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Quantum Integrable Models and Discrete Classical Hirota Equations

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Abstract: The standard objects of quantum integrable systems are identified with elements of classical nonlinear integrable difference equations. The functional relation for commuting quantum transfer matrices of quantum integrable models is shown to coincide with classical Hirota's bilinear difference equation. This equation is equivalent to the completely discretized classical 2D Toda lattice with open boundaries. Elliptic solutions of Hirota's equation give a complete set of eigenvalues of the quantum transfer matrices. Eigenvalues of Baxter's Q-operator are solutions to the auxiliary linear problems for classical Hirota's equation. The elliptic solutions relevant to the Bethe ansatz are studied. The nested Bethe ansatz equations for A_{k-1}-type models appear as discrete time equations of motions for zeros of classical τ-functions and Baker-Akhiezer functions. Determinant representations of the general solution to bilinear discrete Hirota's equation are analysed and a new determinant formula for eigenvalues of the quantum transfer matrices is obtained. Difference equations for eigenvalues of the Q-operators which generalize Baxter's three-term T-Q-relation are derived.

1. Introduction

In spite of the diversity of solvable models of quantum field theory and the vast variety of methods, the final results display dramatic unification: the spectrum of an integrable theory with a local interaction is given by a sum of elementary energies

$$E = \sum_i \varepsilon(u_i)\,, \tag{1.1}$$

where u_i obey a system of algebraic or transcendental equations known as *Bethe equations* [4, 16]. The major ingredients of Bethe equations are determined by the algebraic structure of the problem. A typical example of a system of Bethe equations (related to A_1-type models with an elliptic R-matrix) is

$$e^{-4\eta\nu}\frac{\phi(u_j)}{\phi(u_j-2)} = -\prod_k \frac{\sigma(\eta(u_j-u_k+2))}{\sigma(\eta(u_j-u_k-2))}\,, \tag{1.2}$$

where $\sigma(x)$ is the Weierstrass σ-function and

$$\phi(u) = \prod_{k=1}^{N} \sigma(\eta(u-y_k))\,. \tag{1.3}$$

Entries of these equations which encode information of the model are the function $\varepsilon(u)$ (entering through $\phi(u)$), quasiperiods ω_1, ω_2 of the σ-function, parameters η, ν, y_k and size of the system N. Different solutions of the Bethe equations correspond to different quantum states of the model.

In this paper we show that these equations, which are usually considered as a tool inherent to the quantum integrability, arise naturally as a result of the solution of entirely *classical* non-linear *discrete time* integrable equations. This suggests an intriguing interrelation (if not equivalence) between *integrable quantum field theories* and *classical soliton equations in discrete time*. In forthcoming papers we will show that the Bethe equations themselves may be considered as a discrete integrable dynamical system.

R. Hirota proposed [20] a difference equation which unifies the majority of known continuous soliton equations, including their hierarchies [42, 12]. A particular case of the Hirota equation is a bilinear difference equation for a function $\tau(n,l,m)$ of three discrete variables:

$$\begin{aligned}\alpha\tau(n,l+1,m)\tau(n,l,m+1) + \beta\tau(n,l,m)\tau(n,l+1,m+1)&\\ +\gamma\tau(n+1,l+1,m)\tau(n-1,l,m+1) = 0\,,&\end{aligned} \tag{1.4}$$

where it is assumed that $\alpha+\beta+\gamma = 0$. Different continuum limits at different boundary conditions then reproduce continuous soliton equations (KP, Toda lattice, etc). On the other hand, $\tau(n,l,m)$ can be identified [42] with the τ-function of a continuous hierarchy expressed through special independent variables.

The same equation (with a particular boundary condition) has quite unexpectedly appeared in the theory of *quantum* integrable systems as a fusion relation for the transfer matrix (trace of the quantum monodromy matrix).

The transfer matrix is one of the key objects in the theory of quantum integrable systems [13]. Transfer matrices form a commutative family of operators acting in the Hilbert space of a quantum problem. Let $R_{i,\mathcal{A}}(u)$ be the R-matrix acting in the tensor product of Hilbert spaces $V_i \otimes V_{\mathcal{A}}$. Then the transfer matrix is a trace over the auxiliary space $V_{\mathcal{A}}$ of the monodromy matrix. The latter being the matrix product of N R-matrices with a common auxiliary space:

$$\begin{aligned}\hat{T}_{\mathcal{A}}(u|y_i) &= R_{N,\mathcal{A}}(u-y_N)\ldots R_{2,\mathcal{A}}(u-y_2)R_{1,\mathcal{A}}(u-y_1)\,,\\ T_{\mathcal{A}}(u) &= \mathrm{tr}_{\mathcal{A}}\hat{T}_{\mathcal{A}}(u|y_i)\,.\end{aligned} \tag{1.5}$$

The transfer matrices commute for all values of the spectral parameter u and different auxiliary spaces:

$$[T_{\mathcal{A}}(u),\, T_{\mathcal{A}'}(u')] = 0. \tag{1.6}$$

They can be diagonalized simultaneously. The family of eigenvalues of the transfer matrix is an object of primary interest in an integrable system, since the spectrum of the quantum problem can be expressed in terms of eigenvalues of the transfer matrix.

The transfer matrix corresponding to a given representation in the auxiliary space can be constructed out of transfer matrices for some elementary space by means of the *fusion procedure* [35, 36, 26]. The fusion procedure is based on the fact that at certain values of the spectral parameter u the R-matrix becomes essentially a projector onto an irreducible representation space. The fusion rules are especially simple in the A_1-case. For example, the $R_{1,1}(u)$-matrix for two spin-1/2 representations in a certain normalization of the spectral parameter is proportional to the projector onto the singlet (spin-0 state) at $u = +2$ and onto the triplet (spin-1 subspace) at $u = -2$, in accordance with the decomposition $[1/2] + [1/2] = [0] + [1]$. Then the transfer matrix $T_2^1(u)$ with spin-1 auxiliary space is obtained from the product of two spin-1/2 monodromy matrices $\hat{T}_1^1(u)$ with arguments shifted by 2:

$$T_2^1(u) = \mathrm{tr}_{[1]}\left(R_{1,1}(-2)\hat{T}_1^1(u+1)\hat{T}_1^1(u-1)R_{1,1}(-2)\right).$$

A combination of the fusion procedure and the Yang-Baxter equation results in numerous functional relations (fusion rules) for the transfer matrix [35, 47]. They were recently combined into a universal bilinear form [30, 37]. The bilinear functional relations have the most simple closed form for the models of the A_{k-1}-series and representations corresponding to *rectangular* Young diagrams.

Let $T_s^a(u)$ be the transfer matrix for the rectangular Young diagram of length a and height s. If η can not be represented in the form $\eta = r_1\omega_1 + r_2\omega_2$ with rational r_1, r_2 (below we always assume that this is the case; for models with trigonometric R-matrices this means that the quantum deformation parameter q would not be a root of unity), they obey the following bilinear functional relation:

$$T_s^a(u+1)T_s^a(u-1) - T_{s+1}^a(u)T_{s-1}^a(u) = T_s^{a+1}(u)T_s^{a-1}(u). \tag{1.7}$$

Since $T_s^a(u)$ commute at different $u,\ a,\ s,$ the same equation holds for eigenvalues of the transfer matrices, so we can (and will) treat $T_s^a(u)$ in Eq. (1.7) as number-valued functions. The bilinear fusion relations for models related to other Dynkin graphs were suggested in ref. [37].

Remarkably, the bilinear fusion relations (1.7) appear to be identical to the Hirota equation (1.4). Indeed, one can eliminate the constants α, β, γ by the transformation

$$\tau(n,l,m) = \frac{(-\alpha/\gamma)^{n^2/2}}{(1+\gamma/\alpha)^{lm}}\tau_n(l,m),$$

so that

$$\tau_n(l+1,m)\tau_n(l,m+1) - \tau_n(l,m)\tau_n(l+1,m+1) = \tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1) = 0\,, \tag{1.8}$$

and then change variables from *light-cone* coordinates n, l, m to the *"direct"* variables

$$a = n, \qquad s = l+m, \qquad u = l-m-n,$$
$$\tau_n(l,m) \equiv T_{l+m}^a(l-m-n). \tag{1.9}$$

At least at a formal level, this transformation provides the equivalence between Eqs. (1.7), (1.4) and (1.8). In what follows we call Eq. (1.8) (or (1.7)) Hirota's bilinear difference equation (HBDE).

Leaving aside more fundamental aspects of this "coincidence," we exploit, as a first step, some technical advantages it offers. Specifically, we treat the functional relation (1.7) not as an identity but as a *fundamental equation* which (together with particular boundary and analytical conditions) completely determines all the eigenvalues of the transfer matrix. The solution to HBDE then appears in the form of the Bethe equations. We anticipate that this approach makes it possible to use some specific tools of classical integrability and, in particular, the finite gap integration technique.

The origin of $T_s^a(u)$ as an eigenvalue of the transfer matrix (1.5) imposes specific boundary conditions and, what is perhaps even more important, requires certain analytic properties of the solutions. As a general consequence of the Yang-Baxter equation, the transfer matrices may always be normalized to be *elliptic polynomials* in the spectral parameter, i.e. finite products of Weierstrass σ-functions (as in (1.3)). The problem therefore is stated as finding elliptic solutions of HBDE.

A similar problem appeared in the theory of continuous soliton equations since the works [1, 11], wherein a remarkable connection between the motion of poles of the elliptic solutions to the KdV equation and the Calogero-Moser dynamical system was revealed. Elliptic solutions to Kadomtsev-Petviashvili (KP), matrix KP equations and the matrix 2D Toda lattice (2DTL) were analyzed in Refs. [31, 32, 33], respectively. It was shown, in particular, that poles of elliptic solutions to the abelian 2DTL (i.e. zeros of corresponding τ-functions and Baker-Akhiezer functions) move according to the equations of motion for the Ruijsenaars-Schneider (RS) system of particles [48].

Analytic properties of solutions to HBDE relevant to the Bethe ansatz suggest a similar interpretation of Bethe ansatz equations. We will show that the nested Bethe ansatz for A_{k-1}-type models is equivalent to a chain of Bäcklund transformations of HBDE. The nested Bethe ansatz equations arise as equations of motion for zeros of the Baker-Akhiezer functions in discrete time (discrete time RS system[1]). The discrete time variable is identified with the level of the nested Bethe ansatz.

The paper is organized as follows. In Sect. 2 we review general properties and boundary conditions of solutions to HBDE that yield eigenvalues of quantum transfer matrices. In Sect. 3 the zero curvature representation of HBDE and the auxiliary linear problems are presented. We also discuss the duality relation between "wave functions" and "potentials" and define Bäcklund flows on the set of wave functions. These flows are important ingredients of the nested Bethe ansatz scheme. For illustrative purposes, in Sect. 4, we give a self-contained treatment of the A_1-case, where the major part of the construction contains familiar objects from the usual Bethe ansatz. Section 5 is devoted to the general A_{k-1}-case. We give a general solution to HBDE with the required boundary conditions. This leads to a new type of determinant formulas for eigenvalues of quantum transfer matrices. A sketch of proof of this result is presented in the appendix to Sect. 5. Generalized Baxter's relations (difference equations for $Q_t(u)$) are written in the explicit form. They are used for examining the equivalence to the standard Bethe ansatz results. In Sect. 6 a part of the general theory of elliptic solutions to HBDE is given. Section 7 contains a discussion of the results.

2. General Properties of Solutions to Hirota's Equation Relevant to Bethe Ansatz

2.1. Boundary conditions and analytic properties. HBDE has many different solutions. Not all of them give eigenvalues of the transfer matrix (1.5). There are certain boundary and analytic conditions imposed on the transfer matrix (1.5).

[1] It should be noted that equations of motion for the discrete time RS system were already written down in the paper [43]. However, the relation to elliptic solutions of discrete soliton equations and their nested Bethe ansatz interpretation were not discussed there.

(i) It is known that $T_s^k(u)$, the transfer matrix in the most antisymmetrical representation in the auxiliary space, is a scalar, i.e. it has only one eigenvalue (sometimes called the quantum determinant $\det_q \hat{T}_s(u)$ of the monodromy matrix). It depends on the representation in the quantum space of the model and is known explicitly. In the simplest case of the vector representation (one-box Young diagram) in the quantum space it is [34]:

$$T_s^k(u) = \phi(u-s-k) \prod_{l=0}^{k-1} \prod_{p=1}^{s-1} \phi(u+s+k-2l-2p-2) \prod_{l=1}^{k-1} \phi(u+s+k-2l), \tag{2.1}$$

$$T_s^0(u) = 1. \tag{2.2}$$

These values of $T_s^0(u)$ and $T_s^k(u)$ should be considered as boundary conditions. Let us note that they obey the discrete Laplace equation:

$$T_s^k(u+1)T_s^k(u-1) = T_{s+1}^k(u)T_{s-1}^k(u). \tag{2.3}$$

This leads to the boundary condition (b.c.)

$$T_s^a(u) = 0 \quad \text{as} \quad a < 0 \quad \text{and} \quad a > k \tag{2.4}$$

(with this b.c. Eq. (1.8) is known as the discrete two-dimensional Toda molecule equation [22], an integrable discretization of the conformal Toda field theory [8]).

(ii) The second important condition (which follows, eventually, from the Yang-Baxter equation) is that $T_s^a(u)$ has to be an elliptic polynomial in the spectral parameter u. By elliptic polynomial we mean essentially a finite product of Weierstrass σ-functions. For models with a rational R-matrix it degenerates to a usual polynomial in u.

To give a more precise formulation of this property, let us note that Eq. (1.7) has the gauge invariance under a transformation parametrized by four arbitrary functions χ_i of one variable:

$$T_s^a(u) \to \chi_1(a+u+s)\chi_2(a-u+s)\chi_3(a+u-s)\chi_4(a-u-s)T_s^a(u)\,. \tag{2.5}$$

These transformations can remove all zeros from the characteristics $a \pm s \pm u = \text{const}$. We require that the remaining part of all $T_s^a(u)$ should be an elliptic (trigonometric, rational) polynomial of one and the same degree N, where N is the number of sites on the lattice (see (1.3)).

One can formulate this condition in a gauge invariant form by introducing the gauge invariant combination

$$Y_s^a(u) = \frac{T_{s+1}^a(u)T_{s-1}^a(u)}{T_s^{a+1}(u)T_s^{a-1}(u)}\,. \tag{2.6}$$

We require $Y_s^a(u)$ to be an elliptic function having $2N$ zeros and $2N$ poles in the fundamental domain. This implies that $T_s^a(u)$ has the general form[2]

$$T_s^a(u) = A_s^a e^{\mu(a,s)u} \prod_{j=1}^{N} \sigma(\eta(u - z_j^{(a,s)}))\,, \tag{2.7}$$

where $z_j^{(a,s)}$, A_s^a, $\mu(a,s)$ do not depend on u and the following constraints hold:

[2] This differs from a more traditional expression in terms of Jacobi θ-functions by a simple normalization factor.

$$\sum_{j=1}^{N}(z_j^{(a,s+1)} + z_j^{(a,s-1)}) = \sum_{j=1}^{N}(z_j^{(a+1,s)} + z_j^{(a-1,s)}) \,, \tag{2.8}$$

$$\mu(a, s+1) + \mu(a, s-1) = \mu(a+1, s) + \mu(a-1, s) \,. \tag{2.9}$$

Another gauge invariant combination,

$$X_s^a(u) = -\frac{T_s^a(u+1)T_s^a(u-1)}{T_s^{a+1}(u)T_s^{a-1}(u)} = -1 - Y_s^a(u) \,, \tag{2.10}$$

is also convenient.

As a reference, we point out gauge invariant forms of HBDE [37]:

$$Y_s^a(u+1)Y_s^a(u-1) = \frac{(1+Y_{s+1}^a(u))(1+Y_{s-1}^a(u))}{(1+(Y_s^{a+1}(u))^{-1})(1+(Y_s^{a-1}(u))^{-1})} \,, \tag{2.11}$$

$$X_{s+1}^a(u)X_{s-1}^a(u) = \frac{(1+X_s^a(u+1))(1+X_s^a(u-1))}{(1+(X_s^{a+1}(u))^{-1})(1+(X_s^{a-1}(u))^{-1})} \,. \tag{2.12}$$

It can be shown that the minimal polynomial appears in the gauge

$$T_s^a(u) \to T_s^a(u) \left(\prod_{l=0}^{a-1} \prod_{p=1}^{s-1} \phi(u+s+a-2l-2p-2) \prod_{l=1}^{a-1} \phi(u+s+a-2l) \right)^{-1} \,, \tag{2.13}$$

where all the “trivial" zeros (common for all the eigenvalues) of the transfer matrix are removed (see e.g. [54]). The boundary values at $a = 0, k$ then become:

$$\begin{aligned} T_s^0(u) &= \phi(u+s), \\ T_s^k(u) &= \phi(u-s-k) \,. \end{aligned} \tag{2.14}$$

From now on we adopt this normalization.

(iii) The analyticity conditions and b.c. (2.14) lead to a particular “initial condition" in s. It is convenient, however, to take advantage of it before the actual derivation. The condition reads

$$T_s^a(u) = 0 \quad \text{for any} \quad -k < s < 0, \;\; 0 < a < k \,. \tag{2.15}$$

This is consistent with (1.7), (2.14) and implies

$$T_0^a(u) = \phi(u-a) \tag{2.16}$$

for $0 \le a \le k$.

Under the analyticity conditions (i) and the b.c. (2.14) (and their consequences (2.15), (2.16)) each solution to HBDE (1.7) corresponds to an eigenstate of the A_{k-1}-transfer matrix.

The same conditions are valid for higher representations of the quantum space. However, in that case there are certain constraints on zeros of $\phi(u)$ (they should form “strings"), whence $T_s^a(u)$ acquires extra “trivial" zeros. Here we do not address this question.

2.2. Plücker relations and determinant representations of solutions. Classical integrable equations in Hirota's bilinear form are known to be naturally connected [50, 25, 51], with geometry of Grassmann's manifolds (grassmannians) (see [24, 23, 19]), in general of an infinite dimension. Type of the grassmannian is specified by boundary conditions. Remarkably, the b.c. (2.4) required for Bethe ansatz solutions corresponds to *finite dimensional* grassmannians. This connection suggests a simple way to write down a general solution in terms of determinants and to transmit the problem to the boundary conditions. Numerous determinant formulas may be obtained in this way.

The grassmannian $\mathbf{G}_{n+1}^{r+1}$ is a collection of all $(r+1)$-dimensional linear subspaces of the complex $(n+1)$-dimensional vector space $\mathbf{C}^{n+1}$. In particular, $\mathbf{G}_{n+1}^{1}$ is the complex projective space $\mathbf{P}^n$. Let $X \in \mathbf{G}_{n+1}^{r+1}$ be such a $(r+1)$-dimensional subspace spanned by vectors $\mathbf{x}^{(j)} = \sum_{i=0}^{n} x_i^{(j)}\mathbf{e}^i$, $j = 1, \dots, r+1$, where $\mathbf{e}^i$ are basis vectors in $\mathbf{C}^{n+1}$. The collection of their coordinates form a rectangular $(n+1) \times (r+1)$-matrix $x_i^{(j)}$. Let us consider its $(r+1) \times (r+1)$ minors

$$\det_{pq}(x_{i_p}^{(q)}) \equiv (i_0, i_1, \dots, i_r), \qquad p, q = 0, 1, \dots, r\,, \tag{2.17}$$

obtained by choosing $r+1$ lines $i_0, i_1, \dots, i_r$. These C_{n+1}^{r+1} minors are called *Plücker coordinates* of X. They are defined up to a common scalar factor and provide the *Plücker embedding* of the grassmannian $\mathbf{G}_{n+1}^{r+1}$ into the projective space $\mathbf{P}^d$, where $d = C_{n+1}^{r+1} - 1$ (C_{n+1}^{r+1} is the bimomial coefficient).

The image of $\mathbf{G}_{n+1}^{r+1}$ in $\mathbf{P}^d$ is realized as an intersection of quadrics. This means that the coordinates $(i_0, i_1, \dots, i_r)$ are not independent but obey the *Plücker relations* [23, 19]:

$$(i_0, i_1, ..., i_r)(j_0, j_1, ..., j_r) = \sum_{p=0}^{r} (j_p, i_1, ..., i_r)(j_0, ...j_{p-1}, i_0, j_{p+1}..., j_r) \tag{2.18}$$

for all i_p, j_p, $p = 0, 1, \dots, r$. Here it is implied that the symbol $(i_0, i_1, \dots, i_r)$ is antysymmetric in all the indices, i.e., $(i_0, \dots, i_{p-1}, i_p, \dots, i_r) = -(i_0, \dots, i_p, i_{p-1}, \dots, i_r)$ and it equals zero if any two indices coincide. If one treats these relations as equations rather than identities, then determinants (2.17) would give a solution to Hirota's equations.

The Plücker relations in their general form (2.18) describe fusion rules for transfer matrices corresponding to arbitrary Young diagrams. At the same time these general fusion rules can be recast [40] into the form of higher equations of the discrete KP hierarchy. These are n-term bilinear equations for functions of n variables [12, 44]. In this paper we restrict ourselves to the three-term Hirota equation.

In order to reduce general Plücker relations to 3-term HBDE, one should take $i_p = j_p$ for $p \neq 0, 1$. Then all terms but the first two in the r.h.s. of (2.18) vanish and one is left with the 3-term relation

$$\begin{aligned}(i_0, i_1, \dots, i_r)(j_0, j_1, i_2, \dots, i_r) &= (j_0, i_1, i_2, \dots, i_r)(i_0, j_1, i_2, \dots i_r) \\ &+ (j_1, i_1, i_2, \dots, i_r)(j_0, i_0, i_2, \dots i_r).\end{aligned} \tag{2.19}$$

After substitution of (2.17) these elementary Plücker relations turn into certain determinant identities. For example, choosing $x_{i_0}^{(j)} = \delta_{pj}$, $x_{j_0}^{(j)} = \delta_{qj}$, $q \neq p$, one can recast Eq. (2.19) into the form of the Jacobi identity:

$$D[p|p] \cdot D[q|q] - D[p|q] \cdot D[q|p] = D[p, q|p, q] \cdot D\,, \tag{2.20}$$

where D is the determinant of a $(r+1) \times (r+1)$-matrix and $D[p_1, p_2|q_1, q_2]$ denotes the determinant of the same matrix with $p_{1,2}$-th rows and $q_{1,2}$-th columns removed. Another useful identity contained in Eq. (2.19) connects minors $D[l_1, l_2]$ of a $(r+3)\times(r+1)$ rectangular matrix, where the two rows l_1, l_2 are removed:

$$D[l_1,l_3]\cdot D[l_2,l_4] - D[l_1,l_2]\cdot D[l_3,l_4] = D[l_1,l_4]\cdot D[l_2,l_3]\,, \quad l_1 < l_2 < l_3 < l_4\,. \tag{2.21}$$

Identifying terms in Eq. (2.19) with terms in HBDE (1.8), one obtains various determinant representations of solutions to HBDE. Two of them follow from the Jacobi identity (2.20):

$$\tau_a(l,m) = \det\big(\tau_1(l+i-a, m-j+a)\big), \quad i,j = 1,\dots,a, \quad \tau_0(l,m) = 1 \tag{2.22}$$

or, in “direct" variables

$$T_s^a(u) = \det\big(T^1_{s+i-j}(u+i+j-a-1)\big), \quad i,j = 1,\dots,a, \quad T_s^0(u) = 1\,. \tag{2.23}$$

This representation determines an evolution in a from the initial values at $a = 1$. The size of the determinant grows with a. A similar formula exists for the evolution in s:

$$T_s^a(u) = \det\big(T_1^{a+i-j}(u+i+j-s-1)\big), \quad i,j = 1,\dots,s\,, \quad T_0^a(u) = 1\,. \tag{2.24}$$

The size of this determinant grows with s. Determinant formulas of this type have been known in the literature on quantum integrable models (see e.g. [6]). They allow one to express $T_s^a(u)$ through $T_1^a(u)$ or $T_s^1(u)$.

A different kind of determinant representation follows from (2.21):

$$\begin{aligned} T_s^a(u) &= \det M_{ij}\,, \\ M_{ji} &= \begin{cases} \underline{h}_i(u+s+a+2j) & \text{if } j = 1,...,k-a;\ i = 1,...,k \\ \overline{h}_i(u-s+a+2j) & \text{if } j = k-a+1,...,k;\ i = 1,...,k \end{cases}, \end{aligned} \tag{2.25}$$

where $h_i(x)$ and $\overline{h}_i(x)$ are $2k$ arbitrary functions of one variable. The size of this determinant is equal to k for all $0 \le a \le k$. This determinant formula plays an essential role in what follows.

The determinant representations give a solution to discrete nonlinear equations and expose the essence of the integrability. Let us note that they are simpler and more convenient than their continuous counterparts.

2.3. Examples of difference and continuous A_1-type equations. For illustrative purposes we specialize the Hirota equation to the A_1-case and later study it separately. At $k = 2$ Eq. (1.7) is

$$T_s(u+1)T_s(u-1) - T_{s+1}(u)T_{s-1}(u) = \phi(u+s)\phi(u-s-2) \tag{2.26}$$

with the condition $T_{-1}(u) = 0$ (here we set $T_s(u) \equiv T_s^1(u)$).

This equation is known as a discrete version of the Liouville equation [22] written in terms of the τ-function. It can be recast to a somewhat more universal form in terms of the discrete Liouville field

$$Y_s^1(u) \equiv Y_s(u) = \frac{T_{s+1}(u)T_{s-1}(u)}{\phi(u+s)\phi(u-s-2)} \tag{2.27}$$

(see (2.6)), which hides the function $\phi(u)$ in the r.h.s. of (2.26). The equation becomes

$$Y_s(u-1)Y_s(u+1) = (Y_{s+1}(u)+1)(Y_{s-1}(u)+1)\,. \tag{2.28}$$

(Let us note that the same functional equation but with different analytic properties of the solutions appears in the thermodynamic Bethe ansatz [53, 46].)

In the continuum limit one should put $Y_s(u) = \delta^{-2}\exp(-\varphi(x,t))$, $u = \delta^{-1}x$, $s = \delta^{-1}t$. An expansion in $\delta \to 0$ then gives the continuous Liouville equation

$$\partial_s^2\varphi - \partial_u^2\varphi = 2\exp(\varphi)\,. \tag{2.29}$$

To stress the specifics of the b.c. (2.15) and for further reference let us compare it with the quasiperiodic b.c. Then the A_1-case corresponds to the discrete sine-Gordon (SG) equation [21]:

$$T_s^{a+1}(u) = e^{\alpha}\lambda^{2a}T_s^{a-1}(u-2), \tag{2.30}$$

where α and λ are parameters. Substituting this condition into (1.7), we get:

$$T_s^1(u+1)T_s^1(u-1) - T_{s+1}^1(u)T_{s-1}^1(u) \;=\; e^{\alpha}\lambda^2 T_s^0(u)T_s^0(u-2), \tag{2.31}$$
$$T_s^0(u+1)T_s^0(u-1) - T_{s+1}^0(u)T_{s-1}^0(u) \;=\; e^{-\alpha}T_s^1(u)T_s^1(u+2). \tag{2.32}$$

Let us introduce two fields $\rho^{s,u}$ and $\varphi^{s,u}$ on the square (s,u) lattice

$$T_s^0(u) \;=\; \exp(\rho^{s,u} + \varphi^{s,u}), \tag{2.33}$$
$$T_s^1(u+1) \;=\; \lambda^{1/2}\exp(\rho^{s,u} - \varphi^{s,u}), \tag{2.34}$$

and substitute them into (2.31), (2.32). Finally, eliminating $\rho^{s,u}$, one gets the discrete SG equation:

$$\sinh(\varphi^{s+1,u}+\varphi^{s-1,u}-\varphi^{s,u+1}-\varphi^{s,u-1}) = \lambda\sinh(\varphi^{s+1,u}+\varphi^{s-1,u}+\varphi^{s,u+1}+\varphi^{s,u-1}+\alpha)\,. \tag{2.35}$$

The constant α can be removed by the redefinition $\varphi^{s,u} \to \varphi^{s,u} - \frac{1}{4}\alpha$.

Another useful form of the discrete SG equation appears in variables $X_s^a(u)$ (2.10). Under condition (2.30) one has

$$X_s^{a+1}(u) = X_s^{a-1}(u-2), \qquad \lambda^2 X_s^{a+1}(u+1)X_s^a(u) = 1\,, \tag{2.36}$$

so there is only one independent function

$$X_s^1(u) \equiv x_s(u) = -e^{-\alpha}\lambda^{-1}\exp\left(-2\varphi^{s,u} - 2\varphi^{s,u-2}\right). \tag{2.37}$$

The discrete SG equation becomes [21, 14, 9]:

$$x_{s+1}(u)x_{s-1}(u) = \frac{(\lambda + x_s(u+1))(\lambda + x_s(u-1))}{(1+\lambda x_s(u+1))(1+\lambda x_s(u-1))}\,. \tag{2.38}$$

In the limit $\lambda \to 0$ Eq. (2.38) turns into the discrete Liouville equation (2.28) for $Y_s(u) = -1 - \lambda^{-1}x_s(u)$.

3. Linear Problems and Bäcklund Transformations

3.1. Zero curvature condition. Consider the square lattice in two light cone variables l and m and a vector function $\psi_a(l,m)$ on this lattice. Let $L_{a,a'}(l,m)$ and $M_{a,a'}(l,m)$ be two shift operators in directions l and m:

$$\begin{aligned}\sum_{a'} L_{a,a'}(l,m)\psi_{a'}(l+1,m) &= \psi_a(l,m),\\ \sum_{a"} M_{a,a'}(l,m)\psi_{a'}(l,m+1) &= \psi_a(l,m).\end{aligned} \tag{3.1}$$

The zero curvature condition states that the result of subsequent shifts from an initial point to a fixed final point does not depend on the path:

$$L(l,m)\cdot M(l+1,m) = M(l,m)\cdot L(l,m+1). \tag{3.2}$$

HBDE (1.7) possesses [20, 49] a zero-curvature representation by means of the following two-diagonal infinite matrices:

$$\begin{aligned}L_{a,a'} &= \delta_{a,a'-1} + \delta_{a,a'}V_l^a\,,\\ M_{a,a'} &= \delta_{a,a'} + \delta_{a,a'+1}W_m^a\,,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}V_l^a &= \frac{\tau_a(l+1,m)\tau_{a+1}(l,m)}{\tau_a(l,m)\tau_{a+1}(l+1,m)}\,,\\ W_m^a &= \frac{\tau_{a-1}(l,m+1)\tau_{a+1}(l,m)}{\tau_a(l,m)\tau_a(l,m+1)}\,.\end{aligned} \tag{3.4}$$

More precisely, the compatibility condition of the two linear problems

$$\begin{aligned}\psi_a(l,m) - \psi_{a+1}(l+1,m) &= V_l^a\psi_a(l+1,m)\,,\\ \psi_a(l,m) - \psi_a(l,m+1) &= W_m^a\psi_{a-1}(l,m+1),\end{aligned} \tag{3.5}$$

combined with the b.c. (2.14) yields HBDE (1.8). Introducing an unnormalized “wave function"

$$f_a(l,m) = \psi_a(l,m)\tau_a(l,m)\,, \tag{3.6}$$

we can write the linear problems in the form

$$\begin{aligned}\tau_{a+1}(l+1,m)f_a(l,m) - \tau_{a+1}(l,m)f_a(l+1,m) &= \tau_a(l,m)f_{a+1}(l+1,m)\,,\\ \tau_a(l,m+1)f_a(l,m) - \tau_a(l,m)f_a(l,m+1) &= \tau_{a+1}(l,m)f_{a-1}(l,m+1)\,,\end{aligned} \tag{3.7}$$

or in “direct" variables

$$\begin{aligned}&T_{s+1}^{a+1}(u)F^a(s,u) - T_s^{a+1}(u-1)F^a(s+1,u+1) = T_s^a(u)F^{a+1}(s+1,u)\,,\\ &T_{s+1}^a(u-1)F^a(s,u) - T_s^a(u)F^a(s+1,u-1) = T_s^{a+1}(u-1)F^{a-1}(s+1,u)\,,\end{aligned} \tag{3.8}$$

where $F^a(l+m, l-m-a) \equiv f_a(l,m)$.

An advantage of the light cone coordinates is that they are separated in the linear problems (there are shifts only of l (m) in the first (second) Eq. (3.7)).

The wave function and potential possess a redundant gauge freedom:

$$V_l^a \to \frac{\chi(a-l+1)}{\chi(a-l)}V_l^a, \quad W_m^a \to \frac{\chi(a-l)}{\chi(a-l-1)}W_m^a, \quad \psi_a(l,m) \to \chi(a-l+1)\psi_a \quad (3.9)$$

with an arbitrary function χ.

The b.c. (2.4) implies a similar condition for the object of the linear problems

$$F^a(s,u) = 0 \quad \text{as} \quad a < 0 \quad \text{and} \quad a > k-1 \tag{3.10}$$

so that the number of functions F is one less than the number of T's. Then from the second equation of the pair (3.8) at $a = 0$ and from the first one at $a = k-1$ it follows that $F^0(s,u)$ ($F^{k-1}(s,u)$) depends on one cone variable $u+s$ (resp., $u-s$). We introduce a special notation for them:

$$F^0(s,u) = Q_{k-1}(u+s), \quad F^{k-1}(s,u) = \bar{Q}_{k-1}(u-s). \tag{3.11}$$

Furthermore, it can be shown that the important condition (2.15) relates the functions Q and $\bar{Q}$:

$$\bar{Q}_{k-1}(u) = Q_{k-1}(u-k+1). \tag{3.12}$$

The special form of the functions F^a at the ends of the Dynkin graph ($a = 0, k-1$) reflects the specifics of the “Liouville-type" boundary conditions. This is to be compared with nonlinear equations with the quasiperiodic boundary condition (2.30): in this case all the functions F depend on two variables and obey the quasiperiodic b.c.

3.2. Continuum limit. In the continuum limit $l = -\delta t_+, \ m = -\delta t_-, \ \tau_a \to \delta^{a^2}\tau_a, \ f_a \to \delta^{a^2+a} f_a, \ \delta \to 0$, we recover the auxiliary linear problems for the 2D Toda lattice [52] ($\partial_\pm \equiv \partial/\partial t_\pm$):

$$\begin{aligned} \partial_+\psi_a &= \psi_{a+1} + \partial_+(\log \frac{\tau_{a+1}}{\tau_a})\psi_a, \\ \partial_-\psi_a &= \frac{\tau_{a+1}\tau_{a-1}}{\tau_a^2}\psi_{a-1}, \end{aligned} \tag{3.13}$$

or, in terms of f_a,

$$\begin{aligned} \tau_{a+1}\partial_+ f_a - (\partial_+\tau_{a+1})f_a &= \tau_a f_{a+1}, \\ \tau_a\partial_- f_a - (\partial_-\tau_a)f_a &= \tau_{a+1}f_{a-1}. \end{aligned} \tag{3.14}$$

The compatibility condition of these equations yields the first non-trivial equation of the 2D Toda lattice hierarchy:

$$\partial_+\tau_a\partial_-\tau_a - \tau_a\partial_+\partial_-\tau_a = \tau_{a+1}\tau_{a-1}. \tag{3.15}$$

In terms of

$$\varphi_a(t_+,t_-) = \log \frac{\tau_{a+1}(t_+,t_-)}{\tau_a(t_+,t_-)}$$

it has the familiar form

$$\partial_+\partial_-\varphi_a = e^{\varphi_a-\varphi_{a-1}} - e^{\varphi_{a+1}-\varphi_a}. \tag{3.16}$$

3.3. Bäcklund flow. The discrete nonlinear equation has a remarkable duality between “potentials" T^a and “wave functions" F^a first noticed in [49]. In the continuum version it is not so transparent. Equations (3.8) are symmetric under the interchange of F and T. Then one may treat (3.8) as linear problems for a nonlinear equation on F's. It is not surprising that one again obtains HBDE (1.7):

$$F^a(s,u+1)F^a(s,u-1) - F^a(s+1,u)F^a(s-1,u) = F^{a+1}(s,u)F^{a-1}(s,u)\,. \tag{3.17}$$

Moreover, conditions (3.10)-(3.12) mean that even the b.c. for $F^a(s,u)$ are the same as for $T_s^a(u)$ under a substitution $\phi(u)$ by $Q_{k-1}(u)$. The only change is a reduction of the Dynkin graph: $k \to k-1$. Using this property, one can successively reduce the A_{k-1}-problem up to A_1. Below we use this trick to derive A_{k-1} ("nested") Bethe ansatz equations.

To elaborate the chain of these transformations, let us introduce a new variable $t = 0, 1, \ldots, k$ to mark a level of the flow $A_{k-1} \to A_1$ and let $F_{t+1}^a(s,u)$ be a solution to the linear problem at $(k-t)^{\text{th}}$ level. In this notation, $F_k^a(s,u) = T_s^a(u)$ and $F_{k-1}^a(s,u) = F^a(s,u)$ is the corresponding wave function. The wave function itself obeys the nonlinear equation (3.17), so $F_{k-2}^a(s,u)$ denotes its wave function and so on. For each level t the function $F_t^a(s,u)$ obeys HBDE of the form (3.17) with the b.c.

$$F_t^a(s,u) = 0 \quad \text{as} \quad a < 0 \quad \text{and} \quad a > t\,. \tag{3.18}$$

As a consequence of (3.18), the first and the last components of the vector $F_t^a(s,u)$ obey the discrete Laplace equation (2.3) and under the condition (3.11) are functions of only one of the light-cone variables ($u+s$ and $u-s$ respectively). We denote them as follows:

$$F_t^0(s,u) \equiv Q_t(u+s)\,, \qquad F_t^t(s,u) \equiv \overline{Q}_t(u-s)\,, \tag{3.19}$$

where it is implied that $Q_k(u) = \phi(u)$. It can be shown that ellipticity requirement (ii) and condition (2.14) impose the relation $\overline{Q}_t(u) = Q_t(u-t)$.

In this notation the linear problems (3.8) at level t,

$$F_{t+1}^{a+1}(s+1,u)F_t^a(s,u) - F_{t+1}^{a+1}(s,u-1)F_t^a(s+1,u+1) = F_{t+1}^a(s,u)F_t^{a+1}(s+1,u)\,, \tag{3.20}$$

$$F_{t+1}^a(s+1,u-1)F_t^a(s,u) - F_{t+1}^a(s,u)F_t^a(s+1,u-1) = F_{t+1}^{a+1}(s,u-1)F_t^{a-1}(s+1,u) \tag{3.21}$$

look like bilinear equations for a function of 4 variables. However, Eq. (3.20) (resp., Eq. (3.21)) leaves the hyperplane $u-s+a = \text{const}$ (resp., $u+s+a = \text{const}$) invariant, and actually depends on three variables.

Restricting the variables in Eq. (3.20) to the hyperplane $u-s+a = v$ (where v is a constant), by setting

$$\tau_u(t,a) \equiv F_{k-t}^a(u+a-v,u), \tag{3.22}$$

we reduce Eq. (3.20) to the form of the same HBDE (1.8) in cone coordinates t and a. The b.c. is

$$\tau_u(t,0) = Q_{k-t}(2u-v), \quad \tau_u(t,k-t) = \overline{Q}_{k-t}(v+t-k) = \text{const.} \tag{3.23}$$

Similar equations can be obtained from the second linear problem (3.21) by setting

$$\overline{\tau}_u(b,t) = F_{k-t}^{k-t-b}(\overline{v}+b-u, u+t-k) \tag{3.24}$$

($\overline{v}$ is a constant). This function obeys Eq. (1.8),

$$\overline{\tau}_u(b+1,t)\overline{\tau}_u(b,t+1) - \overline{\tau}_u(b,t)\overline{\tau}_u(b+1,t+1) = \overline{\tau}_{u+1}(b+1,t)\overline{\tau}_{u-1}(b,t+1)\,, \tag{3.25}$$

where t now plays the role of the light cone coordinate m. The b.c. is

$$\bar{\tau}_u(0,t) = \bar{Q}_{k-t}(2u + t - k - \bar{v}), \quad \bar{\tau}_u(k-t,t) = Q_{k-t}(v) = \text{const.} \tag{3.26}$$

It is convenient to visualize this array of τ-functions on a diagram; here is an example for the A_3-case ($k = 4$):

$$\begin{array}{llllll}
0 & 1 & 0 & & & \\
0 & Q_1(u+s) & \bar{Q}_1(u-s) & 0 & & \\
0 & Q_2(u+s) & F_2^1(s,u) & \bar{Q}_2(u-s) & 0 & \\
0 & Q_3(u+s) & F_3^1(s,u) & F_3^2(s,u) & \bar{Q}_3(u-s) & 0 \\
0 & \phi(u+s) & T_s^1(u) & T_s^2(u) & T_s^3(u) & \bar{\phi}(u-s) \quad 0
\end{array} \tag{3.27}$$

Functions in each horizontal (constant t) slice satisfy HBDE (3.17), whereas functions on the $u - s + a$ = const slice satisfy HBDE (1.8) with t, a being light cone variables l, m respectively.

A general solution of the bilinear discrete equation (1.7) with the b.c. (2.14) is determined by $2k$ arbitrary functions of one variable $Q_t(u)$ and $\bar{Q}_t(u)$, $t = 1, ..., k$. The additional requirement (ii) of ellipticity determines these functions through the Bethe ansatz.

3.4. Nested Bethe ansatz scheme. Here we elaborate the nested scheme of solving HBDE based on the chain of successive Bäcklund transformations (Sect. 3.4). This is an alternative (and actually the shortest) way to obtain nested Bethe ansatz equations (3.31). Recall that the function $\tau_u(t,a) = F_{k-t}^a(u+a,u)$ (3.22) (where we put $v = 0$ for simplicity) obeys HBDE in light cone variables:

$$\tau_u(t+1,a)\tau_u(t,a+1) - \tau_u(t,a)\tau_u(t+1,a+1) = \tau_{u+1}(t+1,a)\tau_{u-1}(t,a+1)\,. \tag{3.28}$$

Since $\tau_u(t,0) = Q_{k-t}(2u)$, nested Bethe ansatz equations can be understood as "equations of motions" for zeros of $Q_t(u)$ in discrete time t (level of the Bethe ansatz). The simplest way to derive them is to consider the auxiliary linear problems for Eq. (3.28). Here we present an example of this derivation in the simplest possible form.

Let us assume that $Q_t(u)$ has the form

$$Q_t(u) = e^{\nu_t \eta u} \prod_{j=1}^{M_t} \sigma(\eta(u - u_j^t)) \tag{3.29}$$

(note that we allow the number of roots M_t to depend on t). Since we are interested in dynamics in t at a fixed a, it is sufficient to consider only the first linear equation of the pair (3.7):

$$\tau_{u+1}(t+1,a)f_u(t,a) - \tau_{u+1}(t,a)f_u(t+1,a) = \tau_u(t,a)f_{u+1}(t+1,a)\,. \tag{3.30}$$

An elementary way to derive equations of motion for roots of $\tau_u(t,0)$ is to put u equal to the roots of $f_u(t+1,0)$, $f_u(t,0)$ and $f_{u+1}(t+1,0)$, so that only two terms in (3.30) would survive. Combining relations obtained in this way, one can eliminate f's and obtain the system of equations

$$\frac{Q_{t-1}(u_j^t+2)Q_t(u_j^t-2)Q_{t+1}(u_j^t)}{Q_{t-1}(u_j^t)Q_t(u_j^t+2)Q_{t+1}(u_j^t-2)} = -1 \tag{3.31}$$

as the necessary conditions for solutions of the form (3.29) to exist. In the more detailed notation they look as follows:

$$\prod_{k=1}^{M_{t-1}} \frac{\sigma(\eta(u_j^t - u_k^{t-1}+2))}{\sigma(\eta(u_j^t - u_k^{t-1}))} \prod_{k=1}^{M_t} \frac{\sigma(\eta(u_j^t - u_k^t-2))}{\sigma(\eta(u_j^t - u_k^t+2))} \prod_{k=1}^{M_{t+1}} \frac{\sigma(\eta(u_j^t - u_k^{t+1}))}{\sigma(\eta(u_j^t - u_k^{t+1}-2))}$$

$$= -e^{2\eta(2\nu_t - \nu_{t+1} - \nu_{t-1})}\,. \tag{3.32}$$

With the “boundary conditions”

$$Q_0(u) = 1, \qquad Q_k(u) = \phi(u), \tag{3.33}$$

this system of $M_1 + M_2 + \ldots + M_{k-1}$ equations is equivalent to the nested Bethe ansatz equations for A_{k-1}-type quantum integrable models with Belavin’s elliptic R-matrix. The same equations can be obtained for the right edge of the diagram (3.27) from the second linear equation in (3.7). In Sect. 5 we explicitly identify our Q’s with similar objects known from the Bethe ansatz solution.

Let us remark that the origin of Eq. (3.32) suggests to consider them as equations of motion for the elliptic Ruijsenaars-Schneider model in discrete time. Taking the continuum limit in t (provided $M_t = M$ does not depend on t), one can check that Eqs. (3.32) do yield the equations of motion for the elliptic RS model [48] with M particles. The additional limiting procedure $\eta \to 0$ with finite $\eta u_j = x_j$ yields the well known equations of motion for the elliptic Calogero-Moser system of particles.

However, integrable systems of particles in discrete time seem to have a richer structure than their continuous time counterparts. In particular, the total number of particles in the system may depend on (discrete) time. Such a phenomenon is possible in continuous time models only for singular solutions, when particles can move to infinity or merge to another within a finite period of time.

Remarkably, this appears to be the case for the solutions to Eqs. (3.32) corresponding to eigenstates of the quantum model. It is known that the number of excitations M_t at the t^{th} level of the Bethe ansatz solution does depend on t. In other words, the number of “particles" in the corresponding discrete time RS model is not conserved, though the numbers M_t may not be arbitrary.

In the elliptic case degrees of the elliptic polynomials $Q_t(u)$ are equal to $M_t = (N/k)t$ (provided η is incommensurable with the lattice spanned by ω_1, ω_2 and N is divisible by k). This fact follows directly from Bethe equations (3.31). Indeed, the elliptic polynomial form (3.29) implies that if u_j^t is a zero of $Q_t(u)$, i.e., $Q_t(u_j^t) = 0$, then $u_j^t + 2n_1\omega_1 + 2n_2\omega_2$ for all integers n_1, n_2 are its zeros too. Taking into account the well known monodromy properties of the σ-function, one concludes that this is possible if and only if

$$M_{t+1} + M_{t-1} = 2M_t\,, \tag{3.34}$$

which has a unique solution

$$M_t = \frac{N}{k}t \tag{3.35}$$

satisfying b.c. (3.33). This means that the nested scheme for elliptic A_{k-1}-type models is consistent only if N is divisible by k.

In trigonometric and rational cases the conditions on degrees of Q_t's become less restrictive since some of the roots can be located at infinity. The equality in (3.35) becomes an inequality: $M_t \leq (N/k)t$. A more detailed analysis [28] shows that the following inequalities also hold: $2M_1 \leq M_2, 2M_2 \leq M_1+M_3, \ldots, 2M_t \leq M_{t-1}+M_{t+1}$, $\ldots$, $N = M_k \geq 2M_{k-1} - M_{k-2}$.

4. The A_1-Case: Discrete Liouville Equation

In this section we consider the A_1-case separately. Although in this case the general nested scheme is missing, the construction is more explicit and contains familiar objects from the Bethe ansatz literature.

4.1. General solution. Let us consider a more general functional relation:

$$T_s(u+1)T_s(u-1) - T_{s+1}(u)T_{s-1}(u) = \phi(u+s)\bar{\phi}(u-s), \tag{4.1}$$

where the functions ϕ, $\bar{\phi}$ are independent and $T_s(u) \equiv T^1_s(u)$. The auxiliary linear problems (3.8) acquire the form

$$T_{s+1}(u)Q(u+s) - T_s(u-1)Q(u+s+2) = \phi(u+s)\bar{Q}(u-s-1), \tag{4.2}$$

$$T_{s+1}(u)\bar{Q}(u-s+1) - T_s(u+1)\bar{Q}(u-s-1) = \bar{\phi}(u-s)Q(u+s+2)\,. \tag{4.3}$$

Here we set $Q(u) \equiv Q_1(u)$ and $\phi(u) = Q_2(u)$. Rearranging these equations, we obtain

$$\phi(u-2)Q(u+2) + \phi(u)Q(u-2) = A(u)Q(u), \tag{4.4}$$

$$\bar{\phi}(u)\bar{Q}(u+3) + \bar{\phi}(u+2)\bar{Q}(u-1) = \bar{A}(u)\bar{Q}(u+1) \tag{4.5}$$

with the constraint

$$T_1(u)Q(u) - T_0(u-1)Q(u+2) = \phi(u)\bar{Q}(u-1), \tag{4.6}$$

which follows from Eq. (4.2) at $s = 0$. In these equations,

$$A(u) = \frac{\phi(u-2)T_{s+1}(u-s) + \phi(u)T_{s-1}(u-s-2)}{T_s(u-s-1)}\,, \tag{4.7}$$

$$\bar{A}(u) = \frac{\bar{\phi}(u+2)T_{s+1}(u+s) + \bar{\phi}(u)T_{s-1}(u+s+2)}{T_s(u+s+1)}\,. \tag{4.8}$$

Due to consistency condition (4.1) $A(u)$ and $\bar{A}(u)$ are functions of one variable and do not depend on s. The symmetry between u and s allows one to construct similar objects which in turn do not depend on u. Functions $A(u)$ and $\bar{A}(u)$, in the r.h.s. of (4.4), (4.5) are the conservation laws of the s-dynamics.

Let us note that the connection between ϕ and $\bar{\phi}$, $\bar{\phi}(u) = \phi(u-2)$, and its consequence $T_{-1}(u) = 0$ (see (2.15)), simplifies Eqs. (4.4)-(4.8). Putting $s = 0$ and using the b.c. $T_{-1}(u) = 0$, we find

$$A(u) = \bar{A}(u) = T_1(u)\,. \tag{4.9}$$

Therefore, the following holds

$$T_s(u-1)T_1(u+s) = \phi(u+s-2)T_{s+1}(u) + \phi(u+s)T_{s-1}(u-2), \tag{4.10}$$

$$T_s(u+1)T_1(u-s) = \phi(u-s)T_{s+1}(u) + \phi(u-s-2)T_{s-1}(u+2), \tag{4.11}$$

$$\phi(u-2)Q(u+2) + \phi(u)Q(u-2) = T_1(u)Q(u)\,. \tag{4.12}$$

The first two equalities are known as fusion relations [35, 29, 5] while Eq. (4.12) is Baxter's T-Q-relation [4, 3]. So Baxter's Q function and the T-Q-relation naturally appear in the context of the auxiliary linear problems for HBDE.

A general solution of the discrete Liouville equation (for arbitrary ϕ and $\bar{\phi}$) may be expressed through two independent functions $Q(u)$ and $\bar{Q}(u)$. One may follow the same lines developed for solving the continuous classical Liouville equation (see e.g. [17, 27] and references therein). Let us consider Eq. (4.4) (resp., (4.5)) as a second order linear difference equation, where the function $A(u)$ ($\bar{A}(u)$) is determined from the initial data. Let $R(u)$ (resp., $\bar{R}(u)$) be a second (linearly independent) solution of Eq. (4.4) (resp., (4.5)) normalized so that the wronskians are

$$W(u) = \begin{vmatrix} R(u) & Q(u) \\ R(u+2) & Q(u+2) \end{vmatrix} = \phi(u), \tag{4.13}$$

$$\bar{W}(u) = \begin{vmatrix} \bar{R}(u) & \bar{Q}(u) \\ \bar{R}(u+2) & \bar{Q}(u+2) \end{vmatrix} = \bar{\phi}(u+1), \tag{4.14}$$

and the constraint similar to (4.6) is imposed:

$$T_1(u)R(u) - T_0(u-1)R(u+2) = \phi(u)\bar{R}(u-1). \tag{4.15}$$

Then the general solution of Eq. (4.1) is given in terms of Q and R:

$$T_s(u) = \begin{vmatrix} Q(u+s+1) & R(u+s+1) \\ \bar{Q}(u-s) & \bar{R}(u-s) \end{vmatrix}. \tag{4.16}$$

This formula is a particular case of the general determinant representation (2.25).

Like in the continuous case, this expression is invariant with respect to changing the basis of linearly independent solutions with the given wronskians. The transformation of the basis vectors is described by an element of $SL(2)$. Due to relations (4.6), (4.15) $\bar{Q}$, $\bar{R}$ transform in the same way as Q, R and the invariance of Eq. (4.16) is evident.

For any given $Q(u)$ and $\bar{Q}(u)$ the second solution $R(u)$ and $\bar{R}(u)$ (defined modulo a linear transformation $R(u) \to R(u) + \alpha Q(u)$) can be explicitly found from the first order recurrence relations (4.13), (4.14), if necessary. Let $Q(u_0)$ and $R(u_0)$ be initial values at $u = u_0$. Then, say, for even $r \geq 0$,

$$R(u_0+r) = Q(u_0+r)\left(-\sum_{j=1}^{r/2} \frac{\phi(u_0+2j-2)}{Q(u_0+2j)Q(u_0+2j-2)} + \frac{R(u_0)}{Q(u_0)}\right), \tag{4.17}$$

and so on for other r's and $\bar{R}(u)$.

Finally, one can express the solution to Eq. (4.1) through two independent functions $Q(u)$ and $\bar{Q}(u)$:

$$T_s(u+s-1) = Q(u+2s)\bar{Q}(u-1)\left(\frac{T_0(u-1)}{Q(u)\bar{Q}(u-1)} + \sum_{j=1}^{s} \frac{\phi(u+2j-2)}{Q(u+2j)Q(u+2j-2)}\right), \tag{4.18}$$

where $T_0(u)$ can be found from (4.18) by putting $s = 0$:

$$-\frac{T_0(u-1)}{Q(u)Q(u-1)}+\frac{T_0(u+1)}{Q(u+2)Q(u+1)}=\frac{\phi(u)}{Q(u)Q(u+2)}-\frac{\bar{\phi}(u)}{Q(u-1)Q(u+1)}. \quad (4.19)$$

Note also the following useful representations:

$$A(u)=Q(u+2)R(u-2)-R(u+2)Q(u-2), \quad (4.20)$$

$$\bar{A}(u)=\bar{R}(u+3)\bar{Q}(u-1)-\bar{Q}(u+3)\bar{R}(u-1), \quad (4.21)$$

which are direct corollaries of (4.4), (4.5).

4.2. Equivalent forms of Baxter's equation. The key ingredient of the construction is Baxter's relation (4.12) and its "chiral" versions (4.4), (4.5). For completeness, we gather some other useful forms of them.

Consider first "chiral" linear equations (4.4), (4.5) (thus not implying any specific b.c. in s). Assuming that $T_s(u)$ obeys HBDE (4.1), one can represent Eqs. (4.4), (4.5) in the form

$$\begin{vmatrix} T_s(u) & T_{s+1}(u-1) & Q(u+s+1) \\ T_{s+1}(u+1) & T_{s+2}(u) & Q(u+s+3) \\ T_{s+2}(u+2) & T_{s+3}(u+1) & Q(u+s+5) \end{vmatrix}=0\,, \quad (4.22)$$

$$\begin{vmatrix} T_s(u) & T_{s+1}(u+1) & \bar{Q}(u-s) \\ T_{s+1}(u-1) & T_{s+2}(u) & \bar{Q}(u-s-2) \\ T_{s+2}(u-2) & T_{s+3}(u-1) & \bar{Q}(u-s-4) \end{vmatrix}=0\,, \quad (4.23)$$

respectively. This representation can be straightforwardly extended to the A_{k-1}-case (see Eqs. (5.37), (5.38)).

A factorized form of these difference equations is

$$\left(e^{2\partial_u}-\frac{\phi(u)Q(u-2)}{\phi(u-2)Q(u)}\right)\left(e^{2\partial_u}-\frac{Q(u)}{Q(u-2)}\right)X(u-2)=0\,, \quad (4.24)$$

$$\left(e^{2\partial_u}-\frac{\bar{\phi}(u+2)\bar{Q}(u-1)}{\phi(u)Q(u+1)}\right)\left(e^{2\partial_u}-\frac{\bar{Q}(u+1)}{Q(u-1)}\right)\bar{X}(u-1)=0\,. \quad (4.25)$$

Here e^{∂_u} acts as the shift operator, $e^{\partial_u}f(u)=f(u+1)$, and $X(u)$ ($\bar{X}(u)$) stands for any linear combination of $Q(u)$, $R(u)$ ($\bar{Q}(u)$, $\bar{R}(u)$).

Specifying Eqs. (4.22), (4.23) to the b.c. $T_{-1}(u)=0$ (see (4.9)), we see that both of them turn into the equation

$$\sum_{a=0}^{2}(-1)^a T_1^a(u+a-1)X(u+2a-2)=0, \quad (4.26)$$

that is Baxter's relation (4.12). Furthermore, the difference operator in (4.26) admits a factorization of the form (4.24):

$$\sum_{a=0}^{2}(-1)^a \frac{T_1^a(u+a-1)}{\phi(u-2)} e^{2a\partial_u} = \left(e^{2\partial_u} - \frac{\phi(u)Q(u-2)}{\phi(u-2)Q(u)}\right)\left(e^{2\partial_u} - \frac{Q(u)}{Q(u-2)}\right), \tag{4.27}$$

which is equivalent to the well known formula for $T_1(u)$ in terms of $Q(u)$.

4.3. Double-Bloch solutions to Baxter's equation. In this section we formulate the analytic properties of solutions to Baxter's functional relation (4.4) that are relevant to models on finite lattices.

First let us transform Baxter's relation to a difference equation with *elliptic* (i.e. double-periodic with periods $2\omega_1/\eta$, $2\omega_2/\eta$) coefficients.

The formal substitution

$$\tilde{\Psi}(u) = \frac{Q(u)P(u)}{\phi(u-2)} \tag{4.28}$$

with a (as yet not specified) function $P(u)$ yields

$$\tilde{\Psi}(u+2) + \frac{P(u+2)\phi(u-4)}{P(u-2)\phi(u-2)}\tilde{\Psi}(u-2) = \frac{A(u)P(u+2)}{\phi(u)P(u)}\tilde{\Psi}(u)\,. \tag{4.29}$$

Below we restrict ourselves to the case when the degree N of the elliptic polynomial $\phi(u)$ (1.3) is *even*. Then for any $P(u)$ of the form

$$P(u) = \prod_{j=1}^{N/2} \sigma(\eta(u-p_j)) \tag{4.30}$$

with arbitrary p_j the coefficients in (4.29) are elliptic functions. Indeed, for the coefficient in front of $\tilde{\Psi}(u-2)$ this is obvious. As for the coefficient in the r.h.s. of (4.29), its double-periodicity follows from the "sum rule" (2.8).

Let us represent $\phi(u)$ in the form

$$\phi(u) = \phi_0(u)\phi_1(u)\,, \tag{4.31}$$

where $\phi_0(u)$, $\phi_1(u)$ are elliptic polynomials of degree $N/2$ (of course for $N > 2$ there are many ways to do that). Specifying $P(u)$ as

$$P(u) = \phi_1(u-2)\,, \tag{4.32}$$

we rewrite (4.29) in the form

$$\Psi(u+2) + \frac{\phi_0(u-4)\phi_1(u)}{\phi_0(u-2)\phi_1(u-2)}\Psi(u-2) = \frac{A(u)}{\phi_0(u)\phi_1(u-2)}\Psi(u)\,, \tag{4.33}$$

where

$$\Psi(u) = \frac{Q(u)}{\phi_0(u-2)}\,. \tag{4.34}$$

Now, the coefficients in Eq. (4.33) being double-periodic, it is natural to consider its *double-Bloch solutions*. A meromorphic function $f(x)$ is said to be *double-Bloch* if it obeys the following monodromy properties:

$$f(x+2\omega_\alpha) = B_\alpha f(x), \qquad \alpha = 1,2. \tag{4.35}$$

The complex numbers B_α are called *Bloch multipliers*. It is easy to see that any double-Bloch function can be represented as a linear combination of elementary ones:

$$f(x) = \sum_{i=1}^{M} c_i \Phi(x - x_i, z)\kappa^{x/\eta}, \tag{4.36}$$

where [33]

$$\Phi(x, z) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta)\sigma(x)} \left[\frac{\sigma(z - \eta)}{\sigma(z + \eta)}\right]^{x/(2\eta)}, \tag{4.37}$$

and complex parameters z and κ are related by

$$B_\alpha = \kappa^{2\omega_\alpha/\eta} \exp(2\zeta(\omega_\alpha)(z + \eta)) \left(\frac{\sigma(z - \eta)}{\sigma(z + \eta)}\right)^{\omega_\alpha/\eta} \tag{4.38}$$

($\zeta(x) = \sigma'(x)/\sigma(x)$ is the Weierstrass ζ-function). Considered as a function of z, $\Phi(x, z)$ is double-periodic:

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z).$$

For general values of x one can define a single-valued branch of $\Phi(x, z)$ by cutting the elliptic curve between the points $z = \pm\eta$. In the fundamental domain of the lattice defined by $2\omega_\alpha$ the function $\Phi(x, z)$ has a unique pole at the point $x = 0$:

$$\Phi(x, z) = \frac{1}{x} + O(1)\,.$$

Coming back to the variable $u = x/\eta$, one can formulate the double-Bloch property of the function $\Psi(u)$ (4.34) in terms of its numerator $Q(u)$. It follows from (4.36) that the general form of $Q(u)$ is

$$Q(u) = Q(u; \nu) = e^{\nu\eta u} \prod_{j=1}^{M} \sigma(\eta(u - u_j))\,, \tag{4.39}$$

where $M = N/2$ and ν determines Bloch multipliers.

For the trigonometric and rational degeneration of Eqs. (4.4), (4.33), (4.39) the meaning of ν is quite clear: it plays the role of the “boundary phase" for twisted b.c. in the horizontal (auxiliary) direction. For each ν Eq. (4.12) has a solution of the form (4.39). The corresponding value of $T_1(u) = A(u)$ depends on ν as a parameter: $T_1(u) = T_1(u; \nu)$. If there exist $\nu \neq \nu'$ such that $T_1(u; \nu) = T_1(u; \nu')$, one may put $Q(u) = Q(u, \nu)$, $R(u) = Q(u; \nu')$. In the elliptic case the boundary phase in general is not compatible with integrability and so ν should have a different physical sense which is still unclear.

4.4. Bethe equations. It can be shown that for double-Bloch solutions the relation between ϕ and $\bar\phi$, $\bar\phi(u) = \phi(u - 2)$, implies

$$\bar{Q}(u) = Q(u - 1), \qquad \bar{R}(u) = R(u - 1)\,, \tag{4.40}$$

so that (see (4.16)

$$T_s(u) = \begin{vmatrix} Q(u + s + 1) & R(u + s + 1) \\ Q(u - s - 1) & R(u - s - 1) \end{vmatrix}\,. \tag{4.41}$$

It is clear that if $Q(u)$ and $R(u)$ are elliptic polynomials of degree $N/2$ multiplied by an exponential function (as in (4.39)), $T_s(u)$ has the desired general form (2.7).

Under condition (4.40) Eq. (4.18) yields the familiar result:

$$T_s(u) = Q(u+s+1)Q(u-s-1)\sum_{j=0}^{s} \frac{\phi(u-s+2j-1)}{Q(u-s+2j+1)Q(u-s+2j-1)} \,. \quad (4.42)$$

This formula was obtained in [29, 5] by direct solution of the fusion recurrence relations (4.10), (4.11).

Let u_j and v_j, $j = 1, \ldots, M$, be zeros of $Q(u)$ and $R(u)$, respectively. Then, evaluating (4.13) at $u = u_j$, $u = u_j - 2$ and $u = v_j$, $u = v_j - 2$ we obtain the relations

$$\phi(u_j) = Q(u_j+2)R(u_j)\,, \qquad \phi(u_j-2) = -Q(u_j-2)R(u_j)\,, \quad (4.43)$$

whence it holds

$$\frac{\phi(u_j)}{\phi(u_j-2)} = -\frac{Q(u_j+2)}{Q(u_j-2)}\,, \quad (4.44)$$

$$\frac{\phi(v_j)}{\phi(v_j-2)} = -\frac{R(v_j+2)}{R(v_j-2)}\,. \quad (4.45)$$

Equation (4.44) are exactly the standard Bethe equations (1.2). We refer to Eqs. (4.45) as *complementary Bethe equations*. It is easy to check that Eq. (4.44) ensure cancellation of poles in (4.42). A more standard way to derive Bethe equations (4.44), (4.45) is to substitute zeros of $Q(u)$ (or $R(u)$) directly into Baxter's relation (4.12). However, the wronskian relation (4.13) is somewhat more informative: in addition to Bethe equations for u_j, v_j it provides the connection (4.43) between them. In the next section we derive the system of nested Bethe ansatz equations starting from a proper generalization of Eq. (4.13).

In the elliptic case degrees of the elliptic polynomials $Q(u)$, $R(u)$ (for even N) are equal to $M = N/2$ (provided η is incommensurable with the lattice spanned by ω_1, ω_2). This fact follows directly from Bethe equations (4.44), (4.45) by the same argument as in Sect. 3.5.

In trigonometric and rational cases there are no such strong restrictions on degrees M and $\tilde{M}$ of Q and R respectively. This is because a part of their zeros may tend to infinity thus reducing the degree. Whence M and $\tilde{M}$ can be arbitrary integers not exceeding N. However, they must be complementary to each other: $M + \tilde{M} = N$. The traditional choice is $M \le N/2$. In particular, the solution $Q(u) = 1$ ($M = 0$) corresponds to the simplest reference state ("bare vacuum") of the model.

We already pointed out that the function $Q(u)$ originally introduced by Baxter (see e.g. [4] and references therein) emerged naturally in the context of the auxiliary linear problems. Let us mention that for models with the rational R-matrix this function can be treated as a limiting value of $T_s(u)$ as $s \to \infty$ [35]. Rational degeneration of Eqs. (2.7), (4.39) gives

$$T_s(u) = A_s \prod_{j=1}^{N} (u - z_j^{(s)})\,, \quad (4.46)$$

$$Q(u) = e^{\nu\eta u} \prod_{j=1}^{M} (u - u_j)\,, \quad (4.47)$$

where

$$A_s = \frac{\sinh(2\nu\eta(s+1))}{\sinh(2\nu\eta)} \,. \tag{4.48}$$

(The last expression follows from (4.42) by extracting the leading term as $u \to \infty$.) If the "boundary phase" $-i\nu\eta$ is real and $\nu \neq 0$, one has from (4.41):

$$Q(u) = \pm 2\sinh(2\nu\eta)e^{\nu\eta u} \lim_{s\to\mp\infty} e^{2\nu\eta s}\frac{T_{\mp s-1}(u+s)}{(2s)^{N-M}} \,. \tag{4.49}$$

For each finite $s \geq 0$ $T_s(u)$ has N zeros but in the limit some of them tend to infinity. The degenerate case $\nu = 0$ needs special analysis since the limits $\nu \to 0$ and $s \to \infty$ do not commute.

Another remark on the rational case is in order. Fusion relations (4.10), (4.11) give "Bethe ansatz like" equations for zeros of $T_s(u)$ (4.46). Substituting zeros of $T_s(u \pm 1)$ into (4.10), (4.11) and using (4.48) one finds:

$$\frac{\sinh(2\nu\eta(s+2))}{\sinh(2\nu\eta s)}\frac{\phi(z_j^{(s)}+s-1)}{\phi(z_j^{(s)}+s+1)} = -\prod_{k=1}^{N}\frac{z_j^{(s)}-z_k^{(s-1)}-1}{z_j^{(s)}-z_k^{(s+1)}+1} \,, \tag{4.50}$$

$$\frac{\sinh(2\nu\eta(s+2))}{\sinh(2\nu\eta s)}\frac{\phi(z_j^{(s)}-s-1)}{\phi(z_j^{(s)}-s-3)} = -\prod_{k=1}^{N}\frac{z_j^{(s)}-z_k^{(s-1)}+1}{z_j^{(s)}-z_k^{(s+1)}-1} \,. \tag{4.51}$$

These equations give the discrete dynamics of zeros in s. They are to be compared with dynamics of zeros of rational solutions of classical nonlinear equations [1, 32]. It is an interesting open problem to find elliptic analogues of Eqs. (4.49)-(4.51).

5. The A_{k-1}-Case: Discrete Time 2D Toda Lattice

5.1. General solution. The family of bilinear equations arising as a result of the Bäcklund flow (Sect. 3.4),

$$F_t^a(s,u+1)F_t^a(s,u-1) - F_t^a(s+1,u)F_t^a(s-1,u) = F_t^{a+1}(s,u)F_t^{a-1}(s,u) \,, \tag{5.1}$$

and the corresponding linear problems,

$$F_{t+1}^{a+1}(s+1,u)F_t^a(s,u) - F_{t+1}^{a+1}(s,u-1)F_t^a(s+1,u+1) = F_{t+1}^a(s,u)F_t^{a+1}(s+1,u) \,, \tag{5.2}$$

$$F_{t+1}^a(s+1,u-1)F_t^a(s,u) - F_{t+1}^a(s,u)F_t^a(s+1,u-1) = F_{t+1}^{a+1}(s,u-1)F_t^{a-1}(s+1,u) \,, \tag{5.3}$$

subject to the b.c.

$$F_t^a(s,u) = 0 \quad \text{as } a < 0 \text{ and } a > t. \tag{5.4}$$

They may be solved simultaneously by using the determinant representation (2.25). The set of functions $F_t^a(s,u)$ entering these equations as illustrated by the following diagram:

$$
\begin{array}{cccccc}
0 & 1 & 0 & & & \\
0 & F_1^0 & F_1^1 & 0 & & \\
0 & F_2^0 & F_2^1 & F_2^2 & 0 & \\
\dots & \dots & \dots & \dots & \dots & \\
0 & F_t^0 & F_t^1 & F_t^2 & \cdots & F_t^t \quad 0
\end{array}
\tag{5.5}
$$

(cf. (3.27)). Functions in each horizontal slice satisfy HBDE (5.1). By *level* of Eq. (5.1) we understand the number t. Level 0 is introduced for later convenience. At the moment we do not assume any relations between solutions at different levels.

Determinant formula (2.25) gives the solution to these equations for each level t in terms of t arbitrary holomorphic[3] functions $h_t^{(j)}(u+s)$ and t arbitrary antiholomorphic functions $\overline{h}_t^{(j)}(u-s)$. This is illustrated by the diagrams:

$$
\begin{array}{cccc}
1 & & & \\
h_1^{(1)} & h_1^{(2)} & & \\
h_2^{(1)} & h_2^{(2)} & h_2^{(3)} & \\
\dots & \dots & \dots & \\
h_t^{(1)} & h_t^{(2)} & \cdots & h_t^{(t+1)}
\end{array}
\qquad\qquad
\begin{array}{cccc}
1 & & & \\
\overline{h}_1^{(2)} & \overline{h}_1^{(1)} & & \\
\overline{h}_2^{(3)} & \overline{h}_2^{(2)} & \overline{h}_2^{(1)} & \\
\dots & \dots & \dots & \\
\overline{h}_t^{(t+1)} & \overline{h}_t^{(t)} & \cdots & \overline{h}_t^{(1)}
\end{array}
\tag{5.6}
$$

Then, according to (2.25), the general solution to Eq. (5.1) is

$$
F_{t+1}^a(s,u) =
$$
$$
= \chi_t^a(u+s)\overline{\chi}_t^a(u-s)
\begin{vmatrix}
h_t^{(t+1)}(u+s-a+2) & \cdots & h_t^{(1)}(u+s-a+2) \\
h_t^{(t+1)}(u+s-a+4) & \cdots & h_t^{(1)}(u+s-a+4) \\
\dots & \dots & \dots \\
h_t^{(t+1)}(u+s+a) & \cdots & h_t^{(1)}(u+s+a) \\
\overline{h}_t^{(t+1)}(u-s+a-t) & \cdots & \overline{h}_t^{(1)}(u-s+a-t) \\
\overline{h}_t^{(t+1)}(u-s+a-t+2) & \cdots & \overline{h}_t^{(1)}(u-s+a-t+2) \\
\dots & \dots & \dots \\
\overline{h}_t^{(t+1)}(u-s-a+t) & \cdots & \overline{h}_t^{(1)}(u-s-a+t)
\end{vmatrix},
\tag{5.7}
$$

[3] Here we call *holomorphic* (*antiholomorphic*) a function of $u+s$ (resp., $u-s$).

where $0 \le a \le t{+}1$ and the gauge functions $\chi_t^a(u)$, $\bar{\chi}_t^a(u)$ (introduced for normalization) satisfy the following equations:

$$\begin{aligned} &\chi_t^a(u+1)\chi_t^a(u-1) = \chi_t^{a+1}(u)\chi_t^{a-1}(u)\,,\\ &\bar{\chi}_t^a(u+1)\bar{\chi}_t^a(u-1) = \bar{\chi}_t^{a+1}(u)\bar{\chi}_t^{a-1}(u) \end{aligned} \tag{5.8}$$

(cf. (2.5)). The size of the determinant is $t+1$. The first a rows contain functions $h_i^{(j)}$, the remaining $t-a+1$ rows contain $\bar{h}_i^{(j)}$. The arguments of $h_i^{(j)}$, $\bar{h}_i^{(j)}$ increase by 2, going down a column. Note that the determinant in (5.7) (without the prefactors) is a solution itself. At $a=0$ ($a=t+1$) it is an antiholomorphic (holomorphic) function. The required b.c. (3.19) can be satisfied by choosing appropriate gauge functions χ_t^a, $\bar{\chi}_t^a$.

5.2. Canonical solution. The general solution (5.7) gives the function $T_s^a(u) \equiv F_k^a(s,u)$ in terms of $2k$ functions of one variable h_{k-1}^i and $\bar{h}_{k-1}^i$. However, we need to represent the solution in terms of another set of $2k$ functions $Q_t(u)$ and $\bar{Q}_t(u)$ by virtue of conditions (5.4) in such a way that Eqs. (5.2), (5.3) connecting two adjacent levels are fulfilled. We refer to this specification as the *canonical solution*.

To find it let us notice that at $a=0$ Eq. (5.2) consists of the holomorphic function $Q_t(u+s)$ and a function F^1. According to Eq. (5.7), F^1 is given by the determinant of the matrix with the holomorphic entries $h_t^{(i)}(u+s+1)$ in the first row. Other rows contain antiholomorphic functions only, so $F_t^1(u,s) = \sum_i h_t^{(i)}(u+s+1)\eta_i(u-s)$, where $\eta_i(u-s)$ are corresponding minors of the matrix (5.7) at $a=1$. Substituting this into Eq. (5.2) at $a=0$ and separating holomorphic and antiholomorphic functions one gets relations connecting $h_t^{(i)}, h_{t-1}^{(i)}$ and $Q_t(u), Q_{t+1}(u)$. Similar arguments can be applied to Eq. (5.3) at another boundary $a=t+1$. The general proof is outlined in the appendix to this section. Here we present the result:

$$h_t^{(1)}(u+s) = Q_t(u+s)\,, \qquad \bar{h}_t^{(1)}(u-s) = \bar{Q}_t(u-s) \tag{5.9}$$

and

$$Q_{t+1}(u-2)h_{t-1}^{(i)}(u) = \begin{vmatrix} h_t^{(i+1)}(u-2) & Q_t(u-2) \\ h_t^{(i+1)}(u) & Q_t(u) \end{vmatrix}, \tag{5.10}$$

$$\bar{Q}_{t+1}(u+1)\bar{h}_{t-1}^{(i)}(u+1) = \begin{vmatrix} \bar{h}_t^{(i+1)}(u) & \bar{Q}_t(u) \\ \bar{h}_t^{(i+1)}(u+2) & \bar{Q}_t(u+2) \end{vmatrix}, \tag{5.11}$$

where $1 \le i \le t$. Functions χ, $\bar{\chi}$ in front of the determinant (5.7) are then fixed as follows:

$$\begin{aligned} &\chi_t^a(u) = (-1)^{at}\left(\prod_{j=1}^{a-1} Q_{t+1}(u-a+2j)\right)^{-1}, \quad a \ge 2\,,\\ &\chi_t^0(u) = Q_{t+1}(u)\,, \qquad \chi_t^1(u) = (-1)^t\,, \end{aligned} \tag{5.12}$$

$$\begin{aligned} &\bar{\chi}_t^a(u) = \left(\prod_{j=1}^{t-a} \bar{Q}_{t+1}(u+a-t+2j-1)\right)^{-1}, \quad a \le t-1\,,\\ &\bar{\chi}_t^t(u) = 1\,, \qquad \bar{\chi}_t^{t+1}(u) = \bar{Q}_{t+1}(u)\,. \end{aligned} \tag{5.13}$$

It is easy to check that they do satisfy Eq. (5.8). The recursive relations (5.10), (5.11) allow one to determine functions $h_t^{(i)}$ and $h_t^{(i)}$ starting from a given set of $Q_t(u)$. These formulas generalize wronskian relations (4.13), (4.14) to the A_{k-1}-case.

Let us also note that this construction resembles the Leznov-Saveliev solution [39] to the continuous 2DTL with open boundaries.

5.3. The Bethe ansatz and canonical solution. The canonical solution of the previous section immediately leads to the nested Bethe ansatz for elliptic solutions.

In this case all functions $h_t^{(i)}, \bar{h}_t^{(i)}$ are elliptic polynomials multiplied by an exponential function:

$$h_t^{(i)}(u) = a_t^{(i)} e^{\nu_t^{(i)}\eta u} \prod_{j=0}^{M_t^{(i)}} \sigma(\eta(u - u_j^{t,i})) , \tag{5.14}$$

$$\bar{h}_t^{(i)}(u) = \bar{a}_t^{(i)} e^{\bar{\nu}_t^{(i)}\eta u} \prod_{j=0}^{\bar{M}_t^{(i)}} \sigma(\eta(u - \bar{u}_j^{t,i})) . \tag{5.15}$$

This implies a number of constraints on their zeros.

The determinant in (5.10) should be divisible by $Q_{t+1}(u-2)$ and $h_{t-1}^{(i)}(u)$, whence

$$\frac{h_t^{(i+1)}(u_j^{t+1})}{h_t^{(i+1)}(u_j^{t+1}+2)} = \frac{Q_t(u_j^{t+1})}{Q_t(u_j^{t+1}+2)} , \tag{5.16}$$

$$\frac{h_t^{(i+1)}(u_j^{t-1,i})}{h_t^{(i+1)}(u_j^{t-1,i}-2)} = \frac{Q_t(u_j^{t-1,i})}{Q_t(u_j^{t-1,i}-2)}, \tag{5.17}$$

where $u_j^t \equiv u_j^{t,1}$. Furthermore, it is possible to get a closed system of constraints for the roots of $Q_t(u)$ only. Indeed, choosing $u = u_j^t$, $u = u_j^t + 2$ in (5.10), we get

$$Q_{t+1}(u_j^t - 2) Q_{t-1}(u_j^t) = -Q_t(u_j^t - 2) h_t^{(2)}(u_j^t) , \tag{5.18}$$

$$Q_{t+1}(u_j^t) Q_{t-1}(u_j^t + 2) = Q_t(u_j^t + 2) h_t^{(2)}(u_j^t) . \tag{5.19}$$

Dividing Eq. (5.18) by Eq. (5.19) we obtain the system of nested Bethe equations:

$$\frac{Q_{t-1}(u_j^t+2) Q_t(u_j^t-2) Q_{t+1}(u_j^t)}{Q_{t-1}(u_j^t) Q_t(u_j^t+2) Q_{t+1}(u_j^t-2)} = -1 , \tag{5.20}$$

which coincides with (3.31) from Sect. 3.5.

Similar relations hold true for the h-diagram:

$$\frac{\bar{h}_t^{(i+1)}(\bar{u}_j^{t+1}+1)}{\bar{h}_t^{(i+1)}(\bar{u}_j^{t+1}-1)} = \frac{\bar{Q}_t(\bar{u}_j^{t+1}+1)}{Q_t(\bar{u}_j^{t+1}-1)} , \tag{5.21}$$

$$\frac{\bar{h}_t^{(i+1)}(\bar{u}_j^{t-1,i}+1)}{\bar{h}_t^{(i+1)}(\bar{u}_j^{t-1,i}-1)} = \frac{\bar{Q}_t(\bar{u}_j^{t-1,i}+1)}{\bar{Q}_t(\bar{u}_j^{t-1,i}-1)} , \tag{5.22}$$

$$\bar{Q}_{t+1}(\bar{u}_j^t + 1) \bar{Q}_{t-1}(\bar{u}_j^t + 1) = \bar{Q}_t(\bar{u}_j^t + 2) \bar{h}_t^{(2)}(\bar{u}_j^t) , \tag{5.23}$$

$$\bar{Q}_{t+1}(\bar{u}_j^t - 1)\bar{Q}_{t-1}(\bar{u}_j^t - 1) = -\bar{Q}_t(\bar{u}_j^t - 2)\bar{h}_t^{(2)}(\bar{u}_j^t)\,, \tag{5.24}$$

$$\frac{\bar{Q}_{t-1}(\bar{u}_j^t + 1)\bar{Q}_t(\bar{u}_j^t - 2)\bar{Q}_{t+1}(\bar{u}_j^t + 1)}{Q_{t-1}(\bar{u}_j^t - 1)Q_t(\bar{u}_j^t + 2)Q_{t+1}(\bar{u}_j^t - 1)} = -1\,. \tag{5.25}$$

These conditions are sufficient to ensure that the canonical solution for $T_s^a(u)$ (i.e., for $F_k^a(s,u)$) has the required general form (2.7). To see this, take a generic Q-factor from the product (5.12), $(Q_{t+1}(u-a+2j))^{-1}$. It follows from (5.16) that at its poles the j^{th} and $j+1^{\mathrm{th}}$ rows of the determinant (5.7) become proportional. The same argument repeated for $\bar{Q}$-factors shows that $F_{t+1}^a(s,u)$ has no poles.

Finally, it is straightforward to see from (5.7) that the constraint $\bar{Q}_t(u) = Q_t(u-t)$ leads to condition (2.15) (for $-t \le s \le -1$ two rows of the determinant become equal).

To summarize, the solution goes as follows. First, one should find a solution to Bethe equations (3.31) thus getting a set of elliptic polynomials $Q_t(u)$, $t = 1,\ldots,k-1$, $Q_0(u) = 1$, $Q_k(u) = \phi(u)$ being a given function. To make the chain of equations finite, it is convenient to use the formal convention $Q_{-1}(u) = Q_{k+1}(u) = 0$. Second, one should solve step by step relations (5.10), (5.11) and find the functions $h_t^{(i)}(u)$, $\bar{h}_t^{(i)}(u)$. All these relations are of the same type as the wronskian relation (4.13) in the A_1-case: each of them is a linear inhomogeneous first order difference equation.

5.4. Conservation laws. The solution described in Sects. 5.2 and 5.3 provides compact determinant formulas for eigenvalues of quantum transfer matrices. It also provides determinant representations for conservation laws of the s-dynamics which generalize Eqs. (4.7), (4.8) to the A_{k-1}-case. The generalization comes up in the form of Eqs. (4.22), (4.23) and (4.26). The conservation laws (i.e., integrals of the s-dynamics) follow from the determinant representation (5.7) of the general solution to HBDE.

Let us consider $(C_k^a + 1) \times (C_k^a + 1)$-matrices

$$\mathcal{T}^a_{B,B'}(s,u) \equiv T^a_{s+B+B'}(u - s + B - B'), \qquad B, B' = 1,\ldots,C_k^a + 1\,, \tag{5.26}$$

$$\bar{\mathcal{T}}^a_{B,B'}(s,u) = T^a_{s-B-B'}(u + s + B - B'), \qquad B, B' = 1,\ldots,C_k^a + 1\,, \tag{5.27}$$

where C_k^a is the binomial coefficient. Let $\mathcal{T}^a[P|R](s,u)$ be minors of the matrix (5.26) with row P and column R removed (similarly for (5.27)).

Theorem 5.1. *Let $T_s^a(u)$ be the general solution to HBDE given by Eq. (5.7). Then any ratio of the form*

$$A^{a,R}_{P,P'}(s,u) \equiv \frac{\mathcal{T}^a[P|R](s,u)}{\mathcal{T}^a[P'|R](s,u)} \tag{5.28}$$

does not depend on s. These quantities are integrals of the s-dynamics: $A^{a,R}_{P,P'}(s,u) = A^{a,R}_{P,P'}(u)$. Similarly, minors of the matrix (5.27) give in the same way a complimentary set of conservation laws[4].

A sketch of proof is as follows.

Consider the Laplace expansion of the determinant solution (5.7) with respect to the first a (holomorphic) rows:

$$T_s^a(u) = \sum_{P=1}^{C_k^a} \psi_P^a(u+s)\bar{\psi}_P^a(u-s). \tag{5.29}$$

[4] Compare with (4.7), (4.8).

Here P numbers (in an arbitrary order) sets of indices $(p_1, p_2, \ldots, p_a)$ such that $k \geq p_1 > p_2 > \ldots > p_a \geq 1$, $\psi^a_P(u+s)$ is minor of the matrix in Eq. (5.7) constructed from first a rows and columns $p_1, \ldots, p_a$ (multiplied by $\chi^a_{k-1}(u+s)$), $\psi^a_P(u-s)$ is the complimentary minor (multiplied by $\bar{\chi}^a_{k-1}(u-s)$).

Substitute R^{th} column of the matrix (5.26) by the column vector with components $\psi^a_P(u+2B)$, $B = 1, \ldots, C^a_k + 1$. The matrix obtained this way (let us call it $(\mathcal{T}^{a;R,P})_{B,B'}$) depends on $R = 1, \ldots, C^a_k + 1$, $P = 1, \ldots, C^a_k$ and $a = 1, \ldots, k-1$. The "complementary" matrix $(\bar{\mathcal{T}}^{a;R,P})_{B,B'}$ is defined by the similar substitution of the column vector $\bar{\psi}^a_P(u+2B)$, $B = 1, \ldots, C^a_k + 1$, into the matrix (5.27).

Lemma 5.1. *Determinants of all the four matrices introduced above vanish:*

$$\det(\mathcal{T}^a) = \det(\bar{\mathcal{T}}^a) = \det(\mathcal{T}^{a;R,P}) = \det(\bar{\mathcal{T}}^{a;R,P}) = 0\,. \tag{5.30}$$

The proof follows from the Laplace expansion (5.29). From this representation it is obvious that $C^a_k + 1$ columns of the matrices in (5.30) are linearly dependent. This identity is valid for arbitrary functions $h^{(i)}_t(u+s)$, $\bar{h}^{(i)}_t(u-s)$ in Eq. (5.7).

The conservation laws immediately follow from these identities. Indeed, let us rewrite the determinant of the matrix $\mathcal{T}^{a;R,P}$ as a linear combination of entries of the R^{th} column:

$$\det(\mathcal{T}^{a;R,P}) = \sum_{B'=1}^{C^a_k+1} (-1)^{B'+R} \psi^a_P(u+2B') \mathcal{T}^a[B'|R](s,u) = 0\,. \tag{5.31}$$

Dividing by $\mathcal{T}^a[P'|R](s,u)$, we get, using the notation (5.28):

$$\sum_{B'=1, B'\neq P'}^{C^a_k+1} (-1)^{B'} \psi^a_P(u+2B') A^{a,R}_{B',P'}(s,u) = (-1)^{P'+1} \psi^a_P(u+2P')\,. \tag{5.32}$$

The latter identity is a system of C^a_k linear equations for C^a_k quantities $A^{a,R}_{1,P'}(s,u)$, $A^{a,R}_{2,P'}(s,u)$, $\ldots$, $A^{a,R}_{P'-1,P'}(s,u)$, $A^{a,R}_{P'+1,P'}(s,u)$, $\ldots$, $A^{a,R}_{C^a_k+1,P'}(s,u)$. In the case of the general position the wronskian of the functions $\psi^a_P(u)$ is nonzero, whence system (5.32) has a unique solution for $A^{a,R}_{P,P'}(s,u)$. The coefficients of the system do not depend on s. Therefore, $A^{a,R}_{P,P'}(s,u)$ are s-independent too. Similar arguments areapplied to minors of the matrix (5.27).

Another form of Eq. (5.31) may be obtained by multiplication its l.h.s. by $\bar{\psi}^a_P(u-2s)$ and summation over P. This yields

$$\sum_{B=1}^{C^a_k+1} (-1)^B T^a_{s+B}(u-s+B) \mathcal{T}^a[B|R](s,u) = 0\,, \tag{5.33}$$

which is a difference equation for $T^a_s(u)$ as a function of the "holomorphic" variable $u+s$ with fixed $u-s$.

5.5. Generalized Baxter's relations. Equation (5.31) can be considered as a linear difference equation for a function $\psi^a(u)$ having C^a_k linearly independent solutions $\psi^a_P(u)$. It provides the A_{k-1}-generalization of Baxter's relations (4.4), (4.5). This generalization comes up in the form of Eqs. (4.22), (4.23) and (4.26).

The simplest cases are $a = 1$ and $a = k - 1$. Then there are $k + 1$ terms in the sum (5.31). Furthermore, it is obvious that

$$\psi_i^1(u) = h_{k-1}^{(i)}(u+1), \qquad \bar{\psi}_i^{k-1}(u) = \bar{h}_{k-1}^{(i)}(u)\,. \tag{5.34}$$

Then Eq. (5.31) and a similar equation for antiholomorphic parts read:

$$\sum_{j=1}^{k+1}(-1)^j h_{k-1}^{(i)}(u+2j+1)\mathcal{T}^1[j|k+1](s,u) = 0\,, \tag{5.35}$$

$$\sum_{j=1}^{k+1}(-1)^j \bar{h}_{k-1}^{(i)}(u+2j)\bar{\mathcal{T}}^{k-1}[j|k+1](s,u) = 0\,, \tag{5.36}$$

where we put $R = k + 1$ for simplicity. These formulas may be understood as linear difference equations of order k. Indeed, Eq. (5.35) can be rewritten as the following equation for a function $X(u)$:

$$\begin{vmatrix} T_s^1(u) & T_{s+1}^1(u-1) & \dots & T_{s+k-1}^1(u-k+1) & X(u+s+1) \\ T_{s+1}^1(u+1) & T_{s+2}^1(u) & \dots & T_{s+k}^1(u-k+2) & X(u+s+3) \\ \dots & \dots & \dots & \dots & \dots \\ T_{s+k}^1(u+k) & T_{s+k+1}^1(u+k-1) & \dots & T_{s+2k-1}^1(u+1) & X(u+s+2k+1) \end{vmatrix} = 0\,. \tag{5.37}$$

This equation has k solutions $h_{k-1}^{(i)}(u)$, $i = 1,\dots,k$. One of them is $Q_{k-1} \equiv h_{k-1}^{(1)}(u)$ (see Eq.(5.9)). Similarly Eq. (5.36) for the antiholomorphic parts,

$$\begin{vmatrix} T_s^{k-1}(u) & T_{s-1}^{k-1}(u-1) & \dots & T_{s-k+1}^{k-1}(u-k+1) & \bar{X}(u-s) \\ T_{s-1}^{k-1}(u+1) & T_{s-2}^{k-1}(u) & \dots & T_{s-k}^{k-1}(u-k+2) & \bar{X}(u-s+2) \\ \dots & \dots & \dots & \dots & \dots \\ T_{s-k}^{k-1}(u+k) & T_{s-k-1}^{k-1}(u+k-1) & \dots & T_{s-2k+1}^{k-1}(u+1) & \bar{X}(u-s+2k) \end{vmatrix} = 0\,, \tag{5.38}$$

has k solutions $\bar{h}_{k-1}^{(i)}(u)$, $i = 1,\dots,k$. One of them is $\bar{Q}_{k-1} \equiv \bar{h}_{k-1}^{(1)}(u)$.

Difference equations (5.37), (5.38) can be rewritten in the factorized form. This fact follows from a more general statement. Fix an arbitrary level k and set $T_s^a(u) = F_k^a(s,u)$, $F^a(s,u) = F_{k-1}^a(s,u)$ (as in Sect. 3).

Proposition 5.1. *For each* $j = 0, 1, \dots, k-1$ *it holds:*

$$\big(e^{\partial_s+\partial_u} - R_{j+1}^{(j)}(s,u)\big)\big(e^{\partial_s+\partial_u} - R_j^{(j)}(s,u)\big)\dots\big(e^{\partial_s+\partial_u} - R_1^{(j)}(s,u)\big)F^{k-1-j}(s,u) = 0\,, \tag{5.39}$$

$$\big(e^{\partial_s-\partial_u} - \bar{R}_{j+1}^{(j)}(s,u)\big)\big(e^{\partial_s-\partial_u} - \bar{R}_j^{(j)}(s,u)\big)\dots\big(e^{\partial_s-\partial_u} - \bar{R}_1^{(j)}(s,u)\big)F^{j}(s,u) = 0\,, \tag{5.40}$$

where

$$R_i^{(k-1-j)}(s,u) = \frac{T^j_{s+i-1}(u+i-1)T^{j+i-1}_{s+i-2}(u-1)T^{j+i}_{s+i}(u)}{T^j_{s+i-2}(u+i-2)T^{j+i-1}_{s+i-1}(u)T^{j+i}_{s+i-1}(u-1)}, \tag{5.41}$$

$$\bar{R}_i^{(j)}(s,u) = \frac{T^{j+1}_{s+i-1}(u-l)T^{j-i+1}_{s+l}(u-1)T^{j-i+2}_{s+i-2}(u)}{T^{j+1}_{s+i-2}(u-i+1)T^{j-i+1}_{s+i-1}(u)T^{j-i+2}_{s+i-1}(u-1)}. \tag{5.42}$$

Proof. The proof is by induction. At $j=0$ Eq. (5.39) turns into

$$\left(e^{\partial_s+\partial_u} - \frac{T^k_{s+1}(u)}{T^k_s(u-1)}\right) F^{k-1}(s,u) = 0\,.$$

This means that $F^{k-1}(s,u)$ does not depend on $u+s$. Further,

$$F^a(s+1,u) = -\frac{T^a_s(u-1)}{T^{a-1}_s(u)}\left(e^{\partial_s+\partial_u} - \frac{T^a_{s+1}(u)}{T^a_s(u-1)}\right) F^{a-1}(s,u)\,, \tag{5.43}$$

(see (3.8)). The inductive step is then straightforward. The proof of (5.40) is absolutely identical.

Now, putting $j=k-1$ we get the following difference equations in one variable:

$$\begin{aligned}&\left(e^{2\partial_u+\partial_s} - R_k^{(k-1)}(s,u-s)\right)\left(e^{2\partial_u+\partial_s} - R_{k-1}^{(k-1)}(s,u-s)\right)\\ &\dots\left(e^{2\partial_u+\partial_s} - R_1^{(k-1)}(s,u-s)\right)Q_{k-1}(u) = 0\,,\end{aligned} \tag{5.44}$$

$$\begin{aligned}&(e^{-2\partial_u+\partial_s} - \bar{R}_k^{(k-1)}(s,u+s))(e^{-2\partial_u+\partial_s} - \bar{R}_{k-1}^{(k-1)}(s,u+s))\\ &\dots(e^{-2\partial_u+\partial_s} - \bar{R}_1^{(k-1)}(s,u+s))\bar{Q}_{k-1}(u) = 0\,.\end{aligned} \tag{5.45}$$

Note that operators $e^{\pm\partial_s}$ act only on the coefficient functions in (5.44), (5.45). These equations provide a version of the discrete Miura transformation of generalized Baxter's operators, which is different from the one discussed in Ref. [15] (see also below).

Coming back to Eq. (5.31) and using relations (5.10), (5.11), one finds:

$$\psi_k^{k-1}(u) = h_1^{(1)}(u+k-1) = Q_1(u+k-1)\,, \tag{5.46}$$

$$\bar{\psi}_k^1(u) = \bar{h}_1^{(1)}(u) = \bar{Q}_1(u) \tag{5.47}$$

(for the proof see Lemma 5.2 in the appendix to this section).

Then, in complete analogy with Eqs. (5.37), (5.38), one obtains from (5.31) the following difference equations:

$$\begin{vmatrix} T_s^{k-1}(u) & T_{s+1}^{k-1}(u-1) & \dots & T_{s+k-1}^{k-1}(u-k+1) & X(u+s+k-1) \\ T_{s+1}^{k-1}(u+1) & T_{s+2}^{k-1}(u) & \dots & T_{s+k}^{k-1}(u-k+2) & X(u+s+k+1) \\ \dots & \dots & \dots & \dots & \dots \\ T_{s+k}^{k-1}(u+k) & T_{s+k+1}^{k-1}(u+k-1) & \dots & T_{s+2k-1}^{k-1}(u+1) & X(u+s+3k-1) \end{vmatrix} = 0\,, \tag{5.48}$$

$$\begin{vmatrix} T_s^1(u) & T_{s-1}^1(u-1) & \dots & T_{s-k+1}^1(u-k+1) & \bar{X}(u-s) \\ T_{s-1}^1(u+1) & T_{s-2}^1(u) & \dots & T_{s-k}^1(u-k+2) & \bar{X}(u-s+2) \\ \dots & \dots & \dots & \dots & \dots \\ T_{s-k}^1(u+k) & T_{s-k-1}^1(u+k-1) & \dots & T_{s-2k+1}^1(u+1) & \bar{X}(u-s+2k) \end{vmatrix} = 0 \qquad (5.49)$$

to which $Q_1(u)$ (resp., $\bar{Q}_1(u)$) is a solution. The other $k-1$ linearly independent solutions to Eq. (5.48) (resp., (5.49)) are other algebraic complements of the last (first) line of the matrix in Eq. (5.7) at $a = k-1$ ($a = 1$) multiplied by χ_{k-1}^{k-1} ($\bar{\chi}_{k-1}^{1}$).

Further specification follows from imposing constraints (3.12) which ensure conditions (2.4) forced by the usual Bethe ansatz. One can see that under these conditions Eqs. (5.48) and (5.49) become the same. Further, substituting a particular value of s, $s = -k$, into, say, Eqs. (5.48), (5.37), one gets the following difference equations:

$$\sum_{a=0}^{k}(-1)^a T_1^a(u+a-1)Q_1(u+2a-2) = 0\,, \qquad (5.50)$$

$$\sum_{a=0}^{k}(-1)^a \frac{T_1^a(u-a-1)}{\phi(u-2a-2)}\frac{Q_{k-1}(u-2a)}{\phi(u-2a)} = 0 \qquad (5.51)$$

(we remind the reader that $\phi(u) \equiv Q_k(u)$). The latter equation can be obtained directly from the determinant formula (5.7): notice that under conditions (2.4) the determinants in Eq. (5.7) become minors of the matrix $h_{k-1}^{(i)}(u-2k+2j)$, where i numbers columns running from 1 to k, j numbers lines and runs from 0 to k skipping the value $k-a$. Taking care of the prefactors in Eq. (5.7) and recalling that $h_{k-1}^{(1)}(u) = Q_{k-1}(u)$, one gets Eq. (5.51). These formulas give a generalization of the Baxter equations (4.4), (4.5), (4.12).

At last, we are to identify our Q_t's with Q_t's from the usual nested Bethe ansatz solution. This is achieved by factorization of the difference operators in (5.50) and (5.51) in terms of $Q_t(u)$. Using the technique developed in the appendix to this section, one can prove the following factorization formulas:

$$\sum_{a=0}^{k}(-1)^{a-k}\frac{T_1^a(u+a-1)}{\phi(u-2)}e^{2a\partial_u} = \left(e^{2\partial_u} - \frac{Q_k(u)Q_{k-1}(u-2)}{Q_k(u-2)Q_{k-1}(u)}\right)$$
$$\dots\left(e^{2\partial_u} - \frac{Q_2(u)Q_1(u-2)}{Q_2(u-2)Q_1(u)}\right)\left(e^{2\partial_u} - \frac{Q_1(u)}{Q_1(u-2)}\right), \qquad (5.52)$$

$$\sum_{a=0}^{k}(-1)^{a-k}\frac{T_1^a(u-a-1)}{\phi(u-2a-2)}e^{-2a\partial_u} = \left(e^{-2\partial_u} - \frac{Q_1(u)}{Q_1(u-2)}\right)$$
$$\left(e^{-2\partial_u} - \frac{Q_2(u)Q_1(u-2)}{Q_2(u-2)Q_1(u)}\right)\dots\left(e^{-2\partial_u} - \frac{Q_k(u)Q_{k-1}(u-2)}{Q_k(u-2)Q_{k-1}(u)}\right). \qquad (5.53)$$

Note that these operators are adjoint to each other. The l.h.s. of Eq. (5.52) or (5.53) is known as the generating function for $T_1^a(u)$; $T_s^a(u)$ for $s>1$ can be found with the help of determinant formula (2.24). These formulas for the generating function coincide with the ones known in the literature (see e.g. [5, 38]). They yield $T_1^a(u)$ in terms of elliptic polynomials Q_t with roots constrained by the nested Bethe ansatz equations which ensure cancellation of poles in $T_1^a(u)$.

5.6. Appendix to Section 5. Here we outline the proof of the result of Sect. 5. It is enough to prove that the canonical solution does satisfy Eqs. (5.2), (5.3) connecting adjacent levels. The idea is to show that they are equivalent to the elementary Plücker relation (2.21). We proceed in steps.

First step: Preliminaries. We need the determinant identity

$$\det_{1\le m,n\le k}\left(\begin{vmatrix} a_{m,n} & a_{m,k+1} \\ a_{m+1,n} & a_{m+1,k+1}\end{vmatrix}\right) = \left(\prod_{j=2}^{k} a_{j,k+1}\right) \det_{1\le m,n\le k+1}(a_{m,n}) \tag{5.54}$$

valid for an arbitrary $(k+1)\times(k+1)$-matrix $a_{m,n}$, $1\le m,n\le k+1$. It can be easily proved by induction.

Let us consider minors of the matrices $h_t^{(j)}(u+2i)$, $\bar{h}_t^{(j)}(u+2i)$, $1\le i,j\le t+1$ of size $a\times a$:

$$H_t^{(i_1,i_2,\ldots,i_a)}(u) = \det_{1\le\alpha,\beta\le a}\left(h_t^{(i_\beta)}(u+2\alpha-2)\right) \tag{5.55}$$

and the same expression for $\bar{H}_t$'s through $\bar{h}_t$'s. The following technical lemma follows directly from Eq. (5.54):

Lemma 5.2. *If relations (5.9)-(5.11) hold, then*

$$\frac{H_{t-1}^{(i_1,i_2,\ldots,i_a)}(u+1)}{\prod_{j=1}^{a-1} Q_t(u+2j-1)} = \frac{H_t^{(i_1+1,i_2+1,\ldots,i_a+1,1)}(u-1)}{\prod_{j=1}^{a} Q_{t+1}(u+2j-3)}, \tag{5.56}$$

$$\frac{\bar{H}_{t-1}^{(i_1,i_2,\ldots,i_a)}(u+1)}{\prod_{j=1}^{a-1} \bar{Q}_t(u+2j)} = \frac{\bar{H}_t^{(i_1+1,i_2+1,\ldots,i_a+1,1)}(u)}{\prod_{j=1}^{a} \bar{Q}_{t+1}(u+2j-1)}. \tag{5.57}$$

Relations (5.46), (5.47) are direct corollaries of the lemma.

Second step: From $h_t^{(i)}$'s to q_i's. Let us fix a level k and define the quantities

$$q_i(u) = \frac{H_{k-1}^{(k,k-1,\ldots,\widehat{k-i+1},\ldots,1)}(u-2k+4)}{\prod_{j=1}^{k-2} Q_k(u-2k+2j+2)}, \tag{5.58}$$

$$\bar{q}_i(u) = \frac{\bar{H}_{k-1}^{(k,k-1,\ldots,\widehat{k-i+1},\ldots,1)}(u-k+2)}{\prod_{j=1}^{k-2} \bar{Q}_k(u-k+2j+1)} \tag{5.59}$$

for $1\le i\le k$. The hat means that the corresponding index is skipped. Due to Lemma 5.2 these quantities actually do not depend on the particular value of k used in the definition. More precisely, define $q_i(u)$, $\bar{q}_i(u)$ with respect to any level $k'>k$, then they coincide with those previously defined for $1\le i\le k$.

With this definition, one can prove

Lemma 5.3. *Fix an arbitrary level* $k > 1$. *Let* m_α, $\alpha = 1, 2, \ldots, r$, *be a set of integers such that* $k \geq m_1 > m_2 > \ldots > m_r \geq 1$ *and let* $\tilde{m}_\alpha$, $\alpha = 1, 2, \ldots, k - r$, *be its complement to the set* $1, 2, \ldots, k$ *ordered in the same way:* $k \geq \tilde{m}_1 > \tilde{m}_2 > \ldots > \tilde{m}_{k-r} \geq 1$. *Then the following identities hold:*

$$\det_{1\leq\alpha,\beta\leq r}(q_{m_\beta}(u + 2\alpha - 2)) = \frac{\det_{1\leq\alpha,\beta\leq k-r}(h_{k-1}^{(\tilde{m}_\beta)}(u + 2r - 2k + 2\alpha))}{\prod_{j=1}^{k-r-1} Q_k(u + 2r - 2k + 2j)}, \tag{5.60}$$

$$\det_{1\leq\alpha,\beta\leq r}(\bar{q}_{m_\beta}(u + 2\alpha - 2)) = \frac{\det_{1\leq\alpha,\beta\leq k-r}(\bar{h}_{k-1}^{(\tilde{m}_\beta)}(u + 2r - k + 2\alpha - 2))}{\prod_{j=1}^{k-r-1} \bar{Q}_k(u + 2r - k + 2j - 1)}. \tag{5.61}$$

Let us outline the proof. At $r = 1$, these identities coincide with the definitions of q_i, $\bar{q}_i$. At $r = 2$, they follow from the Jacobi identity (2.20). The inductive step consists in expanding the determinant in the left hand side in the first row and then making use of determinant identities equivalent to the $r + 1$-term Plücker relation.

The identities from Lemma 5.3 allow one to express the canonical solution in terms of q_i, $\bar{q}_i$. The Laplace expansion of the determinant in Eq. (5.7) combined with Eqs. (5.60), (5.61) yields:

$$F_t^a(s, u) = (-1)^{a(t-1)} \begin{vmatrix} q_t(u+s+a) & \cdots & q_1(u+s+a) \\ q_t(u+s+a+2) & \cdots & q_1(u+s+a+2) \\ \ldots & \cdots & \ldots \\ q_t(u+s+2t-a-2) & \cdots & q_1(u+s+2t-a-2) \\ \bar{q}_t(u-s-a+1) & \cdots & \bar{q}_1(u-s-a+1) \\ \bar{q}_t(u-s-a+3) & \cdots & \bar{q}_1(u-s-a+3) \\ \ldots & \cdots & \ldots \\ \bar{q}_t(u-s+a-1) & \cdots & \bar{q}_1(u-s+a-1) \end{vmatrix}. \tag{5.62}$$

In particular, we have:

$$F_t^0(s, u) = Q_t(u + s) = \det_{1\leq i,j\leq t} q_{t+1-j}(u + s + 2i - 2)\,, \tag{5.63}$$

$$F_t^t(s, u) = \bar{Q}_t(u - s) = \det_{1\leq i,j\leq t} \bar{q}_{t+1-j}(u - s - t + 2i - 1)\,. \tag{5.64}$$

Third step: The Plücker relation. Consider the rectangular $(t + 3) \times (t + 1)$-matrix S_{ij}, $i = 1, 2, \ldots, t + 3$, $i = 1, 2, \ldots, t + 1$, given explicitly by

$$\begin{aligned} S_{1j} &= \delta_{1j}\,, \\ S_{ij} &= q_{t+2-j}(u + s + a + 2i - 4)\,, \qquad 2 \leq i \leq t - a + 2\,, \\ S_{ij} &= \bar{q}_{t+2-j}(u - s + a + 2j - 2t - 7)\,, \qquad t - a + 3 \leq i \leq t + 3\,. \end{aligned} \tag{5.65}$$

Applying the determinant identity (2.21) (the elementary Plücker relation) to minors of this matrix, one gets Eq. (5.2) for $l_1 = 1$, $l_2 = 2$, $l_3 = t - a + 2$, $l_4 = t - a + 3$ and Eq. (5.3) for $l_1 = 1$, $l_2 = t - a + 2$, $l_3 = t - a + 3$, $l_4 = t + 1$. This completes the proof.

Remark. Functions $q_i(u)$, $\overline{q}_i(u)$, $i = 1, 2 \dots, k$, are linearly independent solutions to generalized Baxter's equations (5.48), (5.49) respectively. To construct an elliptic polynomial solution for $T^a_s(u)$, it is sufficient to take them to be arbitrary elliptic polynomials of one and the same degree d,

$$q_i(u) = e^{\zeta_i \eta u} \prod_{l=1}^{d} \sigma(\eta(u - v_l^{(i)})), \qquad \overline{q}_i(u) = e^{\overline{\zeta}_i \eta u} \prod_{l=1}^{d} \sigma(\eta(u - \overline{v}_l^{(i)})),$$

with the only conditions that $\zeta_i - \overline{\zeta}_i$, $\sum_{l=1}^{d}(v_l^{(i)} - \overline{v}_l^{(i)})$ do not depend on $i = 1, 2, \dots, k$. It is easy to check that in this case general conditions (2.8), (2.9) are fulfilled.

6. Regular Elliptic Solutions of the HBDE and RS System in Discrete Time

In this section we study the class of elliptic solutions to HBDE for which the number of zeros M_t of the τ-function does not depend on t. We call them *elliptic solutions of the regular type* (or simply *regular elliptic solutions*) since they have a smooth continuum limit. Although it has been argued in the previous section that the situation of interest for the Bethe ansatz is quite opposite, we find it useful to briefly discuss this class of solutions.

It is convenient to slightly change the notation: $\tau^{l,m}(x) \equiv \tau_u(-m, -l), x \equiv u\eta$. HBDE (1.8) acquires the form

$$\tau^{l+1,m}(x)\tau^{l,m+1}(x) - \tau^{l+1,m+1}(x)\tau^{l,m}(x) = \tau^{l+1,m}(x + \eta)\tau^{l,m+1}(x - \eta)\,. \tag{6.1}$$

We are interested in solutions that are elliptic polynomials in x,

$$\tau^{l,m}(x) = \prod_{j=1}^{M} \sigma(x - x_j^{l,m})\,. \tag{6.2}$$

The main goal of this section is to describe this class of solutions in a systematic way and, in particular, to prove that *all* the elliptic solutions of regular type are finite-gap.

The auxiliary linear problems (3.5) look as follows:

$$\Psi^{l,m+1}(x) = \Psi^{l,m}(x + \eta) + \frac{\tau^{l,m}(x)\tau^{l,m+1}(x + \eta)}{\tau^{l,m+1}(x)\tau^{l,m}(x + \eta)}\Psi^{l,m}(x)\,, \tag{6.3}$$

$$\Psi^{l+1,m}(x) = \Psi^{l,m}(x) + \frac{\tau^{l,m}(x - \eta)\tau^{l+1,m}(x + \eta)}{\tau^{l+1,m}(x)\tau^{l,m}(x)}\Psi^{l,m}(x - \eta)\,. \tag{6.4}$$

(The notation is correspondingly changed: $\Psi^{l,m}(u\eta) \equiv \psi_u(-m, -l)$.) The coefficients are elliptic functions of x. Similarly to the case of the Calogero-Moser model and its spin generalizations [31, 32] the dynamics of their poles is determined by the fact that Eqs. (6.3), (6.4) have infinite number of double-Bloch solutions (Sect. 4).

The "gauge transformation" $f(x) \to \tilde{f}(x) = f(x)e^{ax}$ (a is an arbitrary constant) does not change poles of any function and transforms a double-Bloch function into another double-Bloch function. If B_α are Bloch multipliers for f, then the Bloch multipliers for $\tilde{f}$ are $\tilde{B}_1 = B_1 e^{2a\omega_1}$, $\tilde{B}_2 = B_2 e^{2a\omega_2}$, where ω_1, ω_2 are quasiperiods of the σ-function. Two pairs of Bloch multipliers are said to be *equivalent* if they are connected by this relation with some a (or by the equivalent condition that the product $B_1^{\omega_2} B_2^{-\omega_1}$ is the same for both pairs).

Consider first Eq. (6.3). Since l enters as a parameter, not a variable, we omit it for simplicity of the notation (e.g. $x_j^{l,m} \to x_j^m$).

Theorem 6.1. *Equation (6.3) has an infinite number of linearly independent double-Bloch solutions with simple poles at the points x_i^m and equivalent Bloch multipliers if and only if x_i^m satisfy the system of equations*

$$\prod_{j=1}^{M} \frac{\sigma(x_i^m - x_j^{m+1})\sigma(x_i^m - x_j^m - \eta)\sigma(x_i^m - x_j^{m-1} + \eta)}{\sigma(x_i^m - x_j^{m+1} - \eta)\sigma(x_i^m - x_j^m + \eta)\sigma(x_i^m - x_j^{m-1})} = -1\,. \tag{6.5}$$

All these solutions can be represented in the form

$$\Psi^m(x) = \sum_{i=1}^{M} c_i(m, z, \kappa)\Phi(x - x_i^m, z)\kappa^{x/\eta} \tag{6.6}$$

($\Phi(x, z)$ is defined in (4.37)). The set of corresponding pairs (z, κ) are parametrized by points of an algebraic curve defined by the equation of the form

$$R(\kappa, z) = \kappa^M + \sum_{i=1}^{M} r_i(z)\kappa^{M-i} = 0\,. \tag{6.7}$$

Sketch of proof. We omit the detailed proof since it is almost identical to the proof of the corresponding theorem in [33] and only present the part of it which provides the Lax representation for Eq. (6.5).

Let us substitute the function $\Psi^m(x)$ of the form (6.6) into Eq. (6.3). The cancellation of poles at $x = x_i^m - \eta$ and $x = x_i^{m+1}$ gives the conditions

$$\kappa c_i(m, z, \kappa) + \lambda_i(m) \sum_{j=1}^{M} c_j(m, z, \kappa)\Phi(x_i^m - x_j^m - \eta, z) = 0\,, \tag{6.8}$$

$$c_i(m+1, z, \kappa) = \mu_i(m) \sum_{j=1}^{M} c_j(m, z, \kappa)\Phi(x_i^{m+1} - x_j^m, z)\,, \tag{6.9}$$

where

$$\lambda_i(m) = \frac{\prod_{s=1}^{M} \sigma(x_i^m - x_s^m - \eta)\sigma(x_i^m - x_s^{m+1})}{\prod_{s=1,\neq i}^{M} \sigma(x_i^m - x_s^m) \prod_{s=1}^{M} \sigma(x_i^m - x_s^{m+1} - \eta)}\,, \tag{6.10}$$

$$\mu_i(m) = \frac{\prod_{s=1}^{M} \sigma(x_i^{m+1} - x_s^{m+1} + \eta)\sigma(x_i^{m+1} - x_s^m)}{\prod_{s=1,\neq i}^{M} \sigma(x_i^{m+1} - x_s^{m+1}) \prod_{s=1}^{M} \sigma(x_i^{m+1} - x_s^m + \eta)}\,. \tag{6.11}$$

Introducing a vector $C(m)$ with components $c_i(m, z, \kappa)$ we can rewrite these conditions in the form

$$(\mathcal{L}(m) + \kappa I)C(m) = 0\,, \tag{6.12}$$

$$C(m+1) = \mathcal{M}(m)C(m)\,, \tag{6.13}$$

where I is the unit matrix. Entries of the matrices $\mathcal{L}(m)$ and $\mathcal{M}(m)$ are:

$$\mathcal{L}_{ij}(m) = \lambda_i(m)\Phi(x_i^m - x_j^m - \eta, z), \tag{6.14}$$

$$\mathcal{M}_{ij}(m) = \mu_i(m)\Phi(x_i^{m+1} - x_j^m, z). \tag{6.15}$$

The compatibility condition of (6.12) and (6.13),

$$\mathcal{L}(m+1)\mathcal{M}(m) = \mathcal{M}(m)\mathcal{L}(m) \tag{6.16}$$

is the discrete Lax equation.

By the direct commutation of the matrices $\mathcal{L}$, $\mathcal{M}$ (making use of some non-trivial identities for the function $\Phi(x,z)$ which are omitted) it can be shown that for the matrices $\mathcal{L}$ and $\mathcal{M}$ defined by Eqs. (6.14), (6.10) and (6.15), (6.11) respectively, the discrete Lax equation (6.16) holds if and only if the x_i^m satisfy Eqs. (6.5). It is worthwhile to remark that in terms of $\lambda_i(m)$, $\mu_i(m)$ Eqs. (6.5) take the form

$$\lambda_i(m+1) = -\mu_i(m), \qquad i = 1, \ldots, M\,. \tag{6.17}$$

Equation (6.12) implies that

$$R(\kappa, z) \equiv \det(\mathcal{L}(m) + \kappa I) = 0\,. \tag{6.18}$$

The coefficients of $R(\kappa, z)$ do not depend on m due to (6.16). This equation defines an algebraic curve (6.7) realized as a ramified covering of the elliptic curve.

Solutions to Eq. (6.5) are implicitly given by the equation

$$\Theta(\vec{U}x_i^{l,m} + \vec{U}_+ l + \vec{U}_- m + \vec{Z}) = 0\,, \tag{6.19}$$

where the Riemann theta-function $\Theta(\vec{X})$ corresponds to the spectral curve (6.7), (6.18), components of the vectors $\vec{U}$, $\vec{U}_+$, $\vec{U}_-$ are periods of certain dipole differentials on the curve, $\vec{Z}$ is an arbitrary vector. Elliptic solutions are characterized by the following property: $2\omega_i\vec{U}$, $i = 1,\ 2$, belongs to the lattice of periods of holomorphic differentials on the curve. The matrix $\mathcal{L}(m) = \mathcal{L}(l,m)$ is defined by fixing $x_j^{l_0,m_0}, x_j^{l_0,m_0+1}$, $i = 1, \ldots, M$. These Cauchy data uniquely define the curve and the vectors $\vec{U}$, $\vec{U}_+$, $\vec{U}_-$ and $\vec{Z}$ in Eq. (6.19). The curve and vectors $\vec{U}, \vec{U}_+, \vec{U}_-$ do not depend on the choice of l_0, m_0. According to Eq. (6.19), the vector $\vec{Z}$ depends linearly on this choice and its components are thus angle-type variables.

The same analysis can be repeated for the second linear problem (6.4). Now m enters as a parameter and we set $x^{l,m} \to \hat{x}_i^l$ for simplicity. The theorem is literally the same, the equations of motion for the poles being

$$\prod_{j=1}^{M} \frac{\sigma(\hat{x}_i^l - \hat{x}_j^{l+1} + \eta)\sigma(\hat{x}_i^l - \hat{x}_j^l - \eta)\sigma(\hat{x}_i^l - \hat{x}_j^{l-1})}{\sigma(\hat{x}_i^l - \hat{x}_j^{l+1})\sigma(\hat{x}_i^l - \hat{x}_j^l + \eta)\sigma(\hat{x}_i^l - \hat{x}_j^{l-1} - \eta)} = -1\,. \tag{6.20}$$

The corresponding discrete Lax equation is

$$\hat{\mathcal{L}}(l+1)\hat{\mathcal{M}}(l) = \hat{\mathcal{M}}(l)\hat{\mathcal{L}}(l)\,, \tag{6.21}$$

where[5]

$$\hat{\mathcal{L}}_{ij}(l) = \hat{\lambda}_i(l)\Phi(\hat{x}_i^l - \hat{x}_j^l - \eta, z), \tag{6.22}$$

$$\hat{\mathcal{M}}_{ij}(l) = \hat{\mu}_i(l)\Phi(\hat{x}_i^{l+1} - \hat{x}_j^l - \eta, z), \tag{6.23}$$

and

$$\hat{\lambda}_i(l) = \frac{\prod_{s=1}^{M}\sigma(\hat{x}_i^l - \hat{x}_s^l - \eta)\sigma(\hat{x}_i^l - \hat{x}_s^{l+1} + \eta)}{\prod_{s=1,\neq i}^{M}\sigma(\hat{x}_i^l - \hat{x}_s^l)\prod_{s=1}^{M}\sigma(\hat{x}_i^l - \hat{x}_s^{l+1})}\,, \tag{6.24}$$

[5] A very close version of the discrete L-M pair appeared first in the Ref.[43] as an a priori ansatz.

$$\hat{\mu}_i(l) = \frac{\prod_{s=1}^{M} \sigma(\hat{x}_i^{l+1} - \hat{x}_s^{l+1} + \eta)\sigma(\hat{x}_i^{l+1} - \hat{x}_s^{l} - \eta)}{\prod_{s=1,\neq i}^{M} \sigma(\hat{x}_i^{l+1} - \hat{x}_s^{l+1}) \prod_{s=1}^{M} \sigma(\hat{x}_i^{l+1} - \hat{x}_s^{l})} \,. \tag{6.25}$$

All these formulas can be obtained from (6.5), (6.10)-(6.15) by the formal substitutions $x_i^m \to \hat{x}_i^l$, $x_i^{m\pm1} \to \hat{x}_i^{l\pm1} \mp \eta$. According to the comment after Eq. (6.19), the Cauchy data for the l-flow $x_j^{l_0,m_0}$, $x_j^{l_{0+1},m_0}$ are uniquely determined by fixing the Cauchy data $x_j^{l_0,m_0}$, $x_j^{l_0,m_{0+1}}$ for the m-flow and vice versa.

7. Conclusion and Outlook

It turned out that classical and quantum integrable models have a deeper connection than the common assertion that the former are obtained as a “classical limit" of the latter. In this paper we have tried to elaborate perhaps the simplest example of this phenomenon: the fusion rules for quantum transfer matrices coincide with Hirota’s bilinear difference equation (HBDE).

We have identified the bilinear fusion relations in Hirota’s classical difference equation with particular boundary conditions and elliptic solutions of the Hirota equation, with eigenvalues of the quantum transfer matrix. Eigenvalues of the quantum transfer matrix play the role of the τ-function. Positions of zeros of the solution are determined by the Bethe ansatz equations. The latter have been derived from an entirely classical set-up.

We have shown that nested Bethe ansatz equations can be considered as a natural discrete time analogue of the Ruijsenaars-Schneider system of particles. The discrete time t runs over vertices of the Dynkin graph of A_{k-1}-type and numbers levels of the nested Bethe ansatz. The continuum limit in t gives the continuous time RS system [48]. This is our motivation to search for classical integrability properties of the nested Bethe ansatz equations.

In addition we constructed the general solution of the Hirota equation with a certain boundary conditions and obtained new determinant representations for eigenvalues of the quantum transfer matrix. The approach suggested in Sect. 5 resembles the Leznov-Saveliev solution [39] to the 2D Toda lattice with open boundaries. It can be considered as an integrable discretization of the classical W-geometry [18].

We hope that this work gives enough evidence to support the assertion that all spectral characteristics of quantum integrable systems on finite 1D lattices can be obtained starting from classical discrete soliton equations, not implying a quantization. The Bethe ansatz technique, which has been thought of as a specific tool of quantum integrability is shown to exist in classical discrete nonlinear integrable equations. The main new lesson is that solving classical discrete soliton equations one recovers a lot of information about a quantum integrable system.

Soliton equations usually have a huge number of solutions with very different properties. To extract the information about a quantum model, one should restrict the class of solutions by imposing certain boundary and analytic conditions. In particular, elliptic solutions to HBDE give spectral properties of quantum models with elliptic R-matrices.

The difference bilinear equation of the same form, though with different analytical requirements, has appeared in quantum integrable systems in another context. Spin-spin correlation functions of the Ising model obey a bilinear difference equation that can be recast into the form of HBDE [41, 45, 2]. More recently, nonlinear equations for correlation functions have been derived for a more general class of quantum integrable models, by virtue of the new approach of Ref. [10].

Thermodynamic Bethe ansatz equations written in the form of functional relations [53, 46] (see e.g., [7]) appeared to be identical to HBDE with different analytic properties.

All these suggest that HBDE may play the role of a master equation for both classical and quantum integrable systems simultaneously, such that the “equivalence" between quantum systems and discrete classical dynamics might be extended beyond the spectral properties discussed in this paper. In particular, it will be very interesting to identify the quantum group structures and matrix elements of quantum L-operators and R-matrices with objects of classical hierarchies. We do not doubt that such a relation exists.

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