Physics 309 Homework

The Time-Independent Schroedinger Equation

1. Consider a wave function of the form $\Psi(x,t) = A\sin(kx - \omega t)$. Using the definition of the wavelength λ and the period T show

$$k = \frac{2\pi}{\lambda} \qquad\qquad \omega = \frac{2\pi}{T} = 2\pi\nu$$

where $\nu = 1/T$ is the frequency of the wave.

2. To develop a differential equation consistent with Planck's hypothesis ($E = h\nu$) and the DeBroglie equation ($p = h/\lambda$) start with the following expression.

$$\hbar\omega\psi(x,t) = \frac{\hbar^2 k^2}{2m}\psi(x,t) \tag{1}$$

Next, consider a linear superposition of two simple waves.

$$\psi(x,t) = \cos(kx - \omega t) + \gamma\sin(kx - \omega t) \tag{2}$$

Assume the following form of the differential equation

$$\alpha\frac{\partial^2\psi}{\partial x^2} = \beta\frac{\partial\psi}{\partial t} \tag{3}$$

where α and β are constants yet to be determined.

   (a) What value of γ will satisfy Equation 3?

   (b) What would then be the values of α and β that would satisfy Equation 1?

3. We have used a 'plausibility' argument to show that the Schroedinger equation may provide a means to determine the quantum mechanical wave function. We derived in class the time-dependent form of the equation shown below.

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}$$

We now want to show that for the one-dimensional case and for potential energy functions that depend only on position ($i.e.$,$V = V(x)$) that a time-independent form of the Schroedinger equation can be derived as shown below.

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

(a) We will use separation of variables to derive the time-independent form from the time-dependent one. Consider a one-dimensional case where we let $\Psi(x,t) = \psi(x)T(x)$ where $\psi(x)$ depends only on the position and $T(t)$ depends only on time. Take the appropriate derivatives and plug the results into the time-dependent Schroedinger equation. You should then be able to isolate all the pieces that depend on x on side of the result and all the pieces that depend on t on the other side.

(b) You now have an expression that depends only on x on one side on only on t on the other side. The position and time can take on any values, but your result from part 1 implies the two sides of the expression are always equal. In other words we have a situation where $f(x) = g(t)$ for any and all values of x and t. If one varies x and not t, the expression will still hold. What does that imply about what $f(x)$ and $g(t)$ must equal?

(c) Use the results of parts 1-2 to derive the time-independent Schroedinger equation.

(d) Find the equation that $T(t)$ must satisfy and obtain a solution for it.

(e) How is $T(t)$ related to the energy of the particle?

4. In Problem 2 above you assumed that $\psi(x,t) = \cos(kx - \omega t) + \gamma \sin(kx - \omega t)$ and you should have found that $\gamma = \pm i$. If we choose the positive sign and use the Euler relation we can write $\psi(x,t) = e^{i(kx-\omega t)}$. Using the odd and even properties of the sine and cosine show the following for the negative sign.

$$\psi(x,t) = \cos(kx - \omega t) - \gamma \sin(kx - \omega t) = e^{-i(kx-\omega t)}$$