Measurement Magic

"...the principles of quantum mechanics have not been found to fail."

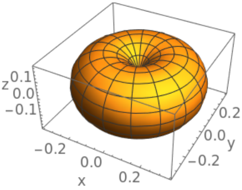
Richard Feynman in
The Feynman Lectures

"On the other hand, I think I can safely say, no one understands quantum mechanics."

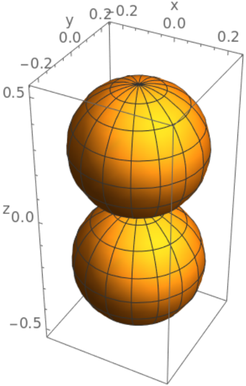
Richard Feynman in
The Character of Physical Law

Measurement Magic 2

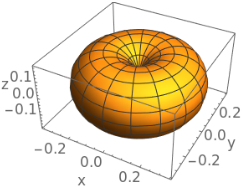
Suppose a rigid rotator is in the eigenstate of $\hat{L}^2$ with $\ell = 1$ and $m_z = -1$ ($Y_1^{-1}(\theta,\phi)$). What is the probability that a measurement of $\hat{L}_x$ finds the respective values of $m_x = 0, \pm 1$? What will a subsequent measurement of $\hat{L}_x$ find? And with what probability? What will a subsequent measurement of $\hat{L}_z$ find? The magnitudes of the spherical harmonics for $\ell = 1$ are shown below.



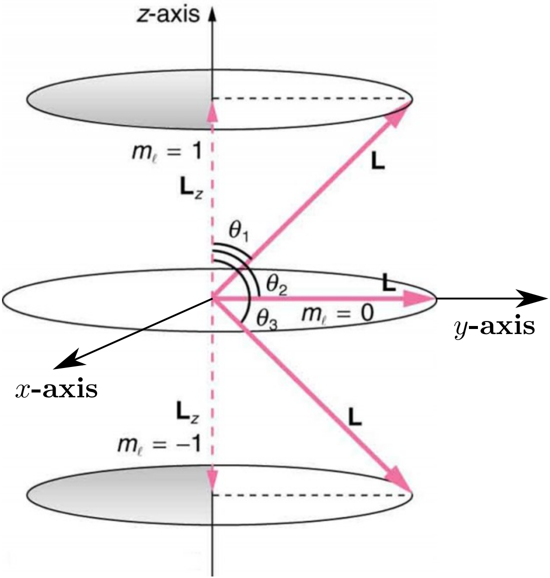
$|Y_1^1|$



$|Y_1^0|$



$|Y_1^{-1}|$



z-axis
$m_\ell = 1$
L_z
L
θ_1
θ_2
L
θ_3
$m_\ell = 0$
y-axis
x-axis
L_z
L
$m_\ell = -1$

1. Each physical, measurable quantity, A, has a corresponding operator, $\hat{A}$, that satisfies the eigenvalue equation $\hat{A}\,\phi = a\phi$ and measuring that quantity yields the eigenvalues of $\hat{A}$. The 'intensity' is proportional to $|\Psi|^2$ and is interpreted as a probability.
2. Measurement of the observable A leaves the system in a state that is an eigenfunction of $\hat{A}$.
3. The state of a system is represented by a wave function Ψ that is continuous, differentiable and contains all possible information about the system. The 'intensity' is proportional to $|\Psi|^2$ and is interpreted as a probability. The average value of any observable A is $\langle A \rangle = \int_{all\ space} \Psi^* \hat{A}\, \Psi d\vec{r}$.
4. The time and spatial dependence of $\Psi(x,t)$ is determined by the time dependent Schroedinger equation.

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) \qquad \mu \equiv \text{reduced mass.}$$

Properties of the Spherical Harmonics

$$\hat{L}^2|\ell, m\rangle = \ell(\ell+1)\hbar^2\,|\ell, m\rangle$$

$$\hat{L}_z|\ell m\rangle = m\hbar\,|\ell m\rangle$$

$$\hat{L}_x|\ell, m\rangle = \frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\;|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\;|\ell, m-1\rangle$$

$$\hat{L}_y|\ell, m\rangle = -\frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\;|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\;|\ell, m-1\rangle$$

$$\hat{L}_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\;|\ell, m\pm1\rangle$$

$$\langle\ell' m'|\ell m\rangle = \int_0^\pi\int_0^{2\pi} Y_{\ell'}^{m'\,*}\,Y_\ell^m\,d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

$$\hat{L}^2|\ell, m\rangle = \ell(\ell+1)\hbar^2\,|\ell, m\rangle$$

$$\hat{L}_z|\ell m\rangle = m\hbar\,|\ell m\rangle$$

$$\hat{L}_x|\ell, m\rangle = \frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\,|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\,|\ell, m-1\rangle$$

$$\hat{L}_y|\ell, m\rangle = -\frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\,|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\,|\ell, m-1\rangle$$

$$\hat{L}_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m\pm 1)}\,|\ell, m\pm 1\rangle$$

$$\langle \ell' m'|\ell m\rangle = \int_0^\pi\int_0^{2\pi} Y_{\ell'}^{m'*}Y_\ell^m d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-fh) - b(di-fg) + c(dh-eg) \qquad |B| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1. Require
$$\hat{L}_x|\phi\rangle = \hat{L}_x X_1 = \alpha\hbar X_1$$
$$\hat{L}_x(aY_1^1 + bY_1^0 + cY_1^{-1}) = \alpha\hbar(aY_1^1 + bY_1^0 + cY_1^{-1})$$
2. Solve this system of equations and get the eigenvalues for
$$\begin{pmatrix} \sqrt{2}\alpha & -1 & 0 \\ -1 & \sqrt{2}\alpha & -1 \\ 0 & -1 & \sqrt{2}\alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
by taking the determinant of the matrix and solving for α. Get $\alpha = 0$ or $\alpha = \pm 1$.
3. Plug each eigenvalue ($\alpha = 0, \pm 1$) into the matrix to get the coefficients a, b, and c. Normalize the results using $\langle X_1^{m_x} | X_1^{m_x} \rangle = 1$. Get the eigenfunctions.
$$X_1^0 = \frac{Y_1^1 - Y_1^{-1}}{\sqrt{2}} \quad X_1^1 = \frac{Y_1^1 + \sqrt{2}Y_1^0 + Y_1^{-1}}{2} \quad X_1^{-1} = \frac{Y_1^1 - \sqrt{2}Y_1^0 + Y_1^{-1}}{2}$$
4. Get the $b_{\ell m_x}$s of $\sum_{\ell=0}^{\infty} \sum_{m_x=-\ell}^{m_x=\ell} b_{\ell m_x} X_\ell^m$ using $b_{\ell m_x} = \langle X_1^{m_x} | \psi \rangle = \langle X_1^{m_x} | Y_1^{-1} \rangle$ here.

The Classical 'Model' of Reality

1. Realism - Regularities in observed phenomena are caused by a physical reality whose existence is independent of human observers.
2. Inductive Inference - Legitimate conclusions can be drawn from consistent observations.
3. Einstein locality - No influence can propagate faster than the speed of light.

This list is often referred to as local realism.

1. The Statistics - A system is described by a wave function ψ where $|\psi|^2$ is the probability distribution of the possible results of an experiment.
2. Calculating observables - Each observable is associated with an operator $\hat{A}$ with eigenfunctions ϕ_i, eigenvalues a_i, and

$$\psi = \sum \alpha_i \phi_i \qquad .$$

3. The Measurement - Doing the experiment 'collapses' the wave function so a well-defined, single result is obtained.

This is the Copenhagen Interpretation associated with Neils Bohr.

1. There are two ways for the quantum wave function to evolve in time.
2. The first is $\Psi(x, t) = \psi(x, t = 0)e^{-i\omega t}$.
3. The second is the impact of a measurement. We write $\psi(x, t = 0) = \sum b_n|\phi_n\rangle$ and say words like "In a measurement a single eigenfunction is picked out of the array of possible potentialities".
4. Both are radically different, but both are necessary.