Physics 309 - Measurement Magic

1. Consider the the raising operator defined as

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

where

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

and

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

Apply $\hat{L}_+$ to the eigenfunction $Y_1^{-1}(\theta,\phi)$ and show you obtain $\sqrt{2}\,Y_1^0(\theta,\phi)$.

2. What is the determinant of the matrices shown below?

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{2}\alpha & -1 & 0 \\ -1 & \sqrt{2}\alpha & -1 \\ 0 & -1 & \sqrt{2}\alpha \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

3. Suppose a rigid rotator is in the eigenstate of $\hat{L}^2$ with $\ell = 1$ and $m_z = -1$ ($Y_1^{-1}(\theta,\phi)$). We want to find the probability of obtaining the values of $m_x = 0, \pm 1$ from a measurement of $\hat{L}_x$. (Hint: Most of the analysis in the Liboff text associated with equation 9.92 involving the expansion of the state Y_1^1 may be applied here.)

   (a) We always measure eigenvalues so to get the results of a measurement of L_x we need to construct the appropriate operator $\hat{L}_x$ which satisfies $\hat{L}_x X = \alpha\hbar X$ where X is an eigenfunction of the $\hat{L}_x$ operator and α is the eigenvalue. To do that we can assume

   $$X = aY_1^1 + bY_1^0 + cY_1^{-1}$$

   since the spherical harmonics form a complete set and we know what they are. We restrict our attention to only $\ell = 1$ states as a consequence of angular momentum conservation. Generate the set of simultaneous equations that the coefficients (a, b, c) and α must satisfy.

   (b) Solve the set of equations from part 2.a and obtain the eigenvalues α. You might find that forming the answer to part 2.a as a matrix and taking its determinant is convenient.

   (c) Insert the eigenvalues α into your matrix or set of simultaneous equations and extract the coefficients a, b, and c for each value of α.

   (d) Normalize each set of coefficients from part 2.c.

(e) Once you have constructed the eigenfunctions $X_\ell^{m_x}$ of $\hat{L}_x$ you can now get the coefficients of the expansion of the initial wave packet in terms of the $X_\ell^{m_x}$'s. In other words get the $b_{\ell m}$'s in the following expression.

$$|\psi(\vec{r}, t = 0)\rangle = Y_1^{-1} = \sum_{\ell=0}^{\infty} \sum_{m_x=-\ell}^{m_x=\ell} b_{\ell m_x} X_\ell^{m_x}$$

Recall from part 2.a that we can restrict our attention to $\ell = 1$. Why?

(f) What will a measurement of $\hat{L}_x$ find? And with what probability?

(g) What will a subsequent measurement of $\hat{L}_x$ find? And with what probability?

(h) What will a subsequent measurement of $\hat{L}_z$ find?

4. Repeat problem 2 with the initial state $|\psi(\vec{r}, t = 0)\rangle = Y_1^0$.