

1)

$$\psi_3 = F e^{+ik_1 x} + G e^{-ik_1 x}$$

$$\underline{\psi}_3 = \psi_3 e^{-i\omega_1 t}$$

$$\hbar \omega_1 = \frac{\hbar^2 k^2}{2m}$$

$$\boxed{\omega_1 = \frac{\hbar k^2}{2m}}$$

$$\begin{aligned} \underline{\psi}_3 &= F e^{i(k_1 x - \omega_1 t)} + G e^{-i(k_1 x + \omega_1 t)} \\ &= F e^{ik_1(x - \frac{\omega_1 t}{k_1})} + G e^{-ik_1(x + \frac{\omega_1 t}{k_1})} \end{aligned}$$

$$\text{let } v_1 = \frac{\omega_1}{k_1}$$

$$= F e^{ik_1(x - v_1 t)} + G e^{-ik_1(x + v_1 t)}$$

functional form
of a wave
traveling to
the right

form of a wave
traveling to the
left

no sources of
waves at large x
so $G = 0$

Rectangular Barrier

2)

$$d_{12} = \frac{1}{2} \begin{pmatrix} 1 + k_2/k_1 & 1 - k_2/k_1 \\ 1 - k_2/k_1 & 1 + k_2/k_1 \end{pmatrix}$$

$$d_{21} = \frac{1}{2} \begin{pmatrix} 1 + k_1/k_2 & 1 - k_1/k_2 \\ 1 - k_1/k_2 & 1 + k_1/k_2 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} e^{-ik_2 2a} & 0 \\ 0 & e^{+ik_2 2a} \end{pmatrix}$$

$$\xi_1 = t_2 \xi_3 = d_{12} p_2 d_{21} \xi_3$$

$$\therefore t_2 = d_{12} p_2 d_{21}$$

$$= \frac{1}{4} \begin{pmatrix} 1 + k_2/k_1 & 1 - k_2/k_1 \\ 1 - k_2/k_1 & 1 + k_2/k_1 \end{pmatrix} \begin{pmatrix} e^{-ik_2 2a} & 0 \\ 0 & e^{+ik_2 2a} \end{pmatrix} \begin{pmatrix} 1 + k_1/k_2 & 1 - k_1/k_2 \\ 1 - k_1/k_2 & 1 + k_1/k_2 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 + k_2/k_1 & 1 - k_2/k_1 \\ 1 - k_2/k_1 & 1 + k_2/k_1 \end{pmatrix} \begin{pmatrix} e^{-ik_2 2a} (1 + k_1/k_2) & e^{-ik_2 2a} (1 - k_1/k_2) \\ e^{+ik_2 2a} (1 - k_1/k_2) & e^{+ik_2 2a} (1 + k_1/k_2) \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} \text{Exp}(-ik_2 2a) (2 + k_2/k_1 + k_1/k_2) + \text{Exp}(-ik_2 2a) (-k_1/k_2 + k_2/k_1 - 1) + \text{Exp}(ik_2 2a) (2 - k_2/k_1 - k_1/k_2) + \text{Exp}(ik_2 2a) (1 + k_1/k_2 - k_2/k_1 - 1) \\ \text{Exp}(-ik_2 2a) (1 + k_1/k_2 - k_2/k_1 - 1) + \text{Exp}(-ik_2 2a) (1 - k_1/k_2 - k_2/k_1 + 1) + \text{Exp}(ik_2 2a) (1 - k_1/k_2 + k_2/k_1 - 1) + \text{Exp}(ik_2 2a) (1 + k_1/k_2 + k_2/k_1 + 1) \end{pmatrix}$$

$$t = \frac{1}{4} \left(\begin{aligned} & \exp(-ik_2 2a) \left(\frac{k_1^2 + 2k_1 k_2 + k_2^2}{k_1 k_2} \right) - \\ & \exp(+ik_2 2a) \left(\frac{k_1^2 - 2k_1 k_2 + k_2^2}{k_1 k_2} \right) + \\ & \exp(-ik_2 2a) \left(\frac{k_1^2 - k_2^2}{k_1 k_2} \right) + \\ & - \exp(+ik_2 2a) \left(\frac{-k_1^2 + k_2^2}{k_1 k_2} \right) + \\ & - \exp(-ik_2 2a) \left(\frac{k_1^2 - 2k_1 k_2 + k_2^2}{k_1 k_2} \right) + \\ & \exp(+ik_2 2a) \left(\frac{k_1^2 + 2k_1 k_2 + k_2^2}{k_1 k_2} \right) \end{aligned} \right)$$

$$= \frac{1}{4} \left(\begin{aligned} & \left(\frac{k_1^2 + k_2^2}{k_1 k_2} \right) (-2i) \sin(2k_2 a) + \cos(2k_2 a) \\ & \left(\frac{k_1^2 - k_2^2}{k_1 k_2} \right) (-2i) \sin 2k_2 a \\ & \frac{k_1^2 + k_2^2}{k_1 k_2} (+2i) \sin 2k_2 a + 2(2 \cos 2k_2 a) \end{aligned} \right)$$

$$t = \frac{1}{4} \left(\begin{aligned} & (-2i) \left(\frac{k_1^2 + k_2^2}{k_1 k_2} \right) \sin 2k_2 a + 4 \cos 2k_2 a \\ & -2i \left(\frac{k_1^2 - k_2^2}{k_1 k_2} \right) \sin 2k_2 a \\ & (+2i) \left(\frac{k_1^2 - k_2^2}{k_1 k_2} \right) \sin 2k_2 a \\ & + 2i \left(\frac{k_1^2 + k_2^2}{k_1 k_2} \right) \sin 2k_2 a + 4 \cos 2k_2 a \end{aligned} \right)$$

3)

$$P_1 = \begin{pmatrix} e^{-i2k_1 a} & 0 \\ 0 & e^{i2k_1 a} \end{pmatrix}$$

$$P_1^{-1} = \begin{pmatrix} e^{i2k_1 a} & 0 \\ 0 & e^{-i2k_1 a} \end{pmatrix}$$

Show $P_1 P_1^{-1} = I$

$$\begin{pmatrix} e^{-i2k_1 a} & 0 \\ 0 & e^{i2k_1 a} \end{pmatrix} \begin{pmatrix} e^{i2k_1 a} & 0 \\ 0 & e^{-i2k_1 a} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} e^0 & 0 \\ 0 & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$4) \quad T = \frac{1}{|t_{11}|^2}$$

See problem 1

$$t_{11} = \frac{1}{4} (-2i) \left(\frac{k_1^2 + k_2^2}{k_1 k_2} \right) \sin 2k_2 a + 4 \cos 2k_2 a$$

$$= -\frac{i}{2} \left(\frac{k_1^2 + k_2^2}{k_1 k_2} \right) \sin 2k_2 a + 4 \cos 2k_2 a$$

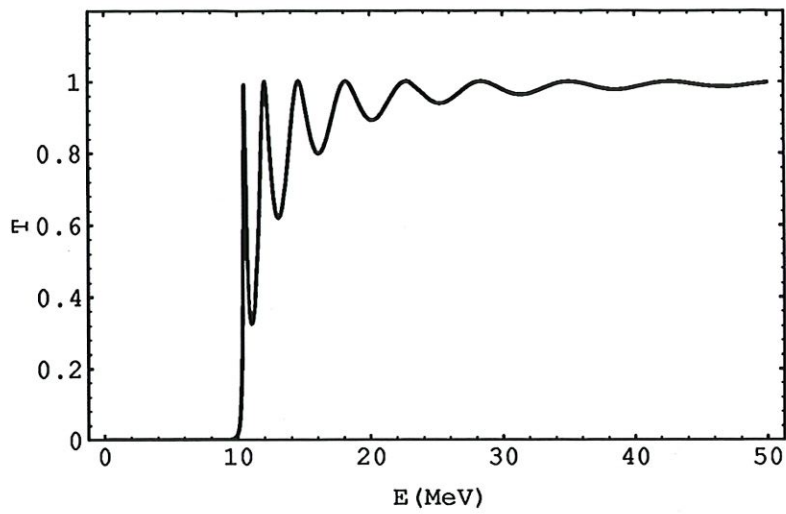
$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

The wave number k_1 is always real, but k_2 can be imaginary when $E < V_0$. The complexity of the expressions for T here and in Liboff invite one to use Mathematica to make the comparison (see accompanying figures).

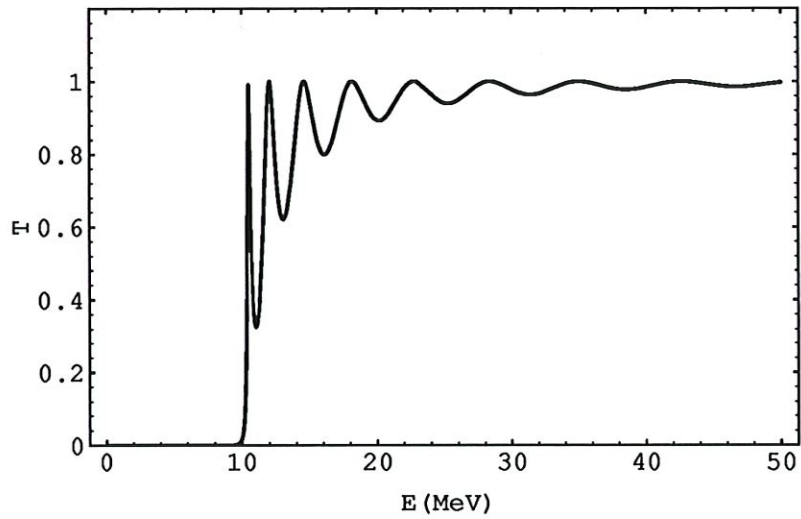
In the first two panels the Transmission Coefficient as a function of incident energy is shown for my calculation and Liboff's result. A more rigorous comparison is made in the third panel where the difference is plotted on an expanded scale. There is no difference between the two curves. The two results are the same.

You can also do much of the algebra in Mathematica.

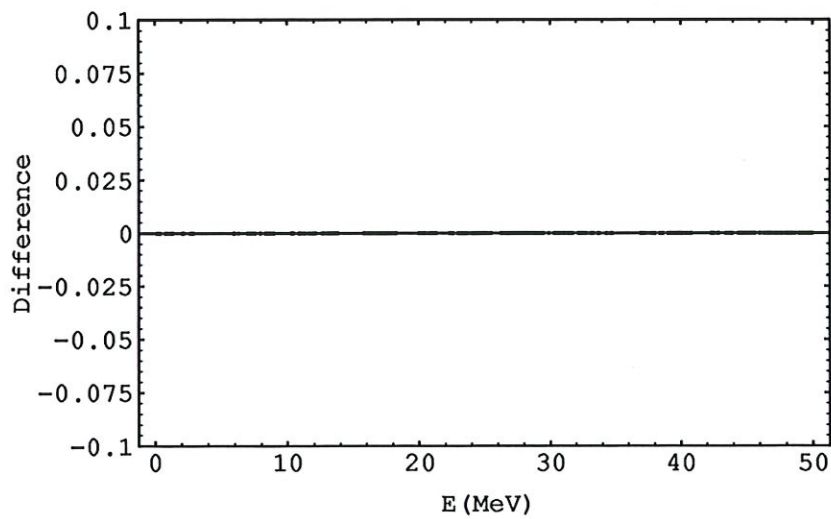
My Result



Liboff's Result



(My Result)-(Liboff's Result)



5) Get the reflection coefficient

$$R = \frac{\text{reflected flux}}{\text{incoming flux}} = \frac{|B|^2 v}{|A|^2 v} = \left| \frac{B}{A} \right|^2$$

Recall $\psi_1 = t \psi_3$

so $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \quad G=0$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} F \\ t_{21} F \end{pmatrix}$$

$$B = t_{21} F \quad \text{and} \quad A = t_{11} F$$

$$\frac{B}{F} = t_{21}$$

$$\frac{B}{A/t_{11}} = t_{21}$$

$$\frac{B}{A} t_{11} = t_{21}$$

$$\frac{B}{A} = \frac{t_{21}}{t_{11}}$$

$$t_{11} = \frac{1}{4} \left[\left(1 - \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) e^{i2ak_2} + \left(1 + \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) e^{-i2ak_2} \right]$$

from mathematics
and by hand.

$$t_{21} = \frac{1}{4} \left[\left(1 + \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) e^{i2ak_2} + \left(1 - \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) e^{-i2ak_2} \right]$$

from Mathematica

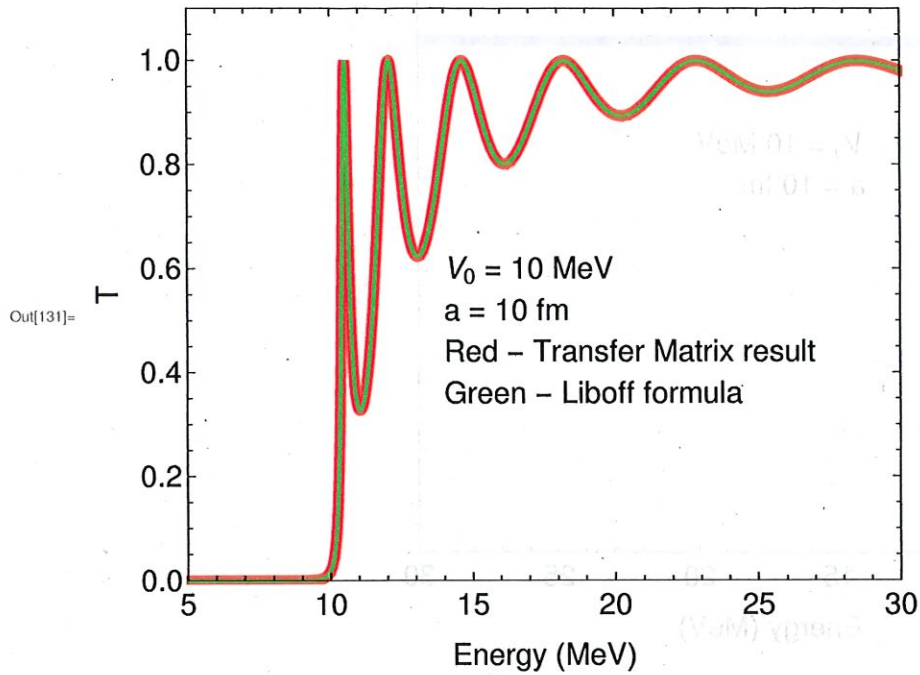
$$R = \left| \frac{B}{A} \right|^2$$

$$= \frac{\left(-1 + e^{i4ak_2}\right)^2 (k_1 - k_2)^2 (k_1 + k_2)^2}{\left[\left(-1 + e^{i4ak_2}\right) k_1^2 - 2 \left(1 + e^{i4ak_2}\right) k_1 k_2 + \left(-1 + e^{i4ak_2}\right) k_2^2 \right]^2}$$

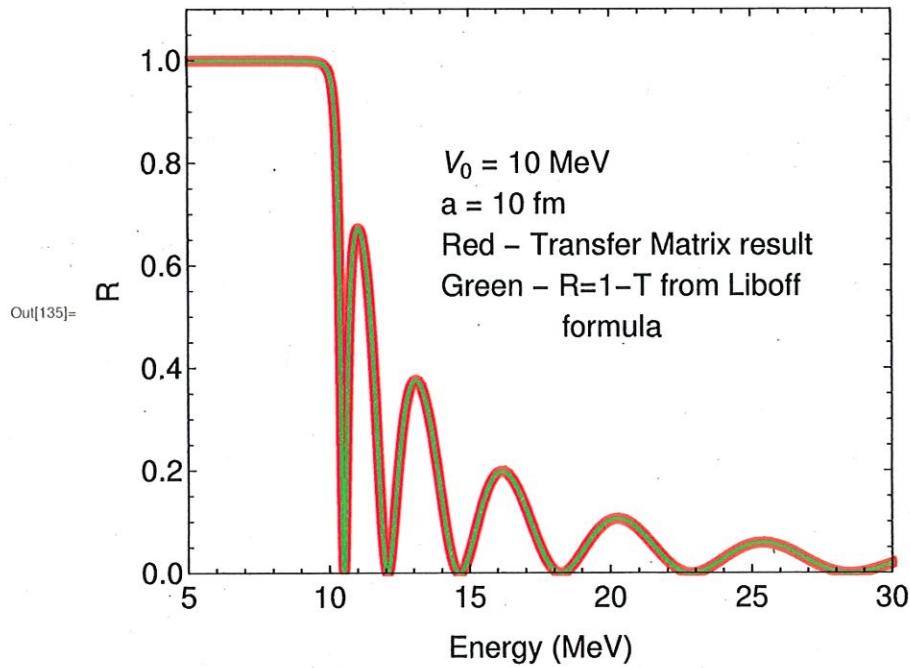
from Mathematica

Test $R+T=1$ numerically

Comparison of Liboff and Transfer Matrix method



Comparison of Liboff and Transfer Matrix method



Square Barrier 3 HW

$$b) \quad \phi_1 = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\phi_2 = Ce^{ik_2x} + De^{-ik_2x}$$

$$\phi_3 = Fe^{ik_3x} + Ge^{-ik_3x}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$T = \frac{|E|^2}{|A|^2} = \frac{1}{|t_{11}|^2}$$

$$A. \quad R = \left| \frac{B}{A} \right|^2$$

$$\xi_1 = \frac{t}{\sim} \xi_3$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11}F + t_{12}G \\ t_{21}F + t_{22}G \end{pmatrix}$$

$G=0$ since no waves are incident from the right

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11}F \\ t_{21}F \end{pmatrix}$$

$$\therefore B = t_{21}F \quad \text{and} \quad A = t_{11}F$$

$$= t_{21} \frac{A}{t_{11}}$$

$$F = \frac{A}{t_{11}}$$

$$\frac{B}{A} = \frac{t_{21}}{t_{11}}$$

$$\therefore R = \left| \frac{t_{21}}{t_{11}} \right|^2$$

To test, get T from the transfer matrix, add with R and you should get one. Check numerically.

B. We have $|F/A|$, $|B/A|$, A and Q so get C and D .

Consider $\xi_1 = d_{12} \xi_2$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} d_{11}C + d_{12}D \\ d_{21}C + d_{22}D \end{pmatrix}$$

Consider each expression separately

$$A = d_{11}C + d_{12}D$$

$$B = d_{21}C + d_{22}D$$

We have two equations and two unknowns.

$$C = \frac{A - d_{12}D}{d_{11}}$$

$$C = \frac{B - d_{22}D}{d_{21}}$$

$$\therefore \frac{A - d_{12}D}{d_{11}} = \frac{B - d_{22}D}{d_{21}}$$

$$\frac{A}{d_{11}} - \frac{B}{d_{21}} = \frac{d_{12}D}{d_{11}} - \frac{d_{22}D}{d_{21}}$$

$$\frac{A}{d_{11}} - \frac{B}{d_{21}} = D \left[\frac{d_{12}d_{21} - d_{22}d_{11}}{d_{11}d_{21}} \right]$$

$$D = \left(\frac{A}{d_{11}} - \frac{B}{d_{21}} \right) \left(\frac{d_{11}d_{21}}{d_{12}d_{21} - d_{22}d_{11}} \right)$$

Now get C

$$C = \frac{A - d_{12}D}{d_{11}}$$

Test it using $R+T=1$. Use the analytical expressions for R and T from the transfer matrix, add them together and you should get one.

assume values for V_0 and the width 'a' of the barrier and make a numerical calculation

Setting the coefficients

$$7) \quad \Phi_1 = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} F \\ t_{21} F \end{pmatrix}$$

$$\checkmark \quad \vec{G} = 0$$

$$\therefore A = 1.0$$

$$T = \left| \frac{E}{A} \right|^2 \text{ see \#5}$$

A is
known

$$\therefore F = \frac{A}{t_{11}}$$

sign of
B is
known

$$B = t_{21} F = t_{21} \frac{A}{t_{11}} = \frac{t_{21}}{t_{11}} A$$

$$\tilde{d}_{12} = \frac{1}{2} \begin{pmatrix} 1 + k_2/k_1 & 1 - k_2/k_1 \\ 1 - k_2/k_1 & 1 + k_2/k_1 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \tilde{d}_{12} \begin{pmatrix} C \\ D \end{pmatrix}$$

①

$$\therefore A = d_{11} C + d_{21} D$$

$$B = d_{12} C + d_{22} D$$

$$C = \frac{A - d_{21} D}{d_{11}}$$

$$C = \frac{B - d_{22} D}{d_{12}}$$

$$\therefore \frac{A - d_{21} D}{d_{11}} = \frac{B - d_{22} D}{d_{12}}$$

$$A - d_{21} D = \frac{d_{11} B - d_{11} d_{22} D}{d_{12}}$$

$$d_{12} A - d_{12} d_{21} D = d_{11} B - d_{11} d_{22} D$$

$$D (d_{11} d_{22} - d_{12} d_{21}) = d_{11} B - d_{12} A$$

$$D = \frac{d_{11} B - d_{12} A}{d_{11} d_{22} - d_{12} d_{21}}$$

from (1)

$$D = \frac{A - d_{11}C}{d_{21}}$$

↓

$$\frac{A - d_{11}C}{d_{21}}$$

$$= \frac{B - d_{12}C}{d_{22}}$$

$$D = \frac{B - d_{12}C}{d_{22}}$$

✓

$$A - d_{11}C = \frac{d_{21}B - d_{12}d_{21}C}{d_{22}}$$

$$d_{22}A - d_{11}d_{22}C = d_{21}B - d_{12}d_{21}C$$

$$C(d_{12}d_{21} - d_{11}d_{22}) = d_{21}B - d_{22}A$$

$$C = \frac{d_{21}B - d_{22}A}{d_{12}d_{21} - d_{11}d_{22}}$$