2,A-J/2)$, $(A-J/2,A+J/2)$, and $(A-J/2,A-J/2)$.

> obs_cross := COSY([A,B],{A},{B});

$$
\begin{aligned}
\text{obs} \text{cross} &:= \sin \left( 2 \pi \frac{W_A}{t_1} \right) \sin \left( \frac{\pi J_{A,B}}{t_1} \right) \cos \left( 2 \pi \frac{W_B}{t_2} \right) \sin \left( \frac{\pi J_{A,B}}{t_2} \right) \\
&+ I \sin \left( 2 \pi \frac{W_A}{t_1} \right) \sin \left( \frac{\pi J_{A,B}}{t_1} \right) \sin \left( 2 \pi \frac{W_B}{t_2} \right) \sin \left( \frac{\pi J_{A,B}}{t_2} \right)
\end{aligned}
$$

$$%1 := W_A - \frac{1}{2} J_{A,B}$$

$$%2 := W_A + \frac{1}{2} J_{A,B}$$
Notice that the coupling modulation is in the sine form, that means that if the $J=0$, the intensity of the signal becomes 0: cross peak will not be present. That is why the coupling between spins A and B is called the active coupling for the cross peak between A and B.

Thus the cross peaks are really due to polarization transfer achieved by through bond coupling between spins A and B.

If we Fourier transform this with the same phase corrections to get the diagonal peaks in phase we get our pure dispersion signals:

\[
\text{spc\_cross} := \text{spectrum(\text{obs\_cross},\pi/2,0)};
\]

\[
\begin{align*}
\text{spc\_cross} := -\frac{1}{8} & \left( D_A \left( W_B + \frac{1}{2} J_{A,B} v_2 \right) + \frac{1}{8} D_A \left( W_B + \frac{1}{2} J_{A,B} v_2 \right) + \frac{1}{8} D_B \left( W_A - \frac{1}{2} J_{A,B} v_1 \right) - \frac{1}{8} D_B \left( W_A - \frac{1}{2} J_{A,B} v_1 \right) \right) \\
& \left( D_A \left( W_B + \frac{1}{2} J_{A,B} v_2 \right) + \frac{1}{8} D_A \left( W_B + \frac{1}{2} J_{A,B} v_2 \right) + \frac{1}{8} D_B \left( W_A - \frac{1}{2} J_{A,B} v_1 \right) - \frac{1}{8} D_B \left( W_A - \frac{1}{2} J_{A,B} v_1 \right) \right)
\end{align*}
\]

This is 4 dispersion signals in both frequency domains, but is a socalled anti-phase pattern; We have two peaks that have a positive amplitude $(A+J/2,A+J/2)$, and $(A-J/2,A-J/2)$, not that a line connecting these two will be parallel to the diagonal, and two with negative amplitude $(A-J/2,B+J/2)$, and $(A+J/2,B-J/2)$.

Let’s make the peaks sharper and look at the separate patterns

\[
\text{T2:=table([\{A\}=1.5,\{B\}=1.5])};
\]

\[
\text{nmrplot2d(spc\_diag,v2=3..7,v1=3..7,numpoints=50*50,title=`diagonal peak COSY`,orientation=[120,60], axes=FRAME)};
\]

These peaks are a little cut off at the top due to the resolution of our grid. We can also plot them using a different procedure that will automatically make sure that the center of the peak lies on a grid point:

\[
\text{super2d(spc\_diag,v2,v1,numpoints=10*10,title=\textasciitilde\text{diagonal peak COSY},axes=FRAME)};
\]
Now the super position of the cross peak looks like this:

```
> super2d(spc_cross,v2,v1,title=`cross peak COSY AB`, axes=FRAME,numpoints=10*10);
```

OK, so superpositioning these is not a good idea because the grids are very broad for that and they overlap (not additive when they overlap). We will have to do a normal surface plot:

```
> nmrplot2d(spc_cross,v2=13..17,v1=3..7,title=`cross peak COSY AB`,orientation=[120,60], axes=FRAME,numpoints=50*50);
```
That still looks disgusting. There is no way around it: the peaks on the diagonal are 90 degrees out of phase in both frequency directions with the cross peaks, so either the cross peaks are pure dispersion and the diagonal peaks are pure absorption, or vice versa.

There is a quick and dirty solution to this: looking at the power spectrum (also called magnitude calculation). In that case one multiplies the complex spectrum with its complex conjugate to get the lengths of the complex vectors as real numbers. This makes that the phase information is lost, so the peaks are all 'in-phase' peaks, but they also are all positive (can't see which coupling is the active one) and in general are broader.

Let's look at a 1D 'power peak'

First let's see what happens to the real and imaginary parts when we do the multiplication:

```latex
> evalc((a+b*I)*conjugate(a+b*I));
\[ a^2 + b^2 \]
```

so we see that we do get the sum of the products of the real part and the imaginary part. In order to scale this properly with the single absorption peak we will take the square root of half of the sum of the squares.

Let's assume that one has a dispersion peak and the other an absorption peak, both a 0Hz with a 1 s relaxation time:

```latex
> p_line:=simplify(sqrt(pA(0,v,1)^2+pD(0,v,1)^2));
\[ p_{\text{line}} := 2 \sqrt{ \frac{1}{1 + 4 \pi^2 v^2} } \]

> plot({p_line,pA(0,v,1)},v=-2..2,title=`Ab and Power lineshapes`);
```
We see that the power line shape is definitely a lot broader than the absorption one. You could of course consider also using a line narrowing data manipulation before you do the magnitude calculation....

The result of that is that the diagonal peak is in total intensity still larger than the cross peaks, so the cross peaks are still a little hard to see. DQF - COSY solves all these problems.
Problems

Chemistry 558

Organic Spectroscopy

Part II: FT-NMR

by

René P. F. Kanters

1994
Organic Spectroscopy
FT-NMR problem set 1

In this problem set we will look a little further into the use of the Fourier transform in NMR. We will also look at a method that makes it a little easier to determine the peak patterns that arises from more complex signals, still within the realm of decaying sine and cosine functions.

Before we start it is useful to have some trigonometric equalities at hand. All these can be derived in a pretty straightforward manner from the exponential expression:

\[ e^{i\alpha} = \cos \alpha + i \sin \alpha \]

The equalities that we may need are

<table>
<thead>
<tr>
<th>products of sines and cosines</th>
<th>products and differences of angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] )</td>
<td>( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta )</td>
</tr>
<tr>
<td>( \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] )</td>
<td>( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta )</td>
</tr>
<tr>
<td>( \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] )</td>
<td></td>
</tr>
<tr>
<td>( \cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] )</td>
<td></td>
</tr>
</tbody>
</table>

1. Using the shorthand notation for the Fourier, cosine, and sine transforms, respectively:

\[ F(\omega) = \text{FT}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt \]
\[ C(\omega) = \text{CosFT}(f(t)) = \int_{-\infty}^{\infty} f(t)\cos \omega t \, dt \]
\[ S(\omega) = \text{SinFT}(f(t)) = \int_{-\infty}^{\infty} f(t)\sin \omega t \, dt \]

show that the Fourier transform can be rewritten as \( F(\omega) = C(\omega) - iS(\omega) \) or in other words \( \text{FT}(f(t)) = C(\omega) - iS(\omega) \).

Notice that the Fourier transform is defined so that it returns a spectrum with angular frequencies, \( \omega \). Keep in mind that it is customary in NMR to have a spectrum expressed in linear frequencies \( \nu \). In order to convert a \( \omega \) axis to a \( \nu \) axis, you will need to divide the axis by \( 2\pi \), since \( \omega = 2\pi \nu \) thus \( \nu = \omega/(2\pi) \).

In order to evaluate the Fourier transform of sine or cosine modulated time signal we will have to evaluate integrals of products of trigonometric functions, at least one trig function from the signal \( f(t) \) and the other one from the Fourier transform itself. From the trigonometric equalities given above, we see that the products can be rewritten into a sum of single trigonometric functions. Thus we only have to consider integrals of single sine and cosine functions, i.e., \( \sin(\omega \pm \omega_0) t \) and \( \cos(\omega \pm \omega_0) t \) over all time.
We should keep in mind that the observed magnetization will decay exponentially. The solution of the Bloch equations told us that relaxation effects will give rise to two lineshape functions, i.e., the dispersion, $\text{Di}(\omega_0)$, and absorption, $\text{Ab}(\omega_0)$ functions. This means that the Fourier transform of an exponentially decaying time signal can be expressed in these lineshape functions also. Let’s put what we know about the line shape functions in a little table:

<table>
<thead>
<tr>
<th>Function</th>
<th>kind</th>
<th>signs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Ab}(\omega) = \frac{2T^2}{1 + [T_2(\omega_0 - \omega)]^2}$</td>
<td>even</td>
<td>$\pm +$</td>
</tr>
<tr>
<td>$\text{Di}(\omega) = \frac{2T^2}{(\omega_0 - \omega)}$</td>
<td>odd</td>
<td>$\pm -$</td>
</tr>
</tbody>
</table>

In order to figure out which integral yields which lineshape, we can look at the signs of the values that we would get if we let $\omega$ approach to $\omega_0$ from above or below. In that case the modulation will be so slow that the exponential decay will have forced the signal to zero before it can change signs, and we can be certain of the sign of the number. Let’s assume that the $\omega_0$ and $\omega$ are only $\varepsilon$ away from each other. Now we can set up the following table for the four integrals that we need to consider:

<table>
<thead>
<tr>
<th>integral</th>
<th>signs</th>
<th>lineshape</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_{-\infty}^{\infty} \sin(\omega_0 + \omega)t e^{-\frac{t}{T_2}} dt$</td>
<td>$\pm +$</td>
<td>$-\text{Di}(-\omega_0)$</td>
</tr>
<tr>
<td>$\int_{-\infty}^{\infty} \cos(\omega_0 + \omega)t e^{-\frac{t}{T_2}} dt$</td>
<td>$+ +$</td>
<td>$\text{Ab}(-\omega_0)$</td>
</tr>
<tr>
<td>$\int_{-\infty}^{\infty} \sin(\omega_0 - \omega)t e^{-\frac{t}{T_2}} dt$</td>
<td>$+ -$</td>
<td>$\text{Di}(\omega_0)$</td>
</tr>
<tr>
<td>$\int_{-\infty}^{\infty} \cos(\omega_0 - \omega)t e^{-\frac{t}{T_2}} dt$</td>
<td>$- +$</td>
<td>$\text{Ab}(\omega_0)$</td>
</tr>
</tbody>
</table>

Thus in short, we can write:
Before we do the Fourier transform of the exponentially decaying complex magnetization we will first find out what the cosine and sine transforms are of exponentially decaying modulation. With all this background information we have accumulated in the previous questions, we can easily show that:

\[
\begin{align*}
\text{CosFT} \left( \cos(\omega_0 \pm \omega) t e^{-t/2} \right) &= \frac{1}{2} \text{Ab}(-\omega_0) + \frac{1}{2} \text{Ab}(\omega_0) + i \text{Di}(-\omega_0) + i \text{Di}(\omega_0) \\
\text{SinFT} \left( \sin(\omega_0 \pm \omega) t e^{-t/2} \right) &= \frac{1}{2} \text{Ab}(-\omega_0) - \frac{1}{2} \text{Ab}(\omega_0) + i \text{Di}(-\omega_0) - i \text{Di}(\omega_0)
\end{align*}
\]

Show that this is indeed true for the last equation.

Now, using the results of our previous questions, we can derive the following Fourier transforms.

\[
\begin{align*}
\text{FT} \left( \cos(\omega_0 t) e^{-t/2} \right) &= \frac{1}{2} \left\{ \text{Ab}(-\omega_0) + \text{Ab}(\omega_0) + i \text{Di}(-\omega_0) + i \text{Di}(\omega_0) \right\} \\
\text{FT} \left( \sin(\omega_0 t) e^{-t/2} \right) &= \frac{1}{2} \left\{ -\text{Di}(-\omega_0) + \text{Di}(\omega_0) + i \text{Ab}(-\omega_0) - i \text{Ab}(\omega_0) \right\}
\end{align*}
\]

Show that this is indeed true for the last equation.

Using the results you derived so far, show that \( \text{FT} \left( e^{i\omega_0 t} e^{-t/2} \right) = \text{Ab}(\omega_0) + i \text{Di}(\omega_0) \).

Derive the expression for the Fourier transform of the phase corrected version of the previous expression, i.e., \( e^{i(\omega_0 t + \varphi)} e^{-t/2} \). What is the result if you have a phase angle of \( \varphi = \frac{\pi}{2} \), and what if \( \varphi = -\frac{\pi}{2} \).
We see now, as you already might have suspected, that it is very tedious to do Fourier transforms algebraically. This is specially the case if you have terms that consist of products of several modulation terms.

Using the trigonometric equalities given, you can show that when you have a carrier signal, with angular frequency of $\omega_0$ that is also a cosine modulated with a frequency of $\omega_m$ the peak is split in two peaks. One at $\omega_0 + \omega_m$ and one at $\omega_0 - \omega_m$, with half the intensity of the ‘original peak’ at $\omega_0$. Analogously an extra sine modulation will split the peak at $\omega_0$ in two peaks, but with opposite intensities. In a mathematical form:

$$\text{FT} \left( \cos \omega_m e^{i \omega_0 t} e^{-\frac{t}{T}} \right) = \frac{1}{2} \left\{ A \text{b} (\omega_0 - \omega_m) + i D \text{i} (\omega_0 - \omega_m) \right\}$$

$$+ \frac{1}{2} \left\{ A \text{b} (\omega_0 + \omega_m) + i D \text{i} (\omega_0 + \omega_m) \right\}$$

$$\text{FT} \left( \sin \omega_m e^{i \omega_0 t} e^{-\frac{t}{T}} \right) = -\frac{1}{2} \left\{ D \text{i} (\omega_0 - \omega_m) - i A \text{b} (\omega_0 - \omega_m) \right\}$$

$$+ \frac{1}{2} \left\{ D \text{i} (\omega_0 + \omega_m) - i A \text{b} (\omega_0 + \omega_m) \right\}$$

a) Show that the last of these two expressions hold. (Tip in order to not get cramps in your hands introduce some short hand notation like $R = e^{-\frac{t}{T}}$, $\omega_0^+ = \omega_0 + \omega_m$, and $\omega_0^- = \omega_0 - \omega_m$ during your derivation.)

Notice that the last expression has the absorption line shapes in the complex part of the spectrum, and dispersion line shapes in the real part. This is not the preferred way. In order to turn these peaks into absorption peaks, you need to perform a phase correction of $\pi/2$. This is equivalent to multiplying with $i$ since

$$e^{i \frac{\pi}{2}} = \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) = i .$$

b) Show what the expression for the spectrum would look like if you did perform this $90^\circ$ phase correction.

The method that utilizes the splitting of a signal by additional sin or cosine modulations is called convolution. The way this works is more easily understood if one expresses cosine and sine terms into complex exponentials. The equations to do that are given by:

$$\cos \alpha = \frac{e^{i \alpha} + e^{-i \alpha}}{2}, \sin \alpha = \frac{e^{i \alpha} - e^{-i \alpha}}{2i}$$

The modulating part of terms in the expression for signals that observe in NMR is:

$$\prod_{c=0}^{n_c} \cos \alpha_c \prod_{s=0}^{n_s} \sin \alpha_s$$

where the $\prod$ stands for the product, and the $\alpha$’s are time dependent angles. We can express this equation in complex exponentials as follows:
\[
\prod_{c=0}^{n_c} \cos \alpha_c \prod_{s=0}^{n_s} \sin \alpha_s = \prod_{c=0}^{n_c} \left( \frac{1}{2} \left( e^{i \alpha_c} + e^{-i \alpha_c} \right) \right) \prod_{s=0}^{n_s} \left( \frac{1}{2} \left( e^{i \alpha_s} - e^{-i \alpha_s} \right) \right)
\]
\[
= \left( \frac{1}{2} \right)^{n_c} \sum_{c=0}^{n_c} \left( \frac{1}{2} \right)^{n_s} \left( \sum \alpha_k \right)
\]

where \( \alpha_k \) are the angles that are the result of the combination of the products of the sum (and difference) of exponentials. Each of these exponents will, with the introduction of an exponential decay, give rise to an absorption peak in the real part of the spectrum and a dispersion peak in the complex part. Thus we can write for the Fourier transform of our expression:

\[
\prod_{c=0}^{n_c} \cos \alpha_c \prod_{s=0}^{n_s} \sin \alpha_s \mathcal{FT}(t) = \left( \frac{1}{2} \right)^{n_c} \sum \left( \sum \alpha_k \right) + i \text{Di}(\alpha_k)
\]

We’ll look at the concrete example, where the peak positions are functions of a chemical shift \( \omega \) and coupling terms. Consider the expression \( e^{i \omega_A t} \sin \pi J_{AB} t \cos \pi J_{AC} t \). Following the outlined procedure we get:

\[
e^{i \omega_A t} \sin \pi J_{AB} t \cos \pi J_{AC} t = e^{i \omega_A t} \left( \frac{e^{i \pi J_{AB} t} - e^{-i \pi J_{AB} t}}{2i} \right) \left( \frac{e^{i \pi J_{AC} t} + e^{-i \pi J_{AC} t}}{2} \right)
\]
\[
= \frac{1}{4i} \left( e^{i (\omega_A + \pi J_{AB} + \pi J_{AC}) t} + e^{i (\omega_A + \pi J_{AB} - \pi J_{AC}) t} - e^{i (\omega_A - \pi J_{AB} + \pi J_{AC}) t} - e^{i (\omega_A - \pi J_{AB} - \pi J_{AC}) t} \right)
\]

Fourier transformation of this expression will give \( \frac{\text{Ab} + \text{Di}}{4i} \) lines for each of the exponential terms above. Thus we see that we need to multiply with i, i.e. introduce a phase correction of \( \pi/2 \), to make that the real part of the spectrum will be in absorption mode. When we Fourier transform this we will also want to convert the angular frequencies into linear frequencies by dividing them by \( 2\pi \). The real part of the Fourier transformation of the above expression, after a phase correction of \( \pi/2 \) (= multiplication with i), is thus:

\[
\frac{1}{4} \text{Ab} \left( \nu_A + \frac{J_{AB}}{2} + \frac{J_{AC}}{2} \right) + \frac{1}{4} \text{Ab} \left( \nu_A + \frac{J_{AB}}{2} - \frac{J_{AC}}{2} \right) - \frac{1}{4} \text{Ab} \left( \nu_A - \frac{J_{AB}}{2} + \frac{J_{AC}}{2} \right) - \frac{1}{4} \text{Ab} \left( \nu_A - \frac{J_{AB}}{2} - \frac{J_{AC}}{2} \right)
\]

where we used that \( \omega = 2\pi \nu \).

The tricky part in all this is the combination of the exponentials that arises from the product of the trigonometric functions. There is a pictorial approach to work out that product. Notice that the cosine splits into two terms with exponents of \( \pm i \alpha \) both with a coefficient of \( 1/2 \). The sine splits into two terms with exponents of \( \pm i \alpha \) one with a coefficient of \( 1/2i \), the other with \(-1/2i \). If we draw this splitting, neglecting for the time being the \( 2 \)'s and the \( i \)'s (only focusing on the signs and the angles), we can picture that as follows:
Notice that we let the angle axis go from low to high going from right to left (just as in a spectrum). If we have a multiplication of sines and/or cosines we can apply this splitting pattern on each of the positions as generated from the previous multiplication, viz.

\[ e^{i\omega_A t} \sin \pi J_{AB} t \cos \pi J_{AC} t \]

If no complex exponential is present, the base frequency is 0, since \( 1 = e^{i0t} \). From this picture we can, by looking at the positions (the powers) and whether the line is up (+e) or whether it is down (-e), immediately tell that the combination of the exponentials is given by:

\[ e^{i(\omega_A + \pi J_{AB} + \pi J_{AC})t} + e^{i(\omega_A + \pi J_{AB} - \pi J_{AC})t} - e^{i(\omega_A - \pi J_{AB} + \pi J_{AC})t} - e^{i(\omega_A - \pi J_{AB} - \pi J_{AC})t} \]

This is indeed the same as we had before. In order to keep track of the line shapes and the intensities, we need to calculate the complex coefficient for the pattern, i.e., 1 for every exponential, 1/2i for every sine, and 1/2 for every cosine. In this case that is \( 1 \frac{1}{2i} \frac{1}{2} = \frac{1}{4i} \).

Thus we can say that the Fourier transform will give rise to a pattern as follows:
where we divided the positions by $2\pi$ to turn them into linear frequencies. Note that we also show the calculation for the line shape and give the phase correction $\phi$ required to turn our peaks into absorption peaks in the real part of the spectrum.

7 Using the convolution approach, determine what the real part of the spectrum resulting from Fourier transformation of the term $e^{i\omega_A t} \sin \pi J_{AB} t \cos \pi J_{AC} t$ would look like. Are this dispersion or absorption peaks? If they aren’t absorption peaks, what could you do to turn them into absorption lines?

8 So far for having mathematical expressions of our observed magnetization. In reality we have a time sampled signal, with an increment time of $\tau$ seconds between samples. We know that $\tau$ is related to the highest frequency that can be obtained by Fourier transformation as given by the Nyquist theorem, $\nu_N = \frac{1}{2\tau}$. Show mathematically that it is impossible to distinguish between signals with a frequency of $\nu_N + \nu$ and $-\nu_N + \nu$. In order to do this you will have to create a time discrete version of the signal. This can be achieved by substituting $t$ with $n\tau$, where $n$ is an integer value ranging from 0 to the total number of sample points.
1. Show that \( F(\omega) = C(\omega) - iS(\omega) \):

\[
F(\omega) = \text{FT}(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(t) (\cos(\omega t) - i\sin(\omega t)) dt
\]

\[
= \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt - i \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt = \text{CosFT}(f(t)) - i\text{SinFT}(f(t))
\]

\[
= C(\omega) - iS(\omega)
\]

2. Show that \( \text{SinFT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right) = -\frac{1}{2}\ Ab(-\omega_0) + \frac{1}{2}\ Ab(\omega_0) \):

\[
\text{SinFT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right) = \int_{-\infty}^{\infty} \sin\omega_0 t e^{-\frac{t}{T_2}} \sin(\omega t) dt = \frac{1}{2} \int_{-\infty}^{\infty} (\cos(\omega_0 - \omega)t - \cos(\omega_0 + \omega)t) e^{-\frac{t}{T_2}} dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \cos(\omega_0 - \omega)t e^{-\frac{t}{T_2}} dt - \frac{1}{2} \int_{-\infty}^{\infty} \cos(\omega_0 + \omega)t e^{-\frac{t}{T_2}} dt = \frac{1}{2} Ab(\omega_0) - \frac{1}{2} Ab(-\omega_0)
\]

3. Show that \( \text{FT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right) = \frac{1}{2}\left\{-\text{Di}(-\omega_0) + \text{Di}(\omega_0) + i\text{Ab}(-\omega_0) - i\text{Ab}(\omega_0)\right\} \)

\[
\text{FT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right) = \text{CosFT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right) - i\text{SinFT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right)
\]

\[
= \left[-\frac{1}{2}\ Di(-\omega_0) + \frac{1}{2}\ Di(\omega_0)\right] - i\left[-\frac{1}{2}\ Ab(-\omega_0) + \frac{1}{2}\ Ab(\omega_0)\right]
\]

\[
= \frac{1}{2}\left\{-\text{Di}(-\omega_0) + \text{Di}(\omega_0) + i\text{Ab}(-\omega_0) - i\text{Ab}(\omega_0)\right\}
\]

4. Show that \( \text{FT}\left(e^{i\omega_0 t} e^{-\frac{t}{T_2}}\right) = \text{Ab}(\omega_0) + i\text{Di}(\omega_0) \)

\[
\text{FT}\left(e^{i\omega_0 t} e^{-\frac{t}{T_2}}\right) = \text{FT}\left[\cos\omega_0 t + i\sin\omega_0 t e^{-\frac{t}{T_2}}\right] = \text{FT}\left(\cos\omega_0 t e^{-\frac{t}{T_2}}\right) + i\text{FT}\left(\sin\omega_0 t e^{-\frac{t}{T_2}}\right)
\]

\[
= \frac{1}{2}\left\{\text{Ab}(-\omega_0) + \text{Ab}(\omega_0) + i\text{Di}(-\omega_0) + i\text{Di}(\omega_0)\right\}
\]

\[
+ \frac{1}{2}\left\{-\text{Di}(-\omega_0) + \text{Di}(\omega_0) + i\text{Ab}(-\omega_0) - i\text{Ab}(\omega_0)\right\}
\]

\[
= \frac{1}{2}\left\{\text{Ab}(-\omega_0) + \text{Ab}(\omega_0) + i\text{Di}(-\omega_0) + i\text{Di}(\omega_0)\right\}
\]

\[
= \frac{1}{2}\left\{\text{Ab}(\omega_0) + i\text{Di}(\omega_0) + i\text{Di}(\omega_0) + \text{Ab}(\omega_0)\right\}
\]

\[
= \text{Ab}(\omega_0) + i\text{Di}(\omega_0)
\]
5 Derive the result for \( FT\left(e^{i(\omega_0^t + \varphi)}e^{-\frac{t}{T^2}}\right)\):

\[
FT\left(e^{i(\omega_0^t + \varphi)}e^{-\frac{t}{T^2}}\right) = FT\left(e^{i\omega_0^t}e^{i\varphi}e^{-\frac{t}{T^2}}\right) = e^{i\varphi}FT\left(e^{i\omega_0^t}e^{-\frac{t}{T^2}}\right)
\]

\[
= [\cos \varphi + i \sin \varphi][Ab(\omega_0) + iDi(\omega_0)]
\]

\[
= \cos \varphi Ab(\omega_0) - \sin \varphi Di(\omega_0) + i \sin \varphi Ab(\omega_0) + i \cos \varphi Di(\omega_0)
\]

with \( \varphi = \pi/2 \) we have \( \cos \varphi = 0 \), \( \sin \varphi = 1 \): \( -Di(\omega_0) + iAb(\omega_0) \)

with \( \varphi = -\pi/2 \) we have \( \cos \varphi = 0 \), \( \sin \varphi = -1 \): \( Di(\omega_0) - iAb(\omega_0) \)

6a Show that

\[
FT\left(\sin \omega_m^t e^{i\omega_0^t} e^{-\frac{t}{T^2}}\right) = -\frac{1}{2}\{Di(\omega_0 - \omega_m) - iAb(\omega_0 - \omega_m)\} + \frac{1}{2}\{Di(\omega_0 + \omega_m) - iAb(\omega_0 + \omega_m)\}
\]

There are two ways to do this. A short one, and a lengthy one.

The lengthy one, using the suggested shorthand notation:

\[
FT\left(\sin \omega_m^t e^{i\omega_0^t} e^{-\frac{t}{T^2}}\right) = FT(\sin \omega_m^t[\cos \omega_0^t + i \sin \omega_0^t]R)
\]

\[
= FT(R \cos \omega_0^t \sin \omega_m^t) + iFT(R \sin \omega_0^t \sin \omega_m^t)
\]

\[
= FT(R \frac{1}{2}[\sin(\omega_0 + \omega_m) - \sin(\omega_0 - \omega_m)]) + iFT(R \frac{1}{2}[\cos(\omega_0 - \omega_m) - \cos(\omega_0 + \omega_m)])
\]

\[
= \frac{1}{4}\left\{FT(R \sin \omega_0^t) - FT(R \sin \omega_0^t) + iFT(R \cos \omega_0^t) - iFT(R \cos \omega_0^t)\right\}
\]

\[
= \frac{1}{2}\left\{-\frac{1}{2}\left[Di(-\omega_0^+) + Di(\omega_0^+) + iAb(-\omega_0^+) - iAb(\omega_0^+)\right]\right\} = \frac{1}{4}\left\{Di(-\omega_0^+)[-1 - i^2] + Di(\omega_0^+)[1 - i^2]\right\}
\]

\[
= \frac{1}{2}\left\{-\frac{1}{2}\left[Di(-\omega_0^-) + Di(\omega_0^-) + iAb(-\omega_0^-) - iAb(\omega_0^-)\right]\right\} + \frac{1}{2}\left\{Di(-\omega_0^-)[1 + i^2] + Di(\omega_0^-)[-1 + i^2]\right\}
\]

\[
= \frac{1}{2}\left\{-\frac{1}{2}\left[Ab(-\omega_0^+) + Ab(\omega_0^+) + iDi(-\omega_0^+) + iDi(\omega_0^+)\right]\right\} = \frac{1}{4}\left\{Ab(-\omega_0^+)[-i + i] + Ab(\omega_0^+)i + i]\right\}
\]

\[
= \frac{1}{4}\left\{2Di(\omega_0^+) - 2iAb(\omega_0^+) - 2Di(\omega_0^-) + 2iAb(\omega_0^-)\right\}
\]

\[
= -\frac{1}{2}\{Di(\omega_0 - \omega_m) - iAb(\omega_0 - \omega_m)\} + \frac{1}{2}\{Di(\omega_0 + \omega_m) - iAb(\omega_0 + \omega_m)\}
\]
The short one:
\[
\text{FT} \left( \sin \omega_m t e^{i\omega_m t} e^{-t T_2} \right) = \text{FT} \left( \frac{e^{i\omega_m t} - e^{-i\omega_m t}}{2i} e^{i\omega_0 t} e^{-t T_2} \right)
\]
\[
= \frac{1}{2i} \text{FT} \left( e^{i\omega_m t} e^{i\omega_0 t} e^{-t T_2} \right) - \frac{1}{2i} \text{FT} \left( e^{-i\omega_m t} e^{i\omega_0 t} e^{-t T_2} \right)
\]
\[
= -\frac{i}{2} \text{FT} \left( e^{i(\omega_0 + \omega_m) t} e^{-t T_2} \right) + \frac{i}{2} \text{FT} \left( e^{i(\omega_0 - \omega_m) t} e^{-t T_2} \right)
\]
\[
= -\frac{i}{2} \left\{ \text{Re} \left( \text{Ab} + i \text{Di} \right) \right\} + \frac{i}{2} \left\{ \text{Re} \left( \text{Ab} - i \text{Di} \right) \right\}
\]

6b A 90° phase correction is equivalent to a multiplication with \(i\), thus we get:
\[
-\frac{i}{2} \left\{ \text{Re} \left( \text{Di}(\omega_0 - \omega_m) - i \text{Ab}(\omega_0 - \omega_m) \right) \right\} + \frac{i}{2} \left\{ \text{Re} \left( \text{Di}(\omega_0 + \omega_m) - i \text{Ab}(\omega_0 + \omega_m) \right) \right\}
\]
\[
= -\frac{i}{2} \left\{ \text{Re} \left( i \text{Di}(\omega_0 - \omega_m) + \text{Ab}(\omega_0 - \omega_m) \right) \right\} + \frac{i}{2} \left\{ \text{Re} \left( i \text{Di}(\omega_0 + \omega_m) + \text{Ab}(\omega_0 + \omega_m) \right) \right\}
\]
\[
= -\frac{i}{2} \left\{ \text{Re} \left( \text{Ab}(\omega_0 - \omega_m) + i \text{Di}(\omega_0 - \omega_m) \right) \right\} + \frac{i}{2} \left\{ \text{Re} \left( \text{Ab}(\omega_0 + \omega_m) + i \text{Di}(\omega_0 + \omega_m) \right) \right\}
\]

7 The Fourier transformation of \(e^{i\omega_m t} \sin \pi J_{AB} t \sin \pi J_{AC} t\) by convolution:
\[
e^{i\omega_m t} \sin \pi J_{AB} t \sin \pi J_{AC} t
\]
\[
e^{i\omega_m t} \sin \pi J_{AB} t
\]
\[
\frac{\text{Re} \left( \text{Ab} + i \text{Di} \right)}{1 \times 2i} = \frac{\text{Ab} + i \text{Di}}{4}
\]

Real part of line shape is -\(\text{Ab}\).
No phase correction needed, but peaks are upside down:
\[
\frac{\text{Re} \left( \text{Ab} + i \text{Di} \right)}{4} = \frac{\text{Ab}}{4}
\]
8 Show that it is mathematically impossible to distinguish between signals at \( \nu_N + \nu \) and 
\(-\nu_N + \nu \), using \( \nu_N = \frac{1}{2\tau} \) and time discretization with \( t = n\tau \).

The question is equivalent to the question: show that \( e^{2\pi i (\nu_N + \nu)t} = e^{2\pi i (-\nu_N + \nu)t} \), which is equivalent to the question: show that \( e^{2\pi i \nu_N t} = e^{-2\pi i \nu_N t} \) for the time discrete version.

From above we see that the time discrete version of \( 2\pi \nu_N t \) is \( 2\pi \frac{1}{2\tau} n\tau = \pi n \). Thus we have to show that \( e^{\pi n} = e^{-\pi n} \). This is easily done:

\[
e^{in\pi} = \cos(\pi n) + i\sin(\pi n) = \cos(\pi n), \quad \text{and} \quad e^{-in\pi} = \cos(-\pi n) + i\sin(-\pi n) = \cos(\pi n)
\]

since \( n \) is an integer number we know that \( \sin(\pi n) = 0 \) thus \( \sin(-\pi n) = \sin(\pi n) = 0 \)

Thus we have shown what we wanted to: a signal that has modulation that is sampled with a too slow sample rate will “roll in” from the negative side.
In this problem set we will look at algebraic transformation rules that describe how the magnetization will change as a function of time.

A transformation expression is really a description of the substitutions that need to be performed in the initial expression as a result of a certain operation. For instance what we did in the previous problem set when we made a complex modulation time discrete, we used the transformation

\[ t_{\text{discrete}} \rightarrow n\tau \]

which means that all occurrences of \( t \) in an expression need to be replaced with \( n\tau \), viz.:

\[ e^{i\omega t_{\text{discrete}}} \rightarrow e^{i\omega n\tau} \]

The time evolution of the state of the spin system is determined by the (spin-) Hamiltonian that is in effect during that period of time. It is customary to divide the transformation rules up into rules that correspond to operators in the Hamiltonian, i.e. the Zeeman term for chemical shift evolution, and the coupling term for coupling evolution. Transformations that describe what happens during an r.f. pulse are separated too, since they work in different time domains than the two evolutions. The Hamiltonian for the a coupled spin system is given by the sum of the Zeeman term and the coupling term for the spins \( k \) and \( l \) present:

\[
H = \sum_k -\gamma_k (1-\sigma_k) \mathbf{B} \cdot \mathbf{l}_k + \frac{2\pi}{\hbar} \sum_{k<l} J_{k,l} \mathbf{l}_k \cdot \mathbf{l}_l
\]

For weakly coupled spin \( I=1/2 \) systems only the z-components of the scalar product of the spins vectors needs to be considered. Thus with having the external field along z, we get for the Hamiltonian operator (remember \( \hat{I} = \hat{\hbar} \hat{I} \)):

\[
H = \sum_k \omega_k \hbar \mathbf{l}_k z + \frac{2\pi \hbar^2}{\hbar} \sum_{k<l} J_{k,l} \mathbf{l}_k z \mathbf{l}_l z = \sum_k \omega_k \hbar \mathbf{l}_k z + 2\pi \hbar \sum_{k<l} J_{k,l} \mathbf{l}_k z \mathbf{l}_l z
\]

\[
\frac{H}{\hbar} = \sum_k \omega_k \mathbf{l}_k z + 2\pi \sum_{k<l} J_{k,l} \mathbf{l}_k z \mathbf{l}_l z
\]

with \( \omega_k = -\gamma_k (1-\sigma_k) B_z \). Notice that the hats on the operators are dropped to keep the notation more simple.

If during a certain period of time more than one of these evolutions occur, we can get the total result by performing the transformations sequentially. This is possible because the time dependent Schrödinger equation gives for the wavefunction as a function of time:

\[
\psi(t) = e^{-\frac{i}{\hbar} \frac{H}{\hbar} t} \psi(0)
\]
And the terms in the Hamiltonian can be split out in separate exponentials.

For the net magnetization caused by spins $k$ projected on the Cartesian axes, $x$, $y$, and $z$, we will use the symbols $I_{kx}$, $I_{ky}$, and $I_{kz}$ respectively, since this is the notation used in the product operator formalism, where it stands for the spin operators in stead of the net magnetization.

The equilibrium state of a spin system is given by the net magnetization of each spin that points along the $+z$ axis, i.e., $\sum_k I_{kz}$. For instance in a CH spin system that would be $I_{Cz} + I_{Hz}$.

We know that the magnetization will spin around a magnetic field, so we need to define the direction of positive rotation around any Cartesian axis. This is given by looking down the rotation axis to the origin and rotating counter clock wise, viz.

1. **Pulses along Cartesian axes**

   From the picture of positive rotations it is easy to see that the transformation of $x$ and $y$ magnetization during a pulse on spin $k$ with tip angle $\beta$ along the $z$-axis, i.e. a positive rotation of magnetization of spins $k$ over an angle $\beta$ around $z$ will turn $I_{kx}$ into $I_{kx} \cos \beta + I_{ky} \sin \beta$ and $I_{ky}$ into $I_{ky} \cos \beta - I_{kx} \sin \beta$. This is usually written in a short hand notation as follows, where $\beta_{kz}$ stands for the $\beta$ $z$-pulse on spins $k$:

   $$I_{kx,ky} \xrightarrow{\beta_{kz}} I_{kx,ky} \cos \beta \pm I_{ky,kx} \sin \beta$$

   The $z$ part of the magnetization will of course not change with a $z$ pulse.

   a) Derive analogously the transformation rules for the $\beta$ $x$-pulse (on $y$ and $z$ magnetization) and $\beta$ $y$ pulse (on $x$ and $z$ magnetization), i.e.,

   $$I_{ky,kz} \xrightarrow{\beta_{kx}}$$

   $$I_{kx,kz} \xrightarrow{\beta_{ky}}$$
b) Using your results, write the effect of a $\pi/2$ $x$ pulse on $H$ in a CH spin system that initially is in equilibrium, i.e.,

$$I_{Cz} + I_{Hz} \xrightarrow{\frac{\pi}{2} H_x}$$

2 Chemical Shift Evolution

The notation used for the chemical evolution of spin $k$ for a time domain, $t$, is the product of the Zeeman term for that spin with the time domain, i.e., $\omega_k t I_{kz}$. The rotation angle is now a function of time, and the axis around which the rotation occurs is off course the z-axis. This makes that the chemical shift evolution is equivalent in its transformation to a z-pulse with this time dependent angle. Give the transformation rule for the chemical shift evolution of initial $x$ and $y$ magnetizations:

$$I_{kx} \xrightarrow{\omega_k t I_{kz}}$$

$$I_{ky} \xrightarrow{\omega_k t I_{kz}}$$

3 Coupling Evolution

The Hamiltonian term for coupling between spins $k$ and $l$ is given by $2\pi J_{k,l} I_{kz} I_{lz}$, and the notation for the coupling evolution for a time $t$ is thus $\pi J_{k,l} t^2 I_{kz} I_{lz}$ (we will see later why the 2 is put immediately before the spin operators).

In the vector model we can not, initially, separate out the chemical shift and the coupling evolution. In this model a coupling between spins $k$ and $l$ will give rise to two resonances for spin $k$ at $\nu = \nu_k \pm J_{k,l}/2$ or equivalently $\omega = 2\pi \nu = \omega_k \pm \pi J_{k,l}$, each with half the original intensity. These two signals will evolve separately, and we can write for the shift and coupling evolution:

$$I_{kx} \xrightarrow{\omega_k t I_{kz} + \pi J_{k,l} t 2 I_{kz} I_{lz}} \frac{1}{2} I_{kx} \cos(\omega_k t + \pi J_{k,l} t) + \frac{1}{2} I_{ky} \sin(\omega_k t + \pi J_{k,l} t)$$

or, with the spin operators combined:

$$I_{kx} \xrightarrow{\omega_k t I_{kz} + \pi J_{k,l} t 2 I_{kz} I_{lz}} \frac{1}{2} I_{kx} \left[ \cos(\omega_k t + \pi J_{k,l} t) + \cos(\omega_k t - \pi J_{k,l} t) \right] + \frac{1}{2} I_{ky} \left[ \sin(\omega_k t + \pi J_{k,l} t) + \sin(\omega_k t - \pi J_{k,l} t) \right]$$

a) show that this is equivalent to

$$I_{kx} \xrightarrow{\omega_k t I_{kz} + \pi J_{k,l} t 2 I_{kz} I_{lz}} \left[ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \right] \cos(\pi J_{k,l} t)$$

It is useful to determine how $\cos(\alpha + \beta) + \cos(\alpha - \beta)$ and $\sin(\alpha + \beta) + \sin(\alpha - \beta)$ can be rewritten, so you can use that immediately for b) too.

b) derive the result of the chemical shift and coupling evolution of $I_{ky}$ with the shift and coupling terms separated out as in question 3a.
From these two equations we see that we can separate the coupling evolution from the chemical shift evolution and can be written as:

\[ I_{kx,ky} \frac{\pi J_{k,l} t^2 I_{kz} I_{lz}}{I_{kx,ky}} \cos(\pi J_{k,l} t) \]

Notice that at the same time transversal spin operators of spin l will evolve too, since you can switch k and l around in the expression and still have the same transformation term over the arrow. So make sure that you don’t forget to do the transformation of the coupling between spins k and l for every transversal term of both spins k and l at the same time, e.g.,

\[ I_{kx} + I_{ly} \frac{\pi J_{k,l} t^2 I_{kz} I_{lz}}{I_{kx} \cos(\pi J_{k,l} t) + I_{ly} \cos(\pi J_{k,l} t)} \]

c) Show now that doing the coupling evolution and chemical shift evolutions can be performed in sequence to give the above stated result (in 3a) for the evolution of the shift and coupling evolution together. Do this only for \( I_{kx} \), i.e., show that

\[ I_{kx} \frac{\omega_k t I_{kz}}{I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t)} \cos(\pi J_{k,l} t) \]

4 Observation

During the observation time domain, both the chemical shift and coupling will evolve. The operation of turning the spin state expression into an observed signal can also be expressed in transformation rules. The signal will be proportional to the terms that only have either x or y magnetization terms present, thus in the case that we have quadrature detection we have to substitute a 1 for the \( I_{kx} \) terms, the complex number i for the \( I_{ky} \) terms, and 0 for the \( I_{kz} \) terms. The obtained expression can then be Fourier transformed. It is often customary to translate the peak positions from rad/s into Hz, which can be done by dividing the peak positions with the factor \( 2\pi \).

a) With all these transformation rules, you can now calculate the resulting signal obtained by quadrature detection in the following pulse experiment:

\[ H: \left( \frac{\pi}{2} \right)_y \text{ FID} \]

The spin system we are interested in consists of two equivalent protons, \( H_{ka} \) and \( H_{kb} \), that are coupled to another single proton, \( H_1 \).

N.B. don’t forget that you will have now two coupling evolutions, \( J_{ka,l} = J_{kb,l} = J \). \( H_{ka} \) and \( H_{kb} \) will not couple with each other and they will have the same chemical shift, \( \omega_k \). Use \( \omega_l \) for the chemical shift of spin \( H_1 \).

Written in the exponential form your observed signal should be:

\[ 2e^{i\omega_k t} \cos(\pi J t) + e^{i\omega_l t} \cos^2(\pi J t) \]

b) Using the convolution method, determine the peak pattern in the spectrum. Show a schematic picture of the real part of the spectrum, with symbolic peak positions in Hz and labels for the height of each peak. Is a phase correction required to obtain your peaks in absorption mode?
You should have noticed that when you determine the peak pattern for the signal with the carrier frequency of $\omega_1$, you did the same thing as when you determine the peak multiplicity by building up the pattern coupling by coupling (1, 1:1, 1:2:1, 1:3:3:1, etc.).
Product Operator Transformation rules

We already saw that the time evolution of our system has something to do with the solution to the time dependent Schrödinger equation. This only worked for a single system. For an ensemble of systems we really need what is called the density operator, \( \hat{\sigma}(t) \). The time evolution of our density operator can be derived from the time dependent Schrödinger equation, since it’s description is based on wavefunctions, and is given by

\[
\frac{\partial}{\partial t} \hat{\sigma}(t) = -i \left[ \hat{H}, \hat{\sigma}(t) \right] = -i \frac{\hat{H}}{\hbar} \hat{\sigma}(t)
\]

where we put in that we have a time independent Hamiltonian \( \hat{H} \), and we introduced \( \hat{H} \) as the so-called superoperator commutator of the Hamiltonian. A commutator superoperator is just an operator that works on another operator and gives the result of the commutator of the two operators, i.e.,

\[
\hat{A}\hat{B} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}
\]

The result is another operator if the two don’t commute, or 0 if they do commute.

The differential equation above has the solution:

\[
\hat{\sigma}(t) = e^{-i \frac{\hat{H}}{\hbar} t} \hat{\sigma}(0) = \sum_n \frac{\left(-i \frac{\hat{H}}{\hbar} t\right)^n}{n!} \hat{\sigma}(0)
\]

where \( \hat{\sigma}(0) \) is the operator expression that gives us the state of our spin system at the beginning of the time domain in which the time independent Hamiltonian describes the evolution of the spin state. In order to evaluate this expression, one needs to re-express the exponential form into a series. This way a power series of the Hamiltonian commutator superoperator is generated.

In the case of spin operators life isn’t too hard, since they have a property called cyclic commutivity. This means that performing the commutator of a spin operator an even number of times on another spin operator, with which it does not commute, you get the original operator back. To see this, we need the commutator rules for spin operators

\[
[\hat{I}_x, \hat{I}_y] = i \hat{I}_z, \quad [\hat{I}_y, \hat{I}_z] = i \hat{I}_x, \quad \text{and} \quad [\hat{I}_z, \hat{I}_x] = i \hat{I}_y
\]

The Hamiltonian operators are z-spin operators only so we will mainly be interested in

\[
\hat{I}_z \hat{I}_x = [\hat{I}_z, \hat{I}_x] = i \hat{I}_y \quad \hat{I}_z \hat{I}_y = [\hat{I}_z, \hat{I}_y] = -[\hat{I}_y, \hat{I}_z] = -i \hat{I}_x
\]
So what happens if we have the commutator superoperator work on a spin operator, with which it does not commute, two times (or in general an even number of times)? We will look at one particular example:

\[\hat{I}_z^2 \hat{I}_x = \hat{I}_z \hat{I}_x \hat{I}_z = \hat{I}_x \hat{I}_z \hat{I}_x = i[\hat{I}_z, \hat{I}_y] = -i[\hat{I}_y, \hat{I}_z] = -i^2 \hat{I}_x = \hat{I}_x\]

So it is easy to see that if the operators do not commute, the complex exponential will split in an even, cosine, function with the original operator that the Hamiltonian worked on, and a -i sine function that still has the commutator in it.

In general one can write both the Hamiltonian and the density operator as a linear combination of cyclic commutating spin operators that can be considered to form an orthogonal basis set of operators, \(\hat{O}\).

Thus in general for the k-th term in the Hamiltonian, \(\frac{\hat{H}_k}{\hbar} = c_k \hat{O}_k\), working on the operator \(\hat{O}_s\), with which \(\hat{O}_k\) doesn’t commute, in the density operator expression for a period of time, t, we have

\[e^{-ic_k \hat{O}_k t} \hat{O}_s = \hat{O}_s \cos(c_k t) - i \left[ \hat{O}_k, \hat{O}_s \right] \sin(c_k t)\]

If the operators do commute, the result of the commutator is 0, and only the first term (n=0) in the series will remain, thus the operator will remain constant.

\[e^{-ic_k \hat{O}_k t} \hat{O}_k = \hat{O}_k + \sum_{n=1}^{\infty} \left( -ic_k \hat{O}_k t \right)^n = \hat{O}_k + \sum_{n=1}^{\infty} \left( -ic_k t \right)^n \hat{O}_k \hat{O}_k = \hat{O}_k\]

where we used the fact that \(\hat{O}_k \hat{O}_k = 0\) and as a result of that we have \(\hat{O}_k^n \hat{O}_k = 0\)

Thus we can write the combination of the two equations we need in one big swoop:

\[e^{-ic_k \hat{O}_k t} \hat{O}_1 = \begin{cases} \hat{O}_1 & \hat{O}_k \hat{O}_1 = 0 \\ \hat{O}_1 \cos(c_k t) - i \hat{O}_k \hat{O}_1 \sin(c_k t) & \hat{O}_k \hat{O}_1 \neq 0 \text{ and } \hat{O}_k^2 \hat{O}_1 = \hat{O}_1 \end{cases}\]

Determining the result of the above two equations for evolution in a time domain, using a set of orthogonal spin operators, you can derive the product operator transformation rules.

In order for the cyclic commutivity to work out properly for products of operators, our basis set of operators for a system consisting of N spin I=1/2 spins, is defined as:

\[\hat{O}_p = 2^{(n-1)} \prod_{k=1}^{N} (\hat{I}_{k\alpha})^{\epsilon_{kp}}\]
where \( n \) is the number of Cartesian spin operators present, \( \alpha = x, y, \) or \( z, \) and \( c_{kp} \) is 1 for \( n \) of the spins, and 0 for the rest. From the expression we see that our basis set will consist of products of spin operators, hence the name product operator formalism.

The basis set seems very complicated but for a single operator \( \hat{I}_{1x} \) we get \( 2^{(1-1)} \hat{I}_{1x} = \hat{I}_{1x} \).
For an operator product of two spins, \( \hat{I}_{1x} \) and \( \hat{I}_{2y} \), we get \( 2^{(2-1)} \hat{I}_{1x} \hat{I}_{2y} = 2 \hat{I}_{1x} \hat{I}_{2y} \). Thus the reason why we put the 2 directly in front of the operator in the coupling evolution notation, is because the operator in our basis set is \( 2 \hat{I}_{1x} \hat{I}_{2y} \), and not \( \hat{I}_{1x} \hat{I}_{2y} \).

Let’s determine the result of the coupling term in the Hamiltonian, \( 2\pi\hat{I}_{k,kz}\hat{I}_{lz} \), on a spin that it can work on, e.g., \( \hat{I}_{kx} \):

\[
e^{-i2\pi J_{k,l}t_{kz}\hat{I}_{lz}}\hat{I}_{kx} = e^{-i\pi J_{k,l}t(2\hat{I}_{kz}\hat{I}_{lz})}\hat{I}_{kx} = \hat{I}_{kx} \cos(\pi J_{k,l}t) - i[2\hat{I}_{kz}\hat{I}_{lz},\hat{I}_{kx}]\sin(\pi J_{k,l}t)
\]

\[
= \hat{I}_{kx} \cos(\pi J_{k,l}t) - 2i[\hat{I}_{kz},\hat{I}_{kx}]\hat{I}_{lz}\sin(\pi J_{k,l}t)
\]

\[
= \hat{I}_{kx} \cos(\pi J_{k,l}t) - 2i\hat{I}_{ky}\hat{I}_{lz}\sin(\pi J_{k,l}t)
\]

Thus in transformation notation, and dropping the hats, we get:

\[
I_{kx} \xrightarrow{\pi J_{k,l}t2\hat{I}_{kz}\hat{I}_{lz}} I_{kx} \cos(\pi J_{k,l}t) + 2I_{ky}\hat{I}_{lz}\sin(\pi J_{k,l}t)
\]

The transformation rules according to the product operator formalism are as follows.

- **pulse**: \( I_{kz} \xrightarrow{\beta_{ky,kx}} I_{kz} \cos \beta \pm I_{kx,kx} \sin \beta \)
- **pulse**: \( I_{kx,ky} \xrightarrow{\beta_{ky,kx}} I_{kx,ky} \cos \beta \mp I_{kz} \sin \beta \)
- **shift**: \( I_{kx,ky} \xrightarrow{\omega_{k}t_{kz}} I_{kx,ky} \cos(\omega_{k}t) \pm I_{kx,kx} \sin(\omega_{k}t) \)
- **coupling**: \( I_{kx,ky} \xrightarrow{\pi J_{k,l}t2\hat{I}_{kz}\hat{I}_{lz}} I_{kx,ky} \cos(\pi J_{k,l}t) \pm 2I_{ky,kx} I_{lz} \sin(\pi J_{k,l}t) \)

Notice again, that at the same time transversal spin operators of spin \( l \) will evolve too. So make sure that you don’t forget to do the transformation of the coupling between spins \( k \) and \( l \) for every transversal term of both spins \( k \) and \( l \) at the same time.

Notice also that these are exactly the same as for the vector model, except for the 2nd term in the coupling evolution. This term consists of a product of spin operators.

Because we have products of operators there are a few more coupling transformations to consider. One for the combination of transversal with longitudinal operators, and for the combination of two transversal operators.
\[ 2I_{kx,ky}I_{lz} \xrightarrow{\pi J_{k,l} I_{lz}} 2I_{kx,ky} \cos(\pi J_{k,l} t) \pm I_{ky,kx} \sin(\pi J_{k,l} t) \]

\[ 2I_{kx,ky,kx,ly,ly,lx} \xrightarrow{\pi J_{k,l} I_{lz}} 2I_{kx,ky,kx,ly,ly,lx} \]

The last one can more easily be put in words as: terms that have two transversal operators for the spins that couple will not change during that coupling evolution.

One important thing to remember is that when performing the transformation of a term, the operators for spins other than the ones present in the transformation are considered to be constant, e.g., in the following chemical shift evolution of spin k, the operator for spin l is a constant:

\[ 2I_{kx}I_{ly} \xrightarrow{\omega_k t I_{kz}} 2I_{kx}I_{ly} \cos(\omega_k t) + 2I_{ky}I_{ly} \sin(\omega_k t) \]

There is a pictorial way to remember these transformation rules, by considering the subspaces through which the Hamiltonian terms will rotate the operators.

Another thing to notice, and keep in mind, is that the number of transversal spin operators will not change during the evaluation of coupling and shift. This is particularly useful to quickly determine which terms will contribute to the observed signal in the observation time domain. The number of transversal spin operators can only be affected by pulses!

Only terms that have a single transversal spin operator can be observed. When at the beginning of the observation time domain, terms are present that consist of a product of a single transversal spin operator with longitudinal spin operators, they can be transformed into observable magnetization if couplings exists between the transversal spin and every
longitudinal spin present in that term. The coupling evolution will turn such a term into an observable operator that has products of the sine modulations of all of these coupling constants, e.g.,

\[
4I_{kx}I_{lz}I_{nmz} \rightarrow 4I_{kx}I_{lz}I_{mz} - \frac{\pi J_{k,m}I_{mz}}{\pi J_{k,l}I_{lz}} \rightarrow 4I_{kx}I_{lz}I_{mz} - \frac{\pi J_{k,l}I_{lz}}{\pi J_{k,m}I_{mz}} \rightarrow \frac{\pi J_{k,l}I_{lz}}{2I_{kz}I_{mz}} \cos(\pi J_{k,l}t) \cos(\pi J_{k,m}t) + \frac{\pi J_{k,l}I_{lz}}{2I_{kz}I_{mz}} \sin(\pi J_{k,l}t) \sin(\pi J_{k,m}t)
\]

coupled spin evolution still needs to be considered, but you can already see that only the last term will give rise to an observable signal.

5. It is clear that one cannot directly observe terms that consist of a product of transversal spin operators. It is possible for this kind of terms to be turned into observable signals. Consider the homonuclear case of two spins \( k \) and \( l \) that are coupled.

a) Show what the result if the term \( 2I_{kx}I_{ly} \) receives a “read-pulse” of \( \pi/2 \) along \( x \), on both spins after which it evolves both the shift and coupling for a time period of length \( t \). In other words what is the result of:

\[
2I_{kx}I_{ly} \rightarrow \frac{\pi J_{k,l}I_{lz}}{2I_{kz}I_{mz}} \cos(\pi J_{k,l}t) \cos(\pi J_{k,m}t) + \frac{\pi J_{k,l}I_{lz}}{2I_{kz}I_{mz}} \sin(\pi J_{k,l}t) \sin(\pi J_{k,m}t)
\]

b) Using the convolution method, determine the peak pattern in the real part spectrum, if it were observed in the time domain \( t \). Show a schematic picture of the real part of the spectrum, with symbolic peak positions in Hz and labels for the height of each peak. Is a phase correction required to obtain your peaks in absorption mode?

In order to make the convolution even more simple, one often only looks at the real, or imaginary part of the observed signal to begin with. The assumption is then that it is possible to determine the absolute sign of the carrier frequency, usually the chemical shift term in the observation time domain. from the fact that quadrature detection was performed. This reduces the amount of terms to look at by a factor of two. You also don't need to re-express the observed signal into rotations in the observation time domain, using the exponential expressions.
1 Pulses along Cartesian axes

\[ I_{kx,kz} \xrightarrow{\beta_{kx}} I_{kx,kz} \cos \beta \pm I_{kz,ky} \sin \beta \]

a)

\[ I_{kx,kz} \xrightarrow{\beta_{ky}} I_{kx,kz} \cos \beta \mp I_{kz,ky} \sin \beta \]

b)

\[ I_{Cz} + I_{Hz} \xrightarrow{\frac{\pi}{2}H_k} I_{Cz} - I_{Hy} \]

2 Chemical shift evolution

\[ I_{kx} \xrightarrow{\omega_k I_{kz}} I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \]

\[ I_{ky} \xrightarrow{\omega_k I_{kz}} I_{ky} \cos(\omega_k t) - I_{kx} \sin(\omega_k t) \]

3 Coupling evolution

First we will use the trigonometric equalities from problem set 1 to derive:

\[ \cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta + \cos \alpha \cos \beta + \sin \alpha \sin \beta = 2 \cos \alpha \cos \beta \]

\[ \sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta = 2 \sin \alpha \cos \beta \]

a) show that

\[ I_{kx} \xrightarrow{\omega_k I_{kz} + \pi J_{k,l} I_{kz} I_{kz}} \{ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]

\[ I_{kx} \xrightarrow{\omega_k I_{kz} + \pi J_{k,l} I_{kz} I_{kz}} \]

\[ \frac{1}{2} I_{kx} \left[ \cos(\omega_k t + \pi J_{k,l} t) + \cos(\omega_k t - \pi J_{k,l} t) \right] + \frac{1}{2} I_{ky} \left[ \sin(\omega_k t + \pi J_{k,l} t) + \sin(\omega_k t - \pi J_{k,l} t) \right] = \]

\[ \frac{1}{2} I_{kx} \left[ 2 \cos(\omega_k t) \cos(\pi J_{k,l} t) \right] + \frac{1}{2} I_{ky} \left[ 2 \sin(\omega_k t) \cos(\pi J_{k,l} t) \right] = \]

\[ \{ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]

b) the result for \( I_{ky} : \)

\[ I_{ky} \xrightarrow{\omega_k I_{kz} + \pi J_{k,l} I_{kz} I_{kz}} \]

\[ \frac{1}{2} I_{ky} \left[ \cos(\omega_k t + \pi J_{k,l} t) + \cos(\omega_k t - \pi J_{k,l} t) \right] - \frac{1}{2} I_{kx} \left[ \sin(\omega_k t + \pi J_{k,l} t) + \sin(\omega_k t - \pi J_{k,l} t) \right] = \]

\[ \frac{1}{2} I_{ky} \left[ 2 \cos(\omega_k t) \cos(\pi J_{k,l} t) \right] - \frac{1}{2} I_{kx} \left[ 2 \sin(\omega_k t) \cos(\pi J_{k,l} t) \right] = \]

\[ \{ I_{ky} \cos(\omega_k t) - I_{kx} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]

c) show that

\[ I_{kx} \xrightarrow{\omega_k I_{kz}} \pi J_{k,l} I_{kz} I_{kz} \{ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]

\[ I_{kx} \xrightarrow{\omega_k I_{kz}} I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \]

\[ \xrightarrow{\pi J_{k,l} I_{kz} I_{kz}} \{ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]

\[ \{ I_{kx} \cos(\omega_k t) + I_{ky} \sin(\omega_k t) \} \cos(\pi J_{k,l} t) \]
4 Observation

a) system containing equivalent protons $H_{ka}$ and $H_{kb}$, and $H_1$, with pulse sequence

$$H: \left( \frac{\pi}{2} \right)_y \text{ FID}$$

This means that we will have to do the transformation of the pulse first, followed by the transformations for the coupling between $H_{ka}$ and $H_1$, coupling between $H_{kb}$ and $H_1$, and the chemical shift evolutions of each of the spins. This gives rise to the following spin state:

$$I_{ka} + I_{kb} + I_{lz} \xrightarrow{\left( \frac{\pi}{2} \right)_y} I_{kax} + I_{kbx} + I_{lx}$$

$$\pi J_{ka,lt} 2I_{ka,lz} \xrightarrow{\pi J_{ka,lt} 2I_{ka,lz}} I_{kax} \cos(\pi Jt) + I_{kbx} \cos(\pi Jt) + I_{lx} \cos(\pi Jt)$$

$$\omega_{ka} I_{kb} \xrightarrow{\omega_{ka} I_{kb}} I_{kax} \cos(\omega_k t) \cos(\pi Jt) + I_{kax} \sin(\omega_k t) \cos(\pi Jt) + I_{kbx} \cos(\pi Jt) + I_{lx} \cos(\pi Jt) \cos(\pi Jt)$$

Observation will turn the $x$ part in 1, $y$ parts in $i$ so that we get for the signal:

$$\cos(\omega_k t) \cos(\pi Jt) + i \sin(\omega_k t) \cos(\pi Jt) + \cos(\omega_k t) \cos(\pi Jt) + i \sin(\omega_k t) \cos(\pi Jt)$$

$$+ \cos(\omega_1 t) \cos^2(\pi Jt) + i \sin(\omega_1 t) \cos^2(\pi Jt)$$

$$= 2(\cos(\omega_k t) + i \sin(\omega_k t)) \cos(\pi Jt) + (\cos(\omega_1 t) + i \sin(\omega_1 t)) \cos^2(\pi Jt)$$

$$= 2 e^{i \omega_k t} \cos(\pi Jt) + e^{i \omega_1 t} \cos^2(\pi Jt)$$

b) convolutions shows that this signal will give rise to the following patterns in the spectrum:

$$2 e^{i \omega_k t} \cos(\pi Jt)$$

$$e^{i \omega_1 t} \cos^2(\pi Jt)$$

No phase correction required, both patterns have absorption line shapes in the real part of the spectrum.
5  a) what is the result of \[2I_{kk}I_{ly} \xrightarrow{\frac{\pi}{2}k_x + \frac{\pi}{2}l_x} \omega_k t_{kz} + \omega_l t_{lz} \xrightarrow{\pi J_{k,l} t_{kz}} \]?

\[
2I_{kk}I_{ly} \xrightarrow{\frac{\pi}{2}k_x + \frac{\pi}{2}l_x} 2I_{kk}I_{lz} \]
\[
\omega_k t_{kz} + \omega_l t_{lz} \xrightarrow{\pi J_{k,l} t_{kz}} \left\{ 2I_{kk}I_{lz} \cos(\pi J_{k,l} t) + I_{ky} \sin(\pi J_{k,l} t) \right\} \cos(\omega_k t) \]
\[+ \{ 2I_{ky}I_{lz} \cos(\pi J_{k,l} t) - I_{kx} \sin(\pi J_{k,l} t) \} \sin(\omega_k t) \]

b) the peak pattern in the real part of the spectrum can be obtained by convolution, if we first make sure that we get the observed signal. Notice that only the second terms between the \{ \} is observable, i.e., contains only a single x or y spin operator. The signal is thus:

\[-\sin(\pi J_{k,l} t) \sin(\omega_k t) + i \sin(\pi J_{k,l} t) \cos(\omega_k t)\]

We should rewrite that in the complex exponential form as follows:

\[-\sin(\omega_k t) + i \cos(\omega_k t) \right\} \sin(\pi J_{k,l} t) = i \left\{ \cos(\omega_k t) + i \sin(\omega_k t) \right\} \sin(\pi J_{k,l} t) =

\[i e^{i\omega_k t} \sin(\pi J_{k,l} t)\]

and convolution gives that the pattern is:

\[i e^{i\omega_k t} \sin(\pi J_{k,l} t)\]

\[e^{i\omega_k t}\]

\[\sin(\pi J_{k,l} t)\]

\[\nu_k + \frac{1}{2}\]

\[\nu_k - \frac{1}{2}\]

\[i \frac{A_b + i D_i}{2} = \frac{A_b + i D_i}{2}\]

The real part of the spectrum has absorption line shapes, so no phase correction is required.
Now that we have products of operators in our spin state expressions we can talk a little about quantum coherences.

As you may know, there are also two step operators, the “raising” and “lowering” operators, $I_+$ and $I_-$, respectively. These will raise or lower the $m_I$ value of a particular spin state, thus they can cause transitions. They can be expressed in Cartesian spin operators as follows:

$$I_+ = I_x + iI_y$$
$$I_- = I_x - iI_y$$

This means that we can also write our transversal operators in step operators:

$$I_x = \frac{1}{2}(I_+ + I_-)$$
$$I_y = \frac{1}{2i}(I_+ - I_-) = -\frac{1}{2}(I_+ - I_-)$$

The coherence order of a operator product is just the sum of the $I_+$ operators minus the sum of the $I_-$ operators that are in the product for any spin, e.g.,

$$I_{kx}I_{ly}I_{mz} = \frac{1}{2}(I_{k+} + I_{k-})\frac{1}{2i}(I_{l+} - I_{l-})I_{mz}$$
$$= \frac{1}{4i}(I_{k+}I_{l+}I_{mz} - I_{k+}I_{l-}I_{mz} + I_{k-}I_{l+}I_{mz} - I_{k-}I_{l-}I_{mz})$$

From left to right this gives \{1+1=2\}, \{1-1=0\}, \{-1+1=0\}, \{-1-1=-2\}

Thus this term has quantum coherence orders of \{2,0,-2\} present. One also talks about these terms as follows: the first and fourth term are double quantum coherences \{-2, 2\} and the other two are zero quantum coherences.

1. Consider the term $4I_{kx}I_{ly}I_{mz}$, where $k$, $l$, and $m$ are homo nuclear spins.

   a) What quantum coherence orders are present in this term? Work this term out into the step operator representation and label each spin operator product term with it’s coherence order.

   b) What can you do to turn this term into one that subsequently will evolve into observable magnetization?
It is possible, by using particular sequences of phases in a pulse sequence, to filter out only terms with a specific quantum coherence order. This is done, for instance, in double quantum filtered correlation spectra (DQF-COSY).

2. Show that performing a double quantum filtering on \( 2I_{kx}I_{ly} \) is equivalent to the following transformation:

\[
2I_{kx}I_{ly} \xrightarrow{\text{DQF}} \frac{1}{2} \left( 2I_{kx}I_{ly} + 2I_{ky}I_{lx} \right)
\]

In order to do that you will have rewrite the operator into step operator notation, select only terms that quantum coherence of +2 and -2, and rewrite these again in Cartesian spin operators.

3. The DQF-COSY pulse sequence is as follows:

\[
\begin{array}{ccc}
\frac{\pi}{2_x}, & t_1, & \frac{\pi}{2_x}, \text{DQF}, \frac{\pi}{2_x}, \text{FID}, \ t_2
\end{array}
\]

In which the DQF stands for a theoretical time period (length 0) in which all quantum coherences that are not of the order of ±2 are eliminated. (In reality the DQF is achieved by the phase cycle that the second \( \pi/2 \) pulse is involved in, and a short constant time interval is used to have the double quantum coherences ‘build up’).

Notice that the \( \pi/2 \) x-pulse that follows is needed to turn these double quantum coherences into observable single quantum coherences.

a. Work out, for a homo nuclear two spin system \( k \) and \( l \), what the effect is of this sequence if we started out with only \( I_{kz} \). This will make that we can only get information about peaks that have the chemical shift modulation for spin \( k \) in time domain \( t_1 \), but we still can figure out the result for the cross-peak and diagonal peak. In order to keep things a little easier on yourself, work out a spin operator tree, as you’ve seen for the INEPT and DEPT experiments for a CH\(_2\) spin system.

b. Perform the two dimensional equivalent of the convolution technique to determine the peak shapes and patterns for the diagonal and off-diagonal peaks. Determine the line-shape of the cross peak if the diagonal peaks are pure absorption.

4. Maple tells us that the modulation that gives rise to a cross-peak between spins \( A \) and \( B \) in a homo nuclear spin system of three spins, ABC, in a COSY experiment is given by

\[
\cos \omega_A t_1 \cos \pi J_{A,C} t_1 \sin \pi J_{A,B} t_1 \sin \omega_B t_2 \cos \pi J_{B,C} t_2 \sin \pi J_{A,B} t_2
\]

Assuming that phase correction has been performed correctly to get pure absorption peaks, i.e. using the chemical shifts as the carriers, draw the peak pattern this gives rise to. In your drawing use that \( J_{A,B} > J_{B,C} > J_{A,C} \).

You should have noticed that the cross peaks show a sine modulation of the homo nuclear coupling of the spins that it has the chemical shifts for (\( J_{A,B} \)). This coupling is called the active coupling. If it were not present the cross peak would not exist, since the whole odd pattern would collapse on top of it’s center thus canceling everything out. The other homo
nuclear couplings that are present in the cross peaks are passive couplings and occur as cosine modulations ($J_{A,C}$ and $J_{B,C}$ above). Absence of these couplings ($J=0$) would not make the cross peak disappear.
1a Label that quantum coherence orders present in the term $I_{xx}I_{ly}I_{mx}$:

$$I_{xx}I_{ly}I_{mx} = \frac{1}{2}(I_{k+} + I_{k-})I_{l+}I_{m+} + I_{l+}I_{m-} - I_{l-}I_{m+} - I_{l-}I_{m-}$$

$$= \frac{1}{8i}(I_{k+} + I_{k-})(I_{l+}I_{m+} + I_{l+}I_{m-} - I_{l-}I_{m+} - I_{l-}I_{m-})$$

$$= \frac{1}{8i}\left(I_{k+}I_{l+}I_{m+} + I_{k+}I_{l+}I_{m-} - I_{k+}I_{l-}I_{m+} - I_{k+}I_{l-}I_{m-} + I_{k-}I_{l+}I_{m+} - I_{k-}I_{l+}I_{m-} - I_{k-}I_{l-}I_{m+} - I_{k-}I_{l-}I_{m-}\right)$$

Thus the coherence orders are (top left to bottom right) 3, 1, 1, -1, 1, -1, -1, -3.

So the quantum coherences of orders {-3,-1,1,3} are present, and the operator term is a combination of single and triple quantum coherences.

Notice that the highest quantum coherence present is just the total number of transversal spin operators, the other coherences present differ from that in multiples of 2.

1b In order to turn the term into something that can evolve into observable magnetization, we need to turn it into a single quantum coherence. The coupling will then make sure that the $I_j$ terms will vanish for some of them. To turn it into a single quantum coherence we can give a $\pi/2$ pulse on all of the spins (homo nuclear set of spins). The axis along which the pulse must be given is the $y$-axis:

$$4I_{xx}I_{ly}I_{mx} \xrightarrow{\frac{\pi}{2}ky.ly.my} 4I_{xx}I_{ly}I_{mx}$$

The coupling between $k$ and $l$, and the one between $m$ and $l$ makes that the sine parts of those transformation will loose the $z$ spin operators. Only a transversal operator for spin $y$ will remain, viz., focusing on the sine parts only

$$4I_{xx}I_{ly}I_{mx} \xrightarrow{\pi I_{kl}t2I_{lx}I_{lz}} -2I_{lx}I_{mx} \sin(\pi I_{kl}t) - \pi I_{lm}t2I_{lx}I_{nz} \xrightarrow{I_{ly} \sin(\pi I_{kl}t) \sin(\pi I_{lm}t)}$$

This term will remain observable when the shift also evolves in time domain $t$.

2 Show that double quantum filtering of $2I_{xx}I_{ly}$ is equivalent to the transformation

$$2I_{xx}I_{ly} \xrightarrow{DQF} \frac{1}{2}(2I_{xx}I_{ly} + 2I_{ky}I_{lx})$$

$$2I_{xx}I_{ly} = 2\left(I_{k+} + I_{k-}\right)I_{l+}I_{l-} = \frac{1}{2i}\left(I_{k+}I_{l+}I_{l+} - I_{k+}I_{l+} + I_{k-}I_{l+} - I_{k-}I_{l+}\right)$$

$$\xrightarrow{DQF} \frac{1}{2i}\left(I_{k+}I_{l+} - I_{k-}I_{l+}\right) = \frac{1}{2i}\left(\left(I_{kx} + iI_{ky}\right)\left(I_{lx} + iI_{ly}\right) - \left(I_{kx} - iI_{ky}\right)\left(I_{lx} - iI_{ly}\right)\right)$$

$$= \frac{1}{2i}\left(2I_{kx}I_{lx} + 2iI_{ky}I_{lx} + 2I_{kx}I_{ly} + 2I_{ky}I_{ly} - I_{kx}I_{lx} - I_{ky}I_{lx} - I_{kx}I_{ly} - I_{ky}I_{ly}\right)$$

Notice that performing a nQF filter on a nQC term (where $n$ is an integer) will make that the resulting combination of the product of all raising and all lowering operators remain.

An nQF filter will make that a kQC (with $k < n$) will be completely lost, but filtering a (n+2k)QC will make that some terms will remain.
3a The resulting spin state expression for a DQF COSY on a homonuclear two spin system, k and l, starting out with only the $I_{kz}$ is as determined as follows:

\[
\begin{align*}
\frac{\pi}{2} & \text{DQF} \\
\frac{\pi}{2} & \text{shift evolution left, filter out observables (only 1 transversal spin operator)} \\
\pi J_1 & 2I_{kz}I_{lz} \\
\omega_k t_1 & I_{kz} \\
\frac{\pi}{2} & \text{DQF} \\
\pi J_2 & 2I_{kz}I_{lz} \\
\omega_k t_2 & I_{kz} \\
\omega_l t_2 & I_{lz}
\end{align*}
\]

Thus the final expression for observable part of the spin state is given by:

\[
\frac{1}{2} \left\{ I_{kx} \sin \omega_k t_2 \sin \pi J_2 \cos \omega_k t_1 + I_{lx} \sin \omega_l t_2 \sin \pi J_2 \cos \omega_k t_1 \sin \pi J_1 \right\}
\]

\[
= \frac{1}{2} \left\{ I_{kx} \sin \omega_k t_2 + I_{lx} \sin \omega_l t_2 - I_{ky} \cos \omega_k t_2 - I_{ly} \cos \omega_l t_2 \right\} \sin \pi J_2 \cos \omega_k t_1 \sin \pi J_1
\]

3b Observation of this spin state will give for our signal:

\[
\frac{1}{2} \left\{ \sin \omega_k t_2 + \sin \omega_l t_2 - i \cos \omega_k t_2 - i \cos \omega_l t_2 \right\} \sin \pi J_2 \cos \omega_k t_1 \sin \pi J_1
\]

\[
= -\frac{i}{2} \left\{ i \sin \omega_k t_2 + i \sin \omega_l t_2 + \cos \omega_k t_2 + \cos \omega_l t_2 \right\} \sin \pi J_2 \cos \omega_k t_1 \sin \pi J_1
\]

\[
= \frac{-i}{2} \left\{ e^{i \omega_k t_2} + e^{i \omega_l t_2} \right\} \sin \pi J_2 \cos \omega_k t_1 \sin \pi J_1
\]

From this we see that the first term gives rise to a diagonal peak pattern, and the second term to a cross peak pattern. From the fact that the modulation terms for the cross and diagonal peaks are exactly equal (except for the chemical shift argument in $t_2$ of course) it is obvious that the line shapes are the same, i.e. when we perform phase correction so that the diagonal peaks are pure absorption, the cross peaks will be pure absorption also.

Notice that the $i$ present in the expression results from the $t_2$ observation, so we must use this term there in the Fourier transformation of the $t_2$ domain. Convolution gives:
Exactly the same pattern is obtained for the cross peak.

Thus we see that the problem that we had with a normal COSY, the different line shapes for the diagonal peaks and cross peaks (i.e. if one is pure absorption the other one is pure dispersion) is solved. Both peak patterns are pure absorption with the same phase correction. You may notice that the peak pattern is upside down compared to the COSY pattern. This is easily solved by doing a -\(\pi/2\) phase shift for the \(v_1\) dimension, in stead of the \(\pi/2\) one.
4 Perform the convolution, assuming proper phase corrections to obtain pure absorption lines, for the term \( \cos \omega_A t_1 \cos \pi J_{A,C} t_1 \sin \pi J_{A,B} t_1 \sin \omega_B t_2 \cos \pi J_{B,C} t_2 \sin \pi J_{A,B} t_2 \).

\[
\cos \omega_A t_1 \cos \pi J_{A,C} t_1 \sin \pi J_{A,B} t_1 \sin \omega_B t_2 \cos \pi J_{B,C} t_2 \sin \pi J_{A,B} t_2
\]

\( J_{A,B} > J_{B,C} > J_{A,C} \): do couplings in this order to avoid crossing of arrows, but shifts first.

Notice that you can distinguish between the active coupling (sine makes peaks go + and −, in this case coupling between A and B) and the passive coupling (cosine makes that peaks remain in the same direction, in this case the coupling between A and C in \( v_1 \) and B and C in \( v_2 \)).
A student is asked to determine the $^{195}\text{Pt}$ chemical shift of a new platinum di-hydride compound. She will be using a machine that operates at 300 MHz for protons. Use the following gyro-magnetic constants: $\gamma_H = 26.75 \times 10^7 \text{rad/(T.s)}$, and $\gamma_{\text{Pt}} = 5.75 \times 10^7 \text{rad/(T.s)}$.

a) At how many MHz does this instrument operate for $^{195}\text{Pt}$?

b) What should the maximum pulse length be if no zero intensities in a 2000 ppm window of frequency distributions comprised in the pulse are to be present?

c) What should the size of the acquisition buffer be if a sweep width of 1000 ppm and a resolution of 10 Hz/pt is required? Quadrature detection is used.

Comparable compounds have chemical shifts of $\approx -4000$ to $-5000$ ppm, and she decides to do a phase sensitive 1 pulse experiment, with the pulse carrier at $-4500$ ppm and a total frequency range of 1000 ppm. The spectrum she obtains shows some peaks around $-4490$ ppm. When she reduces the sweep width to 400 ppm, the peaks have moved to $-4690$ ppm.

d) What is the cause of this seemingly strange behavior?

e) (extra credit question) Can you determine what the chemical shift may be for the $^{195}\text{Pt}$ in this compound? (tip: you can set up an equation for the real chemical shift with two, whole-number unknowns. This means that there will be more than one answer possible. Pick the one that is closest to $-4500$ ppm.)

f) In order to determine the $T_1$ relaxation time, an inverse recovery experiment is performed (gray pulse is a $\pi$ pulse, other one is a $\pi/2$ pulse):

$$^{195}\text{Pt} \quad \boxed{x \quad \tau \quad x} \quad \text{FID, t}$$

The peak intensities at different $\tau$ values, as listed in the table, were measured. What is the $T_1$ for the platinum nucleus in the compound?

<table>
<thead>
<tr>
<th>$\tau$ (sec)</th>
<th>Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>-27.3</td>
</tr>
<tr>
<td>0.015</td>
<td>-11.3</td>
</tr>
<tr>
<td>0.030</td>
<td>0.0</td>
</tr>
<tr>
<td>0.045</td>
<td>8.0</td>
</tr>
<tr>
<td>0.060</td>
<td>13.6</td>
</tr>
</tbody>
</table>

Since the compound is a platinum di-hydride, it is possible to use a refocused-INEPT experiment to get enhancement of the sensitivity in the $^{195}\text{Pt}$ spectrum. The scalar coupling constant between the $^{195}\text{Pt}$ and the two equivalent $^1\text{H}$’s is determined from the proton NMR to $J_{\text{PtH}} = 200$ Hz. The refocused INEPT sequence is as follows (gray shaded pulses are $\pi$ pulses, other pulses are $\pi/2$ pulses)

$$^1\text{H} \quad \boxed{x \quad \frac{1}{4J} \quad x \quad \frac{1}{4J} \quad y \quad \Delta \quad x \quad \Delta} \quad \text{BB}$$

$$^{195}\text{Pt} \quad \boxed{x \quad \boxed{x \quad x \quad \boxed{x}}} \quad \text{FID, t}$$

g) What is the sensitivity enhancement possible due to polarization transfer between $^1\text{H}$ and $^{195}\text{Pt}$?

h) What is the optimum length of the value $\Delta$ for this specific di-hydride compound?
2 Consider a homo nuclear COSY experiment for the spin system $A_2M$.
   a) Calculate the signal giving rise to the cross peak at $(\nu_1, \nu_2) = (\nu_A, \nu_M)$. To keep it a little easier start out with the $z$ spin operator of only one of the two $A$’s present.
   b) Perform the 2D convolution. Give the phase correction necessary so that the peaks will be pure absorption peaks in the real-real part of the spectrum. Draw a clearly labeled, schematic representation of the cross peak pattern.

3 The sequence for a triple quantum filtered homo nuclear shift correlation (TQF-COSY) experiment is analogous to that of a DQF-COSY, only the phase cycling required for the particular quantum coherence filter is different. The pulse sequence is:
   \[ \text{FID} \]  
   \[ \frac{\pi}{2} x t_1 \frac{\pi}{2} x \text{TQF} \frac{\pi}{2} x \text{FID} t_2 \]

Determine the requirements for the presence of the cross peak at $(\nu_1, \nu_2) = (\nu_k, \nu_l)$ for a three spin system of spins $k$, $l$, and $m$. In other words: which couplings and how many (minimally) should be present for this cross peak to exist. (tip: in order to do this it is not necessary to do the convolution. You only need to determine the modulations present in the signal that gives rise to the cross peak itself and draw conclusions from that.)

4 Homo nuclear $J$ correlation spectroscopy (J-Res) is just a 2D version of the spin echo experiment, and is pulse sequence is given by:
   \[ \text{FID} t_1 \frac{\pi}{2} x t_1 \frac{\pi}{2} x \text{FID} t_2 \]

It can easily be shown that for a coupled two spin system, $k$ and $l$, the observed signal with only chemical shift modulations for spin $k$ during the observation time domain (with a $-\pi/2$ phase correction) is given by:
   \[ e^{i\omega k t_2} (\cos \pi J t_2 \cos \pi J t_1 - \sin \pi J t_2 \sin \pi J t_1) \]

a) Why is there no chemical shift information present in $t_1$?

The two terms are $90^\circ$ phase shifted with respect to each other in both $t_1$ and $t_2$. This means that there will be pure absorption peaks in the imaginary-imaginary ($\Im\Im$) buffer due to one of the two, and pure absorption peaks in the real-real ($\Re\Re$) buffer due to the other one. Thus it is useful to combine both the $\Re\Re$ and $\Im\Im$ parts in the observed spectrum. In order to do that you will perform the Fourier transforms completely mathematically (not by the pictorial convolution):

b) Express the observed signal in complex exponentials, and simplify the expression.

c) Perform the Fourier transforms of the complex exponentials, using the notation $Ab(\alpha, \nu_1) + iDi(\alpha, \nu_1)$, where $\alpha$ is the peak position and $i = 1$ or 2 depending on the number of the domain that was transformed to obtain the peak, e.g., $Ab(\nu_H + J/2, \nu_1) + iDi(\nu_H + J/2, \nu_1)$. Work the products out and then take only the real part of the result.
d) Draw a schematic representation of the real part of your resulting spectrum expression. Just mark the positions where peaks will be present with a circle.

e) Is the line shape of the peaks(s) pure dispersion, pure absorption, or neither?
1a (1 pt) The spectrometer frequency for $^{195}$Pt on this NMR is:

$$E_H = h\nu_H = \gamma_H B_0$$

$$E_{Pt} = h\nu_{Pt} = \gamma_{Pt} B_0$$

Filling out the numbers:

$$\nu_{Pt} = \frac{5.75 \times 10^7 \text{ rad/(Ts)}}{26.75 \times 10^7 \text{ rad/(Ts)}} \times 300 \text{MHz} = 64.5 \text{MHz}$$

1b (1 pt) What is the maximum pulse length in order not to have zero intensity in the frequencies present in the pulse within a spectral window of 2000 ppm.

$$2000 \text{ ppm} = \pm 1000 \text{ ppm} = \pm 1000 \text{ ppm} \times 64.5 \text{ MHz} = 64,500 \text{ Hz}.$$  

The intensity distribution of the angular frequencies present in a pulse of length $\tau$ is:

$$2 \sin^2 \left(\frac{1}{2} \omega \tau\right) \omega.$$  

Zeros for this function when $\frac{1}{2} \omega \tau = \frac{1}{2} 2 \pi \nu \tau = \pi \nu \tau = k \pi$, with $k = 1, 2, 3, \ldots$ In our case we have $\nu = 64,500 \text{ Hz}$, and $k = 1$ so that $\tau = \frac{k \pi}{\pi \nu} = \frac{1}{64,500 \text{Hz}} = 15.5 \mu\text{s}$

1c (1 pt) The size of the acquisition buffer with a sweep of 1000 ppm and a resolution of 10 Hz/pt should be:

$$1000 \text{ ppm} = 64,500 \text{ Hz}, \ 10 \text{ Hz/pt} : \frac{1000 \text{ ppm} \times 64.5 \text{ MHz}}{10 \text{ Hz/pt}} = 6,450 \text{ pt}.$$  

This means 8K large.

1d (1 pt) The peak must lie outside of both spectral windows and roll back into the spectrum.

1e (bonus 1 pt) Rolling back means that the real peak position must be $\nu_{obs} + nSW$ with $n$ a positive or negative number. We can set up two equations (one for each spectral window and position) and find a set of solutions as follows:

$$\nu_{real} = \nu_1 + n_1SW_1 = \nu_2 + n_2SW_2$$

$$n_2 = \frac{\nu_1 - \nu_2 + n_1SW_1}{SW_2}$$

we have $SW_1 = 1000$, ppm $\nu_1 = -4490$ ppm, and $SW_2 = 400$, ppm $\nu_2 = -4690$ ppm, so we get:

$$n_2 = \frac{-4490 - (-4690) + n_11000}{400} = \frac{200 + 1000n_1}{400} = \frac{1 + 5n_1}{2}$$

to get the solution closest to -4500 ppm, we have that $n_1$ should be -1, so that $n_2 = -2$, and the real chemical shift is $-4490$ ppm–1000 ppm = -5490 ppm.

1f (1 pt) To find the $T_1$ we can use the fact that the intensity of the peak in a $T_1$IR experiment is 0 when $\tau = T_1 \ln(2)$ thus $T_1 = \frac{\tau_{Int=0}}{\ln 2} = \frac{.030 \text{ sec}}{\ln 2} = 43.3 \text{ msec}$.

1g (1 pt) The enhancement is given by $\frac{\gamma_H}{\gamma_{Pt}} = \frac{26.75 \times 10^7 \text{ rad/(Ts)}}{5.75 \times 10^7 \text{ rad/(Ts)}} = 4.65$

1h (1 pt) The optimum $\Delta$ value for PtH$_2$ is equivalent to that of the CH$_2$ given by $\Delta = \frac{1}{8J}$, which in this case gives $\Delta = 1/(1600 \text{Hz}) = 0.625 \text{ msec}$.
2 (6 pts) The cross peak pattern of a COSY experiment working on a homo nuclear A₂M system:

\[
\begin{align*}
\frac{\pi}{2} & A_{1x}, A_{2x}, M_x \\
\pi J_1 & 2I_{A_{1z}} I_{Mz} \\
\pi J_1 & 2I_{A_{2z}} I_{Mz} \\
\omega A_{12} & I_{A_{1z}} \\
\frac{\pi}{2} & A_{1x}, A_{2x}, M_x \\
\end{align*}
\]

Select only SQC's

\[
\begin{align*}
0 & \text{ diagonal peak} \\
0 & \text{ cross peak} \\
\end{align*}
\]

Observed signal from 2 A spins:

\[
2 e^{i\omega M t_2} \cos \pi J_2 \sin \pi J_1 \sin \omega A t_1 \sin \pi J_1 \\
e^{i\omega M t_2} \sin 2\pi J_2 \sin \omega A t_1 \sin \pi J_1
\]
3 (6 pts) The requirements for the existence of a crosspeak between spins k and l in a homo nuclear TQF COSY of the spin system klm.

\[
\frac{\pi}{2} \left[ \frac{\pi}{2} \pi \frac{\pi}{2} \right]_{\text{TQF}}
\]

\[
\pi J_{kl} t_1 2 I_{kz} I_{lz} - I_{ky} 2 I_{kx} I_{lz}
\]

\[
\pi J_{km} t_1 2 I_{kz} I_{mz}
\]

\[
\pi J_{lm} t_2 2 I_{lz} I_{mz}
\]

\[
\omega_{kl} t_1 I_{kz}
\]

\[
\omega_{lm} t_2 I_{lz}, \text{signal}
\]

\[
\frac{\pi}{2} \left[ \frac{\pi}{2} \pi \frac{\pi}{2} \right]_{\text{ZQC SQC DQC SQC SQC TQC}}
\]

\[
\text{not observable} \quad \text{diagonal peak} \quad \text{cross peak with l} \quad \text{cross peak with m}
\]

\[
-4 I_{kx} I_{lx} I_{my}
\]

\[
= -4 \left( \frac{I_k + I_{-1}}{2i} \right) \left( \frac{I_l + I_{-1}}{2i} \right) \left( \frac{I_m + I_{-1}}{2i} \right)
\]

\[
\text{TQF} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}
\]

\[
\frac{1}{4} e^{i \omega_{k1} t_2} \sin \pi J_{lm} t_2 \sin \pi J_{kl} t_1 \sin \pi J_{km} t_1 \sin \pi J_{kl} t_1
\]

\[
\text{observed cross peak signal}
\]

We used for the TQF of the TQC term that was generated:

\[
-4 I_{kx} I_{lx} I_{my} = -4 \left( \frac{I_k + I_{-1}}{2i} \right) \left( \frac{I_l + I_{-1}}{2i} \right) \left( \frac{I_m + I_{-1}}{2i} \right)
\]

\[
\frac{1}{4} \left( \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \right)
\]

\[
\text{TQF} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}
\]

\[
\frac{1}{4} e^{i \omega_{k1} t_2} \sin \pi J_{lm} t_2 \sin \pi J_{kl} t_1 \sin \pi J_{km} t_1 \sin \pi J_{kl} t_1
\]
4a (1 pt) No shift modulation present in $t_1$ since we have a $\pi$ pulse in the center of the $t_1$ time domain. The effect of this pulse is that both the chemical shift and the hetero nuclear couplings would refocus and the signal would not have the chemical shifts nor hetero nuclear couplings as modulations.

4b (2 pts) rewriting the observed signal as a complex exponential gives:

$$e^{i\omega_k t_2} \left( \cos \pi t_2 \cos \pi J_1 t_1 - \sin \pi t_2 \sin \pi J_1 t_1 \right)$$

$$= e^{i\omega_k t_2} \left( \frac{e^{i\pi t_2} + e^{-i\pi t_2}}{2} e^{i\pi J_1 t_1} + e^{-i\pi J_1 t_1} - \frac{e^{i\pi t_2} - e^{-i\pi t_2}}{2i} e^{i\pi J_1 t_1} - e^{-i\pi J_1 t_1} \right)$$

$$= \frac{1}{4} e^{i\omega_k t_2} \left( e^{i\pi t_2} e^{i\pi J_1 t_1} + e^{-i\pi t_2} e^{-i\pi J_1 t_1} + e^{-i\pi J_1 t_1} e^{i\pi J_1 t_1} + e^{-i\pi J_1 t_1} \right)$$

$$= \frac{1}{2} e^{i\omega_k t_2} \left( e^{i\pi t_2} e^{i\pi J_1 t_1} + e^{-i\pi J_1 t_2} e^{-i\pi J_1 t_1} \right)$$

4c (2 pts) In order to do the Fourier transform for both time domains, we can do them one by one (since the other time variable is constant), so we have:

$$\frac{1}{2} e^{i\omega_k t_2} \left( e^{i\pi t_1} \left( Ab(v_k + \frac{1}{2}, v_2) + iDi(v_k + \frac{1}{2}, v_2) \right) \right.$$  

$$\left. + e^{-i\pi t_1} \left( Ab(v_k - \frac{1}{2}, v_2) + iDi(v_k - \frac{1}{2}, v_2) \right) \right)$$

$$\frac{1}{2} \left( \left( Ab(+\frac{1}{2}, v_1) + iDi(+\frac{1}{2}, v_1) \right) \left( Ab(v_k + \frac{1}{2}, v_2) + iDi(v_k + \frac{1}{2}, v_2) \right) \right.$$  

$$\left. \right.$$  

$$\left( Ab(-\frac{1}{2}, v_1) + iDi(-\frac{1}{2}, v_1) \right) \left( Ab(v_k - \frac{1}{2}, v_2) + iDi(v_k - \frac{1}{2}, v_2) \right) \right)$$

$$\frac{1}{2} \left( \left. \left( Ab(+\frac{1}{2}, v_1) Ab(v_k + \frac{1}{2}, v_2) + iAb(+\frac{1}{2}, v_1) Di(v_k + \frac{1}{2}, v_2) \right) \right.$$  

$$\left. + iDi(+\frac{1}{2}, v_1) Ab(v_k + \frac{1}{2}, v_2) - Di(v_k + \frac{1}{2}, v_2) Di(v_k + \frac{1}{2}, v_2) \right)$$

$$\left. \right.$$  

$$\left. \left| Ab(-\frac{1}{2}, v_1) Ab(v_k - \frac{1}{2}, v_2) + iAb(-\frac{1}{2}, v_1) Di(v_k - \frac{1}{2}, v_2) \right) \right.$$  

$$\left. + iDi(-\frac{1}{2}, v_1) Ab(v_k - \frac{1}{2}, v_2) - Di(-\frac{1}{2}, v_1) Di(v_k - \frac{1}{2}, v_2) \right) \right)$$

Taking the real part of the expression for the spectrum gives:

$$\frac{1}{2} \left( \left( Ab(+\frac{1}{2}, v_1) Ab(v_k + \frac{1}{2}, v_2) - Di(v_k + \frac{1}{2}, v_2) Di(v_k + \frac{1}{2}, v_2) \right) \right.$$  

$$\left. \right.$$  

$$\left( Ab(-\frac{1}{2}, v_1) Ab(v_k - \frac{1}{2}, v_2) - Di(-\frac{1}{2}, v_1) Di(v_k - \frac{1}{2}, v_2) \right) \right)$$

From which we see that we have phase twisted peaks at $\left( \frac{1}{2}, v_k + \frac{1}{2} \right)$ and at $\left( -\frac{1}{2}, v_k - \frac{1}{2} \right)$.
4d (1 pt) A graphical representation of the observed peak patterns:

4e (1 pt) Line shapes are phase twisted peaks, since the line are AbAb-DiDi.