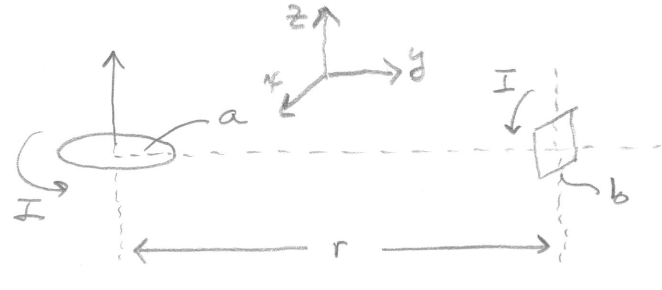
6-1)



$$\vec{\tau} = \vec{m} \times \vec{B} \qquad \vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3}\left[3(\vec{m}_\ell \cdot \hat{r})\hat{r} - m_\ell\right]$$

for circular loop $\quad \vec{m}_\ell = I \int d\vec{A}$ $\qquad$ see Prob 5-33

$$= I \pi a^2 \hat{z}$$

$$\hat{r} = \hat{y} \text{ here}$$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3}\left[3\left(I\pi a^2 \overbrace{\hat{z} \cdot \hat{y}}^{0}\right)\hat{r} - I\pi a^2 \hat{z}\right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3}\left(-I\pi a^2 \hat{z}\right)$$

$$= -\frac{\mu_0}{4\pi} \frac{I\pi a^2}{r^3} \hat{z}$$

for square loop $\quad \vec{m}_s = I \int d\vec{A} = I b^2 \hat{y}$

$$\vec{\tau} = \vec{m}_s \times \vec{B}$$

$$= I b^2 \hat{y} \times \left(-\frac{\mu_0}{4\cancel{\pi}} \frac{I \cancel{\pi} a^2}{r^3} \hat{z}\right)$$

$$\boxed{\vec{\tau} = \frac{-\mu_0 I^2 a^2 b^2}{4 r^3} \hat{x}}$$

It will rotate clockwise looking in toward the origin along the x axis. It will reach equilibrium when $\vec{m}_s = -I b^2 \hat{z}$

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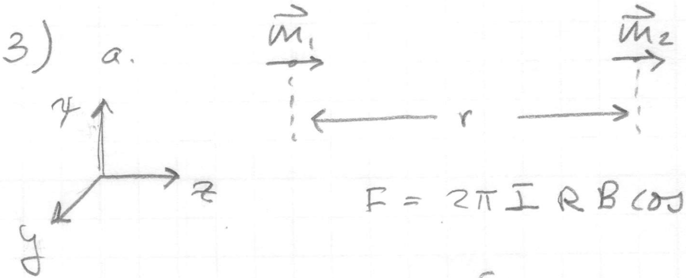
6-3) a.



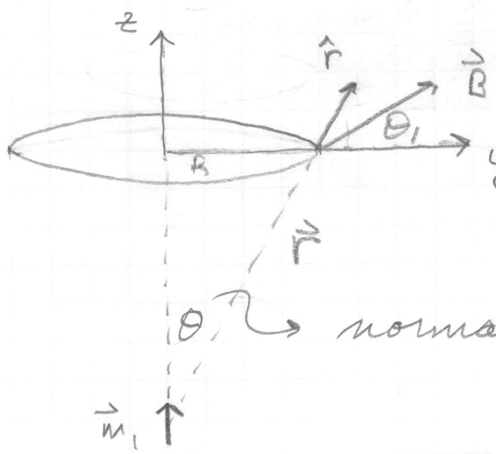
$\vec{F}_{12} = ?$

$$F = 2\pi I R B \cos\theta_1$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[(3\vec{m}_1\cdot\hat{r})\hat{r} - \vec{m}_1\right]$$

$B\cos\theta_1 = \vec{B}\cdot\hat{y}$ see figure below.

$$\vec{B}\cdot\hat{y} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[(3\vec{m}_1\cdot\hat{r})\,\hat{r}\cdot\hat{y} - \vec{m}_1\cdot\hat{y}\right]$$



The angle θ_1 is NOT the usual spherical coordinate angle so the subscript is added to distinguish it.

→ normal spherical coordinates θ

$\vec{m}_1 = m_1\hat{z}$ $\quad\vec{m}_2 = m_2\hat{z}$

$\vec{m}_1\cdot\hat{y} = 0$

$\hat{r}\cdot\hat{y} = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\vec{m}_1\cdot\hat{r} = m_1\cos\theta$

$$B\cos\theta_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3m_1\cos\theta\sin\theta\right]$$

$$\therefore\; F = 2\pi I R\,\frac{\mu_0}{4\pi}\frac{1}{r^3}\,3m_1\cos\theta\sin\theta$$

$$\sin\theta = \frac{R}{r} \qquad \cos\theta = \frac{\sqrt{r^2 - R^2}}{r}$$

$$F = 2\pi I R \frac{\mu_0}{4\pi} \frac{1}{r^3} 3m_1 \frac{R}{r} \frac{\sqrt{r^2 - R^2}}{r}$$

for $R \ll r$, $r^2 - R^2 \sim r^2$

$$= 2\pi I R \frac{\mu_0}{4\pi} \frac{1}{r^3} 3m_1 \frac{R}{r} \cancel{\frac{r}{r}}^{\,1}$$

$$= \cancel{2} \underbrace{\pi R^2 I}_{m_2} \frac{\mu_0}{\cancel{4}\pi}_{2} \frac{1}{r^4} 3m_1$$

$$\boxed{F = \frac{3}{2} \frac{\mu_0}{\pi} \frac{m_1 m_2}{r^4}}$$

b. $\vec{F} = \nabla(\vec{m} \cdot \vec{B})$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[(3\vec{m}_1 \cdot \hat{r}) \hat{r} - \vec{m}_1 \right]$$

$$\vec{m} = \vec{m}_2 = m_2 \hat{z} \qquad \vec{m}_1 = m_1 \hat{z}$$

$$\vec{m}_2 \cdot \vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[(3m_1 \cos\theta) m_2 \hat{z} \cdot \hat{r} - \vec{m}_2 \cdot \vec{m}_1 \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[(3m_1 \cos\theta) m_2 \cos\theta - m_1 m_2 \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} m_1 m_2 (3\cos^2\theta - 1)$$

$$\nabla(\vec{m}_2 \cdot \vec{B}) = \frac{\partial}{\partial r}(\vec{m}_2 \cdot \vec{B})\,\hat{r} + \frac{1}{r}\frac{\partial(\vec{m}_2 \cdot \vec{B})}{\partial \theta}\,\hat{\theta}$$

$$= \frac{\mu_0}{4\pi} m_1 m_2 \Big[-\frac{3}{r^4}(3\cos^2\theta - 1)\hat{r} +$$

$$\frac{1}{r^4} 3\cdot 2\cos\theta(-\sin\theta)\hat{\theta} \Big]$$

$$= \frac{\mu_0}{4\pi} \frac{m_1 m_2}{r^4} 3 \Big[(1 - 3\cos^2\theta)\hat{r} +$$

$$- 2\cos\theta \sin\theta\,\hat{\theta} \Big]$$

For the vector in the brackets use

$$\cos\theta = \frac{\sqrt{r^2 - R^2}}{r} \sim 1$$

$$\therefore \quad (1 - 3\cos^2\theta)\hat{r} - 2\cos\theta\sin\theta\,\hat{\theta} \cong$$

$$= -2\hat{r} - 2\sin\theta\,\hat{\theta}$$

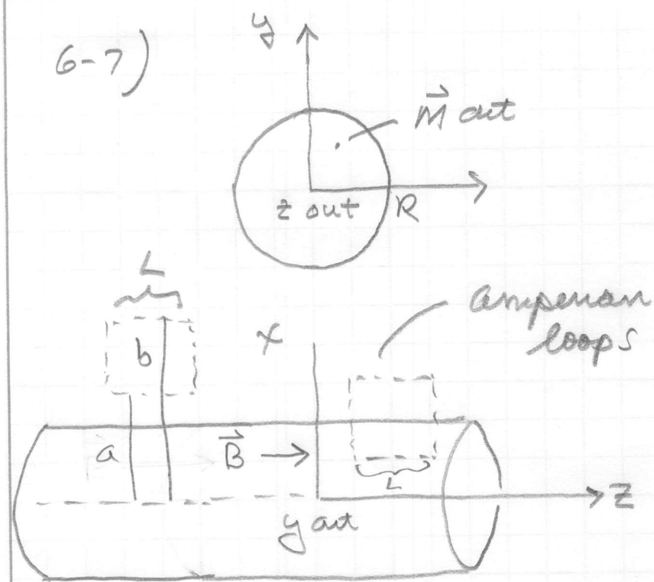
$$= -2(\underbrace{\hat{r} - \sin\theta\,\hat{\theta}}_{\sim \hat{z}})$$

$$= -2\hat{z}$$

$$\nabla(\vec{m}_2 \cdot \vec{B}) = \frac{\mu_0}{\cancel{4}_{2}\pi} \frac{3 m_1 m_2}{r^4} (-\cancel{2}\hat{z})$$

$$= -\frac{3}{2}\frac{\mu_0}{\pi}\frac{m_1 m_2}{r^4}\hat{z} \quad \text{as before.}$$

6-7)



$$\vec{M} = M_0 \hat{z}$$

$$\vec{J}_b = \nabla \times \vec{M} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{s} = M_0 \hat{z} \times \hat{s} = M_0 \hat{\phi}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\int \nabla \times \vec{B} \cdot d\vec{A} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$\vec{B} = B(s)\hat{z}$ by symmetry

$d\vec{l} = d\hat{z}$ on top and bottom of loops

$d\vec{l} = d\hat{s}$ on sides

$d\hat{s} \cdot d\hat{z} = 0$ on sides

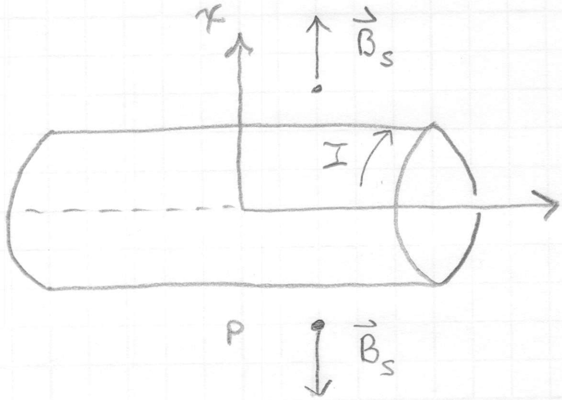
for $s > R$ $\quad B(a) \cancel{L} - B(b) \cancel{L} = 0$

$$B(a) = B(b)$$

But $B(s) = 0$ at $s = \infty$

$$\boxed{\therefore \quad \vec{B}_{outside} = 0}$$

Why is B_s zero in a solenoid?



① Imagine $\vec{B}_s$ exist at P and P' symmetrically located around the solenoid so the two fields $|B_s|$ are equal.

② Reverse the current. By Biot-Savart, the fields should now be inward

③ Put the current back to its original direction.

④ Flip the solenoid 180°. What does the current look like?

It looks like ②. The current is reversed.

What happens to $\vec{B}_s$ at P and P' after the flip?

They still point outward!

This contradicts the current reversal effect of the 180° flip

⑤ The only way out is $\underline{B_s = 0}$.

---

another way: Put a coaxial cylindrical surface inside or outside the solenoid. If $B_s \neq 0$, then

$$\int \vec{B} \cdot d\vec{A} = B_s 2\pi s' L$$

but $\int \vec{B} \cdot d\vec{A} = \int \nabla \cdot \vec{B} \, d\tau = \int 0 \, d\tau = 0$

$\therefore B_s = 0$

for $s < R$ $\quad B\ell - 0 \cdot L = \mu_0 K_b \ell$

$$B = \mu_0 M_0$$

$$\boxed{\therefore \vec{B} = \mu_0 M_0 \hat{z}}$$

6-8) $\vec{M} = ks^2\hat{\varphi}$

$$\vec{J}_b = \nabla\times\vec{M} = \left(\frac{1}{s}\frac{\partial M_z}{\partial\varphi} - \frac{\partial M_\varphi}{\partial z}\right)\hat{s} + \left(\frac{\partial M_s}{\partial z} - \frac{\partial M_z}{\partial s}\right)\hat{\varphi} +$$

$$\frac{1}{s}\left(\frac{\partial s M_\varphi}{\partial s} - \frac{\partial M_s}{\partial\varphi}\right)\hat{z}$$

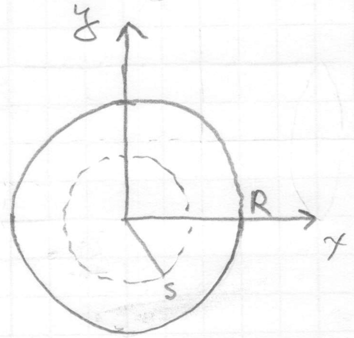
$$= \frac{1}{s}\frac{\partial}{\partial s} ks^3\,\hat{z}$$

$$\vec{J}_b = 3ks\,\hat{z}$$

$$\vec{K}_b = \vec{M}\times\hat{n} = kR^2\hat{\varphi}\times\hat{s} = -kR^2\hat{z}$$

$s > R$ $\quad \vec{B} = 0 \quad$ See 6-7.

$s < R$



$$\nabla\times\vec{B} = \mu_0\vec{J}$$

$$\int\nabla\times\vec{B}\cdot d\vec{A} = \mu_0\int\vec{J}\cdot d\vec{A}$$

$$\int\vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$\vec{B} = B(s)\hat{\varphi}$ by symmetry

$$B\,2\pi s = \mu_0\int_0^{2\pi}\int_0^s 3ks'\,s'\,d\varphi'\,ds'$$

$$= \mu_0 3k(2\pi)\int_0^s s'^2\,ds'$$

$$= \mu_0 6\pi k\,\frac{s'^3}{3}\Big|_0^s$$

$$B\,\cancel{2\pi}\cancel{s} = \cancel{2\pi}\,k\mu_0 s^{\cancel{3}\,2}$$

$$\boxed{\vec{B} = \mu_0 ks^2\hat{\varphi}} \qquad \boxed{\vec{B} = \mu_0\vec{M}}$$

6-12) $\vec{M} = ks\,\hat{z}$ $\quad J_f = 0$

a. $\vec{J}_b = \nabla \times \vec{M}$

$= -\dfrac{\partial}{\partial s} M_z \,\hat{\phi}$

$= -k\,\hat{\phi}$

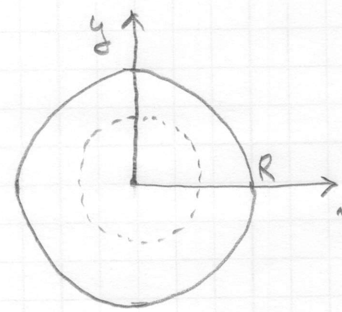
$\vec{K}_b = \vec{M} \times \hat{n}$

$= kR\,\hat{z} \times \hat{s}$

$= kR\,\hat{\phi}$

for $s > R$ $\quad \vec{B} = 0$ $\quad$ see 6-7

for $s < R$

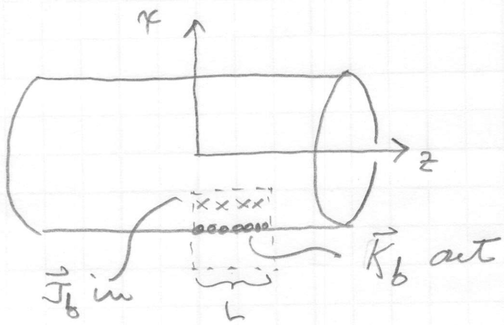


$\nabla \times \vec{B} = \mu_0 \vec{J}$

$\int \nabla \times \vec{B} \cdot d\vec{A} = \mu_0 \int \vec{J} \cdot d\vec{A}$

$\int \vec{B} \cdot d\vec{l} = 0$

$\therefore B_\phi = 0$



$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$\vec{B} = B(s)\,\hat{z}$ by symmetry

$B(s)L = \mu_0 \left[kRL - kL(R-s) \right]$

$= \mu_0 kLs$

$\vec{B} = \mu_0 k s\,\hat{z}$

b. $\oint \vec{H} \cdot d\vec{l} = I_{f\,enc}$ $\quad \vec{H} = H(s)\,\hat{z}$ by symmetry

for $s > R$ $\quad \oint \vec{H} \cdot d\vec{l} = I_{f\,enc} = 0$

$$\therefore \vec{H} = 0 = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\therefore \vec{B} = \mu_0 \vec{M}$$ but $\vec{M} = 0$ outside the cylinder

$$\therefore \vec{B} = 0$$

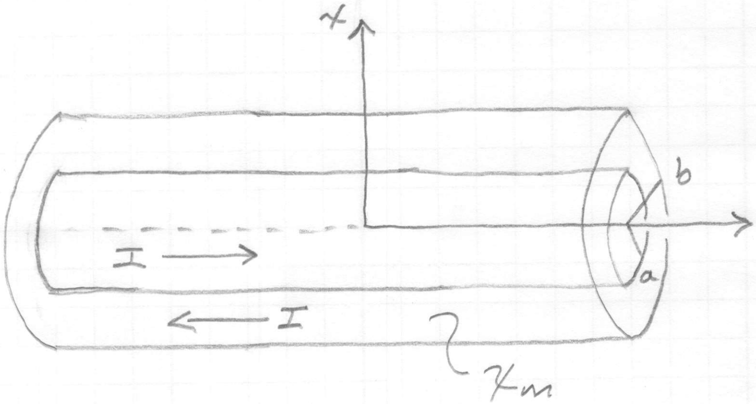
for $s < R$ $\quad \oint \vec{H} \cdot d\vec{l} = 0$

$$\therefore \vec{H} = 0 = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\vec{B} = \mu_0 \vec{M}$$

$$\boxed{\vec{B} = \mu_0 k s \hat{z}}$$

6-16)



$\vec{B} = ?$ $a < s < b$

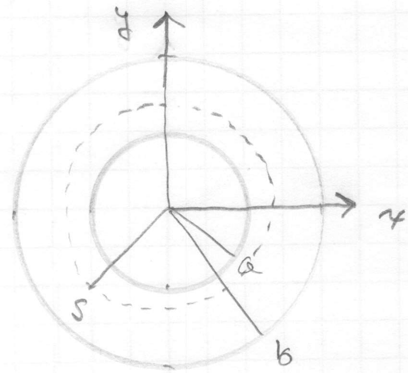
$\vec{J}_b = ?$

$\vec{K}_b = ?$

$$\vec{M} = \chi_m \vec{H} \qquad \vec{B} = \mu \vec{H} = \mu_0 (1 + \chi_m) \vec{H}$$

$$\vec{J}_b = \nabla \times \vec{M}$$

$$\vec{K}_b = \vec{M} \times \hat{n}$$



$$\nabla \times \vec{H} = \vec{J}_f$$

$$\int \nabla \times \vec{H} \cdot d\vec{A} = \int \vec{J}_f \cdot d\vec{A}$$

$$\int \vec{H} \cdot d\vec{l} = I$$

$$2\pi s H = I$$

$$\vec{H} = \frac{I}{2\pi s} \hat{\phi}$$

$$\vec{B} = \mu_0 (1 + \chi_m) \frac{I}{2\pi s} \hat{\phi}$$

$$\vec{M} = \chi_m \vec{H} = \frac{\chi_m I}{2\pi s} \hat{\phi}$$

$$\vec{J}_b = \nabla \times \vec{M} = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\chi_m I}{2\pi s} \right) \hat{z} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = \frac{\chi_m I}{2\pi s} \hat{\phi} \times \hat{s} = -\frac{\chi_m I}{2\pi s} \hat{z}$$

at $s = a$ $\vec{K}_b = + \frac{\chi_m I}{2\pi a} \hat{z}$ $\Big\}$ at $s = b$ $\vec{K}_b = - \frac{\chi_m I}{2\pi b} \hat{z}$

confirm the field calculation for $a < s < b$.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int (\vec{J}_f + \vec{J}_b) \cdot d\vec{A}$$

$$B\, 2\pi s = \mu_0 I + \mu_0 K_b\, 2\pi a$$

$$= \mu_0 I + \mu_0 \frac{\chi_m I}{\cancel{2\pi a}} \cancel{2\pi a}$$

$$= \mu_0 I (1 + \chi_m)$$

$$\boxed{\vec{B} = \mu_0 (1 + \chi_m) \frac{I}{2\pi s} \hat{\phi}}$$ as before

6-21) a. $dW = \underbrace{F_{\perp} r}_{\tau}\, d\theta$

$$\boxed{dW = \tau\, d\theta}$$

- potential energy

What is potential energy?

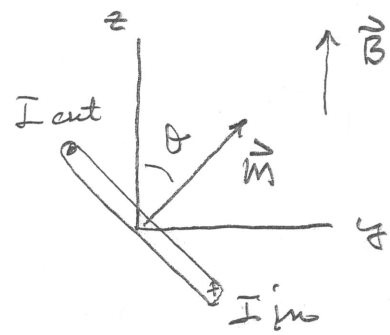
ΔPE = work done

$$\Delta PE = \int_{\vec{r}_1}^{\vec{r}_2} dW$$

$$= \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

for our loop

Discuss direction of $\vec{m}$



$\vec{\tau} = \vec{m} \times \vec{B}$

$= mB\sin\theta$

$$\Delta PE = \int_{\theta_1}^{\theta_2} mB\sin\theta\, d\theta$$

$$= -mB\cos\theta \Big|_{\theta_1}^{\theta_2}$$

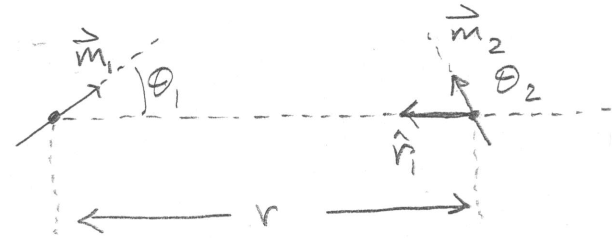
let $\theta_1 = \frac{\pi}{2}$ by convention.

$$\Delta PE = -mB\cos\theta_f$$

$$\boxed{\Delta PE = -\vec{m}\cdot\vec{B}}$$

b. What is the interaction energy between two magnetic dipoles?

Recall: We have hypothesized that the 21-cm line is from the hyperfine interaction. The hyperfine interaction is the force of the intrinsic magnetic field of the proton with the electron's intrinsic magnetic field.



What is the $\vec{B}$ field of a dipole?

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

prob 5-33

What is the energy of a dipole in a field $\vec{B}$?

$$\Delta PE = U = -\vec{m}\cdot\vec{B}$$

Bring both pieces together

$$U_{12} = -\vec{m}_1 \cdot \vec{B}_2$$ potential of $\vec{m}_1$ in field of $\vec{m}_2$

$\hat{r}_1$ points from $\vec{m}_2$ to $\vec{m}_1$

$$= -\vec{m}_1 \cdot \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_2\cdot\hat{r}_1)\hat{r}_1 - \vec{m}_2\right]$$

$$= +\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_2\cdot\hat{r}_1)(\vec{m}_1\cdot\hat{r}_1)\right]$$

Put the answer in terms of θ_1 and θ_2.

$$\vec{m}_1\cdot\vec{m}_2 = m_1 m_2 \cos(\theta_1 - \theta_2)$$

$$\vec{m}_1\cdot\hat{r}_1 = m_1 \cos(\pi - \theta_1)$$
$$= -m_1 \cos\theta_1$$

$$\vec{m}_2\cdot\hat{r}_1 = m_2 \cos(\pi - \theta_2)$$
$$= -m_2 \cos\theta_2$$

$$\therefore U_{12} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[m_1 m_2 \cos(\theta_1-\theta_2) - 3 m_1 m_2 \cos\theta_1\cos\theta_2\right]$$

$$\boxed{U_{12} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\, m_1 m_2 \left[\cos(\theta_1-\theta_2) - 3\cos\theta_1\cos\theta_2\right]}$$

7. Can we account for the hyperfine interaction?

$$m_p = 2.79\,\mu_N$$

$$\mu_N = \frac{e\hbar}{2m_p} = \frac{(1.6\times10^{-19}\,C)(1.054\times10^{-34}\,J\text{-}s)}{2(1.672\times10^{-27}\,kg)}$$

units $\frac{J\text{-}s\text{-}C}{kg} = \frac{J}{kg/s\text{-}C} = \frac{J}{T}$

$$= 5.05\times10^{-27}\,J/T = 3.15\times10^{-8}\,eV/T$$

c. $$U_{12} = \frac{\mu_0}{4\pi}\frac{1}{r^3} m_1 m_2 \left[\cos(\theta_1 - \theta_2) - 3\cos\theta_1 \cos\theta_2\right]$$

Get minimum for U_{12} by requiring

$$\frac{\partial U_{12}}{\partial \theta_1} = \frac{\partial U_{12}}{\partial \theta_2} = 0$$

$$\frac{\partial U_{12}}{\partial \theta_1} = -\sin(\theta_1 - \theta_2) + 3\sin\theta_1 \cos\theta_2 = 0$$

$$3\sin\theta_1 \cos\theta_2 = \sin(\theta_1 - \theta_2) \qquad (1)$$

$$\frac{\partial U_{12}}{\partial \theta_2} = -\sin(\theta_1 - \theta_2)(-1) + 3\cos\theta_1 \sin\theta_2 = 0$$

$$3\cos\theta_1 \sin\theta_2 = -\sin(\theta_1 - \theta_2) \qquad (2)$$

Add (1) and (2)

$$3\sin\theta_1 \cos\theta_2 + 3\cos\theta_1 \sin\theta_2 = 0$$

$$\sin\theta_1 \cos\theta_2 = -\cos\theta_1 \sin\theta_2$$

$$\frac{\sin\theta_1}{\cos\theta_1} = -\frac{\sin\theta_2}{\cos\theta_2}$$

$$\tan\theta_1 = -\tan\theta_2$$

$$\theta_1 = \text{Arctan}(-\tan\theta_2)$$

A plot of this shows the $\theta_1 = -\theta_2$ (see plot).

Use this result in (1)

$$3\sin(-\theta_2)\cos\theta_2 = \sin(-\theta_2 - \theta_2)$$

$$+3\sin\theta_2 \cos\theta_2 = +\sin 2\theta_2$$

$$3 \cancel{\sin\theta_2} \cancel{\cos\theta_2} = 2 \cancel{\sin}\,\theta_2 \cancel{\cos}\theta_2$$

$3 = 2$ <u>Fascinating!</u>

This requires $\sin\theta_2 = 0$ or $\cos\theta_2 = 0$

$\therefore\ \theta_2 = 0, \pi$ or $\theta_2 = \frac{\pi}{2}, \frac{3\pi}{2}$

Looking at the surface plot of the angular dependence, the energy will be a minimum for

$\theta_2 = 0,\ \theta_1 = 0$

and nowhere else.

$\boxed{\therefore \text{ They all line up.}}$