4. Apply the Spherical Harmonics

What do we measure?

eigenvalues

If we want to measure eigenvalues of $\hat{L}_x$, then we need to find the eigenfunctions

∘ We want

$$\hat{L}_y |\varphi\rangle = \alpha\hbar |\varphi\rangle$$

$$\hat{L}_y X = \alpha\hbar X$$

What are the possible values of α?

Our mission

What are their probabilities?

Our other mission

We have to construct X, the eigenfunctions of $\hat{L}_x$.

Let $X = aY_1^1 + bY_1^0 + cY_1^{-1}$

Why use Y_ℓ^ms? They form a complete set.

Why only $\ell = 1$? Conservation of $\hat{L}$

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5. Eigenvalues (use 'Properties of Spherical Harmonics')

5. $\hat{L}_y \chi = \hat{L}_y (a Y_1^1 + b Y_1^0 + c Y_1^{-1})$

$$= \frac{\hbar}{2} \left\{ a\left[0 + \sqrt{2\cdot 1}\, Y_1^0\right] + b\left[\sqrt{2}\, Y_1^1 + \sqrt{2}\, Y_1^{-1}\right] \right.$$
$$\left. c\left[\sqrt{2\cdot 1}\, Y_1^0\right]\right\}$$

$$= \frac{\hbar}{2}\sqrt{2}\left[a Y_1^0 + b\left(Y_1^1 + Y_1^{-1}\right) + c Y_1^0\right]$$

$$= \frac{\hbar}{\sqrt{2}}\left[b Y_1^1 + (a+c) Y_1^0 + b Y_1^{-1}\right]$$

HW # 1.a

$\alpha \hbar (a Y_1^1 + b Y_1^0 + c Y_1^{-1}) =$

Bring everything to the left side and group by Y_ℓ^m

$$\therefore \left(\sqrt{2}\,\alpha a - b\right) Y_1^1 + \left[\sqrt{2}\,\alpha b - (a+c)\right] Y_1^0 +$$

$$\left(\sqrt{2}\,\alpha c - b\right) Y_1^{-1} = 0$$

The coefficient on each Y_ℓ^m must vanish. Write it out like a matrix.

$$\sqrt{2}\alpha a - b + 0c = 0$$
$$- a + \sqrt{2}\alpha b - c = 0$$
$$0\cdot a - b + \sqrt{2}\alpha c = 0$$

Rewrite the set of simultaneous equations as an eigenfunction problem in matrix form.

$$\underset{\sim}{M}(\alpha) \rightarrow \begin{pmatrix} \sqrt{2}\alpha & -1 & 0 \\ -1 & \sqrt{2}\alpha & -1 \\ 0 & -1 & \sqrt{2}\alpha \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\vec{C}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{0}}$$

Properties of the Spherical Harmonics 5

$$\hat{L}^2|\ell, m\rangle = \ell(\ell+1)\hbar^2\,|\ell, m\rangle$$

$$\hat{L}_z|\ell m\rangle = m\hbar\,|\ell m\rangle$$

$$\hat{L}_x|\ell, m\rangle = \frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\,|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\,|\ell, m-1\rangle$$

$$\hat{L}_y|\ell, m\rangle = -\frac{\hbar}{2}\sqrt{(\ell-m)(\ell+m+1)}\,|\ell, m+1\rangle + \frac{\hbar}{2}\sqrt{(\ell+m)(\ell-m+1)}\,|\ell, m-1\rangle$$

$$\hat{L}_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\,|\ell, m\pm1\rangle$$

$$\langle\ell' m'|\ell m\rangle = \int_0^\pi \int_0^{2\pi} Y_{\ell'}^{m'*} Y_\ell^m d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

$$\underset{\sim}{m}(\alpha)\vec{c} = \vec{0}$$

Take the determinant and set it to zero.

$$\begin{vmatrix} \sqrt{2}\,\alpha & -1 & 0 \\ -1 & \sqrt{2}\,\alpha & -1 \\ 0 & -1 & \sqrt{2}\,\alpha \end{vmatrix} = 0$$

$$\sqrt{2}\,\alpha\left(2\alpha^2 - 1\right) + (1)\left(-\sqrt{2}\,\alpha - 0\right) + 0 = 0$$

$$\sqrt{2}\,\alpha\left(2\alpha^2 - 1\right) - \sqrt{2}\,\alpha = 0$$

$$\sqrt{2}\,\alpha\left[2\alpha^2 - 1 - 1\right] = 0$$

$$\sqrt{2}\,\alpha\,(2)(\alpha^2 - 1) = 0$$

$\therefore \quad \alpha = 0 \quad$ or $\quad \alpha^2 - 1 = 0$

$$\alpha = \pm 1$$

HW # 1.b

6. Get eigenfunction for each eigenvalue.

for $\alpha = 0$, $\underset{\sim}{m}(0)$ becomes

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -b \\ -a-c \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad b = 0 \qquad -a - c = 0$$
$$a = -c$$

$$\therefore \quad X_1^0 = \left(Y_1^1 - Y_1^{-1} \right) A_0$$ normalization constant

get A_0 $\quad \langle X_1^0 | X_1^0 \rangle = 1$

$$A_0^2 \langle Y_1^1 - Y_1^{-1} | Y_1^1 - Y_1^{-1} \rangle = 1$$

$$A_0^2 \Big[\underbrace{\cancelto{1}{\langle Y_1^1 | Y_1^1 \rangle} - \cancelto{0}{\langle Y_1^1 | Y_1^{-1} \rangle} - \cancelto{0}{\langle Y_1^{-1} | Y_1^1 \rangle}}_{\text{use } \langle Y_{\ell'}^{m'} | Y_\ell^{\ell} \rangle = \delta_{\ell\ell'} \delta_{mm'}} + \cancelto{1}{\langle Y_1^{-1} | Y_1^{-1} \rangle} \Big] = 1$$

$$A_0^2 \left[1 - 0 - 0 + 1 \right] = 1$$

$$A_0 = \frac{1}{\sqrt{2}}$$

$$\therefore \quad X_1^0 = \frac{Y_1^1 - Y_1^{-1}}{\sqrt{2}}$$

HW 1.c 1.d

for $\alpha = +1$ $\underset{\sim}{m}(+1)$ becomes

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & 0 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1st row $\quad \sqrt{2}\, a - b = 0$

$$b = \sqrt{2}\, a$$

2nd row $\quad -a + \sqrt{2}\, b = 0$

$$a = \sqrt{2}\, b$$

3rd row $\quad -b + \sqrt{2}\, c = 0$

$$b = \sqrt{2}\, c$$

Picks rows to get relations between a, b, c. First and third look easiest.

$$b = \sqrt{2}\, a = \sqrt{2}\, c \quad \therefore a = c$$

$$X_1^1 = \underbrace{A_+}_{\text{normalization constant}} \left(Y_1^1 + \sqrt{2}\, Y_1^0 + Y_1^{-1} \right]$$

$$\langle X_1^1 | X_1^1 \rangle = 1$$

$$A_+^2 \left(\langle Y_1^1 | + \sqrt{2} \langle Y_1^0 | + \langle Y_1^{-1} | \right) \left(|Y_1^1\rangle + \sqrt{2} |Y_1^0\rangle + |Y_1^{-1}\rangle \right) = 1$$

use $\langle l'm' | lm \rangle = \delta_{ll'} \delta_{mm'}$

$$A_+^2 \left(\langle Y_1^1 | Y_1^1 \rangle + 2 \langle Y_1^0 | Y_1^0 \rangle + \langle Y_1^{-1} | Y_1^{-1} \rangle \right) = 1$$

$$A_+^2 (1 + 2 + 1) = 1$$

$$A_+ = \frac{1}{2}$$

$$\therefore X_1^1 = \frac{Y_1^1 + \sqrt{2}\, Y_1^0 + Y_1^{-1}}{2}$$

HW 1.c 1.d

for $\alpha = -1$ $\underset{\sim}{M}(-1)$ becomes

$$\begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & 0 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1st row $\quad -\sqrt{2}\,a - b = 0 \qquad b = -\sqrt{2}\,a$

2nd row $\quad -a - \sqrt{2}\,b = 0 \qquad a = -\sqrt{2}\,b$

3rd row $\quad -b - \sqrt{2}\,c = 0 \qquad b = -\sqrt{2}\,c$

HW 1.c 1.d

Pick rows 1 and 3

$$b = -\sqrt{2}\,a = -\sqrt{2}\,c \quad \therefore \quad a = c$$

$$X_1^{-1} = A_-\left(Y_1^1 - \sqrt{2}\,Y_1^0 + Y_1^{-1} \right)$$

$\hookrightarrow$ normalization

$$\langle X_1^{-1} | X_1^{-1} \rangle = 1$$

$$A_-^2 \left(\langle Y_1^1 | - \sqrt{2} \langle Y_1^0 | + \langle Y_1^{-1} | \right) \left(| Y_1^1 \rangle - \sqrt{2}\, | Y_1^0 \rangle + | Y_1^{-1} \rangle \right) = 1$$

$$\langle l'm' | l m \rangle = \delta_{l'l}\, \delta_{m'm}$$

$$A_-^2 \left(\langle Y_1^1 | Y_1^1 \rangle + 2 \langle Y_1^0 | Y_1^0 \rangle + \langle Y_1^{-1} | Y_1^{-1} \rangle \right) = 1$$

$$A_-^2 \left(1 + 2 + 1 \right) = 1$$

$$A_- = \frac{1}{2}$$

$$\therefore \quad X_1^{-1} = \frac{| Y_1^1 \rangle - \sqrt{2}\, | Y_1^0 \rangle + | Y_1^{-1} \rangle}{2}$$

7. We have the eigenfunctions now (X_ℓ^m). Get coefficients ($b_{\ell m}$'s) of superposition.

$$|\psi\rangle = Y_1^{-1} = \sum b_{\ell m} X_\ell^m$$

the $b_{\ell m}$'s

$$\langle \ell' m' | \psi \rangle = b_{\ell' m'}$$

$$\int_0^\pi \int_0^{2\pi} X_{\ell'}^{m'*} Y_1^{-1} \, d\Omega = b_{\ell' m'}$$

$\sin\theta \, d\theta \, d\phi$

- if $\ell' \neq 1$, you get zero
- if $m' = 1$

$$\langle X_1^1 | Y_1^{-1} \rangle = \langle \frac{Y_1^1 + \sqrt{2}\, Y_1^0 + Y_1^{-1}}{2} | Y_1^{-1} \rangle$$

$$b_{11} = \frac{1}{2}$$

- if $m' = 0$

$$\langle X_1^0 | Y_1^{-1} \rangle = \langle \frac{Y_1^1 - Y_1^{-1}}{\sqrt{2}} | Y_1^{-1} \rangle = b_{10}$$

$$b_{10} = -\frac{1}{\sqrt{2}}$$

- if $m' = -1$

$$\langle X_1^{-1} | Y_1^{-1} \rangle = \langle \frac{Y_1^1 - \sqrt{2}\, Y_1^0 + Y_1^{-1}}{2} | Y_1^{-1} \rangle$$

$$b_{1-1} = \frac{1}{2}$$

$$\boxed{\therefore \; |\psi\rangle = |Y_1^{-1}\rangle = \frac{1}{2}|X_1^1\rangle - \frac{1}{\sqrt{2}}|X_1^0\rangle + \frac{1}{2}|X_1^{-1}\rangle}$$

HW # 1.e

$\therefore\ |\psi\rangle = |Y_1^{-1}\rangle$

$$= \frac{1}{2} X_1^{1} - \frac{1}{\sqrt{2}} X_1^{0} + \frac{1}{2} X_1^{-1}$$

HW 1.f

$\therefore$ probability of getting $l_x = 1, m_x = 1$ $= \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

for $l_x = 1$, $m_x = 0$ $= \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

for $l_x = 1$, $m_x = -1$ $= \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

subsequent measurement of $\hat{L}_x$ will find the same result as the first measurement with 100% probability — HW 1.g

subsequent measurement of $\hat{L}_z$ will find $m_z = 0, \pm 1$ with probabilities determined like the process above. — HW 1.h