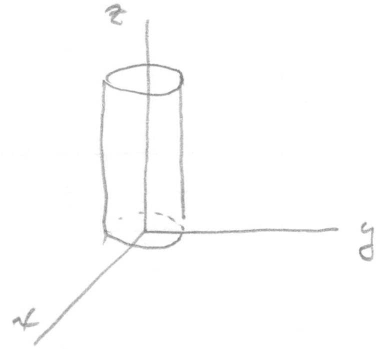
1) 

$\vec{P} = \rho_0 s \hat{s}$

$\sigma_b = ?$ $\rho_b = ?$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

top: $\hat{n} = \hat{z}$

$$\sigma_b = \rho_0 s \, \hat{s} \cdot \hat{z} = 0$$

bottom: $\hat{n} = -\hat{z}$ $\quad \sigma_b = -\rho_0 s \, \hat{s} \cdot \hat{z} = 0$

sides: $\boxed{\sigma_b = \rho_0 s \, \hat{s} \cdot \hat{s} = \underline{\rho_0 R}}$

$$\rho_b = -\nabla \cdot \vec{P}$$

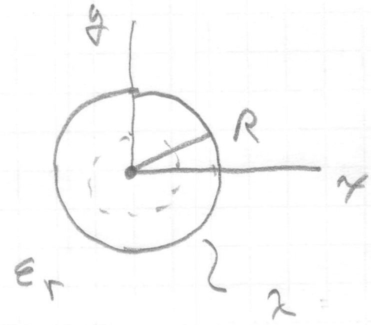
$$\rho_b = -\left[\frac{1}{s} \frac{\partial (s \rho_0 s)}{\partial s} + \cancel{\frac{1}{s} \frac{\partial 0}{\partial \phi}}^{\,0} + \cancel{\frac{\partial 0}{\partial z}}^{\,0} \right]$$

$$= -\frac{1}{s} \frac{\partial}{\partial s} \rho_0 s^2$$

$$= -\frac{1}{s} (2 \rho_0 s)$$

$$\boxed{\rho_b = -2\rho_0}$$

2)



$\vec{P} = ?$ $\sigma_b = ?$ $\rho_b = ?$

for $s < R$ $\nabla \cdot \vec{D} = \rho_f$

$$\int \nabla \cdot \vec{D}\, d\tau = \int \rho_f \, d\tau$$

$$\oint \vec{D} \cdot d\vec{A} = 0$$

$$\vec{D} = D\hat{s} \qquad d\vec{A} = dz\, s\, d\theta\, \hat{s}$$

$$D\, 2\pi s L = 0$$

$$\vec{D} = 0$$

$\therefore \vec{E} = 0$ since $\vec{D} = \varepsilon \vec{E}$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\boxed{\therefore \vec{P} = 0}$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$\boxed{\therefore \rho_b = 0}$$

for $s > R$ $\oint \vec{D} \cdot d\vec{A} = \lambda L$

$$D\, 2\pi s \cancel{L} = \lambda \cancel{L}$$

$$D = \frac{\lambda}{2\pi s} = \varepsilon E = \varepsilon_0 \varepsilon_r E$$

$\vec{D} = \frac{\lambda}{2\pi s}\hat{s}$ $\therefore \vec{E} = \frac{\lambda}{2\pi \varepsilon_0 \varepsilon_r s}\hat{s}$

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$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$$

$$= \frac{\lambda}{2\pi s}\hat{s} - \frac{\cancel{\varepsilon_0}\lambda}{2\pi \cancel{\varepsilon_0}\varepsilon_r s}\hat{s}$$

$$\boxed{\vec{P} = \frac{\lambda}{2\pi s}\left(1 - \frac{1}{\varepsilon_r}\right)\hat{s}}$$

$$\chi_e = \varepsilon_r - 1$$

$$\varepsilon_r = 1 + \chi_e$$

$$\therefore \vec{P} = \frac{\lambda}{2\pi s}\left(1 - \frac{1}{1+\chi_e}\right)\hat{s}$$

$$= \frac{\lambda}{2\pi s}\left(\frac{\cancel{1} + \chi_e - \cancel{1}}{1+\chi_e}\right)\hat{s}$$

$$= \frac{\lambda}{2\pi s}\frac{\chi_e}{1+\chi_e}\hat{s}$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$= -\left[\frac{1}{s}\frac{\partial}{\partial s}\left(\cancel{s}\frac{\lambda}{2\pi \cancel{s}}\left(1 - \frac{1}{\varepsilon_r}\right)\right) + 0 + 0\right]$$

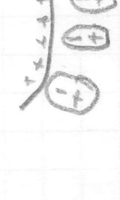
$$\boxed{\rho_b = 0}$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$$= \frac{\lambda}{2\pi s}\left(1 - \frac{1}{\varepsilon_r}\right)\hat{s} \cdot \hat{s}\,\Big|_{s=R}$$

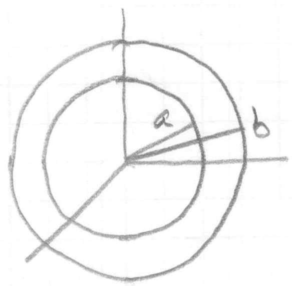
$$\boxed{\sigma_b = -\frac{\lambda}{2\pi R}\left(1 - \frac{1}{\varepsilon_r}\right)}$$

If free charge λ is positive, surface polarization charge will point inward



3) $\nabla^2 V = 0$

$V_a = 0V \quad V_b = 100V$



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) +$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

$$V = R\Theta\cancel{\Phi}^{\to 0}$$

azimuthal symmetry

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial R\Theta}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R\Theta}{\partial\theta}\right) = 0\right]\frac{r^2}{R\Theta}$$

$$\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = l(l+1) = -\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$$

$$R = Ar^l + \frac{B}{r^{l+1}}$$

$$\Theta = P_l(\cos\theta)$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V(r=a,\theta) = 0$$

$$\therefore \quad A_l a^l + \frac{B_l}{a^{l+1}} = 0$$

$$A_l = -\frac{B_l}{a^{l+1}} \cdot \frac{1}{a^l}$$

$$A_l = -\frac{B_l}{a^{2l+1}}$$

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty} B_l \left(-\frac{r^l}{a^{2l+1}} + \frac{1}{r^{l+1}} \right) P_l(\cos\theta)$$

$$V(r=b, \theta) = V_b = 100V$$

$$V_b = \sum_{l=0}^{\infty} B_l \left(\frac{1}{b^{l+1}} - \frac{b^l}{a^{2l+1}} \right) P_l(\cos\theta)$$

$$\int_{-1}^{1} V_b \, P_{l'}(\cos\theta) \, d\cos\theta =$$

$$\int_{-1}^{1} \sum_{l=0}^{\infty} B_l \left(\frac{1}{b^{l+1}} - \frac{b^l}{a^{2l+1}} \right) P_l P_{l'} \, d\cos\theta$$

---

l.h.s. $= 2V_b \quad l' = 0$

$= 0$ otherwise.

$$\text{r.h.s.} = B_\ell \left(\frac{1}{b^{\ell+1}} - \frac{b^\ell}{a^{2\ell+1}} \right) \frac{2}{2\ell+1} \qquad \ell' = \ell$$

$$= 0 \qquad \ell' \neq \ell$$

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for $\ell' = \ell = 0$

$$2V_b = B_0 \left(\frac{1}{b} - \frac{1}{a} \right)(2)$$

$$\boxed{B_0 = \frac{V_b}{\frac{1}{b} - \frac{1}{a}}}$$

for $\ell' = \ell \neq 0$

$$\boxed{B_\ell = 0}$$

$$\therefore \quad V(r,\theta) = \frac{V_b}{\frac{1}{b} - \frac{1}{a}} \left(\frac{1}{r} - \frac{1}{a} \right)$$

$$\vec{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial r} \hat{r}$$

$$= -\frac{V_b}{\frac{1}{b} - \frac{1}{a}} \left(-r^{-2} + 0 \right) \hat{r}$$

$$\boxed{\vec{E} = \frac{V_b}{\frac{1}{b} - \frac{1}{a}} \frac{1}{r^2} \hat{r}}$$

Check it.

$$\vec{E} = \frac{100\,V}{\frac{1}{2m} - \frac{1}{0.5m}} \frac{1}{r^2}\hat{r}$$

$a = 0.5m$

$b = 2m$

$V_b = 100V$

$$= \frac{100\,V\text{-}m}{0.5 - 2} \frac{1}{r^2}\hat{r}$$

$$= -67\,V\text{-}m \frac{1}{r^2}\hat{r}$$

$\frac{200}{3}$ ✓

$$V = \frac{67V\text{-}m}{\frac{1}{2m} - \frac{1}{0.5m}} \left(\frac{1}{r} - \frac{1}{0.5m}\right)$$

$$= \left(-67\,V\text{-}m\right)\left(\frac{1}{r} - 2m^{-1}\right) = -\frac{200}{3r}\,V\text{-}m + \frac{400}{3}\,V$$

---

at $r = a = 0.5\,m$

$$V = \left(-67\,V\text{-}m\right)\left(\frac{1}{0.5m} - 2m^{-1}\right) = 0 \checkmark$$

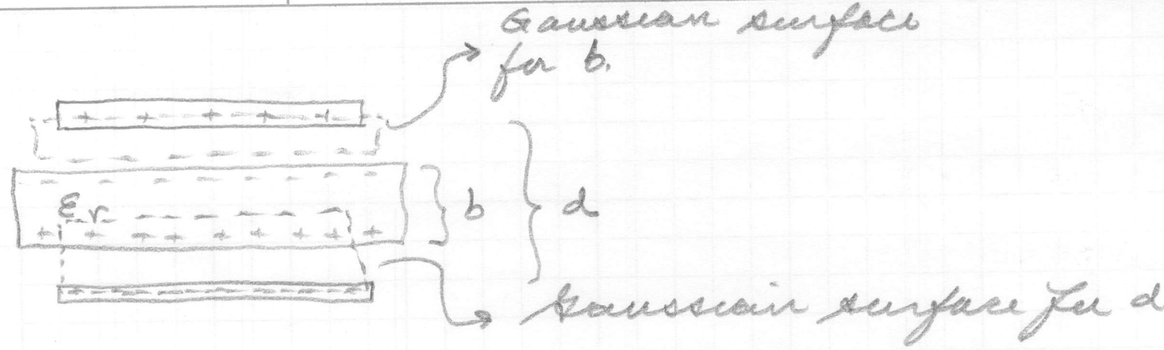
at $r = b = 2m$

$$V = \left(-67\,V\text{-}m\right)\left(\frac{1}{2m} - 2m^{-1}\right)$$

$$= \left(-67\,V\text{-}m\right)\left(0.5m^{-1} - 2m^{-1}\right)$$

$$\boxed{V = 100\,V}$$

4)



a. $C_0 = ?$ $\quad C_0 = \frac{Q}{V} = \frac{\sigma A}{\frac{\sigma}{\varepsilon_0} d} = \frac{\varepsilon_0 A}{d}$

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z}$$

$$V_0 = -\int \vec{E}\cdot d\vec{l}$$

$$= Ed$$

$$= \frac{\sigma}{\varepsilon_0} d$$

$$\boxed{C_0 = \frac{\varepsilon_0 A}{d}}$$

b. $\int \nabla\cdot\vec{E}\, d\tau = \int \frac{\sigma}{\varepsilon_0} d\tau$ $\quad V_0 = \frac{\sigma d}{\varepsilon_0}$

$$\oint \vec{E}\cdot dA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{\sigma = \frac{\varepsilon_0 V_0}{d}}$$

$$E \cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

c. $\boxed{\vec{E}_0 = -\frac{\sigma}{\varepsilon_0}\hat{z} = -\frac{V_0}{d}\hat{z}}$ see a + b.

d.

$$\int \nabla\cdot\vec{D}\, d\tau = \int \rho_f\, d\tau$$

$$\oint \vec{D}\cdot d\vec{A} = \sigma A$$

$$DA = \sigma A$$

using the lower negative plane.

$$\therefore \vec{D} = -\sigma \hat{z} = \varepsilon \vec{E}_1$$

$$\boxed{\vec{E}_1 = -\frac{\sigma}{\varepsilon_0 \varepsilon_r}\hat{z} = -\frac{\varepsilon_0 V_0}{\varepsilon_0 \varepsilon_r d}\hat{z} = -\frac{V_0}{\varepsilon_r d}\hat{z}}$$

e. $V_1 = -\int \vec{E}\cdot d\vec{l}$

$$= -\left[\int_0^{-(d-b)/2} \vec{E}_0 \cdot d\vec{l} + \int_{-\frac{d-b}{2}}^{d-\frac{d-b}{2}} \vec{E}_1 \cdot d\vec{l} + \int_{d-\frac{d-b}{2}}^{d} \vec{E}_0 \cdot d\vec{l}\right]$$

$$d\vec{l} = dz\,\hat{z}$$

$$= -\left[-\frac{\sigma}{\varepsilon_0}\left(\frac{d-b}{2}\right) - \frac{\sigma}{\varepsilon_0 \varepsilon_r}(b) - \frac{\sigma}{\varepsilon_0}\left(\frac{d-b}{2}\right)\right]$$

$$= \frac{\sigma}{\varepsilon_0}(d-b) + \frac{\sigma}{\varepsilon_0 \varepsilon_r} b$$

$$\boxed{V_1 = \frac{\sigma}{\varepsilon_0}\left(d - b + \frac{b}{\varepsilon_r}\right)}$$

$$= \frac{\sigma}{\varepsilon_0} d - \frac{\sigma}{\varepsilon_0}\left(b - \frac{b}{\varepsilon_r}\right) = V_0 - V_0\left(\frac{b}{d} - \frac{b}{d\varepsilon_r}\right)$$

$$\boxed{V_1 = V_0 - \frac{\sigma}{\varepsilon_0} b\left(1 - \frac{1}{\varepsilon_r}\right) < V_0} \quad = V_0\left(1 - \frac{b}{d}\left(1 - \frac{1}{\varepsilon_r}\right)\right)$$

f. $$C_1 = \frac{Q_1}{V_1} = \frac{\sigma A}{\frac{\sigma}{\varepsilon_0}\left(d - b + \frac{b}{\varepsilon_r}\right)} = \frac{\varepsilon_0 A}{d - b + \frac{b}{\varepsilon_r}} = \frac{\varepsilon_0 A}{d - b\left(1 - \frac{1}{\varepsilon_r}\right)} > C_0$$

$$= \frac{C_0}{1 - \frac{b}{d}\left(1 - \frac{1}{\varepsilon_r}\right)} > C_0$$