3-18) $V_0 = k\cos 3\theta \qquad V = ? \qquad \sigma = ?$

for $r < R \qquad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi V_0(\theta)\,P_\ell(\cos\theta)\sin\theta\,d\theta$$

$$V_0 = k\cos 3\theta$$

$$V_0 = k\left(4\cos^3\theta - 3\cos\theta\right)$$

Math Handbook, p. 16

$$P_1(\cos\theta) = \cos\theta$$

$$V_0 = k\left(4\cos^3\theta - 3P_1\right)$$

$$P_3 = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)$$

$$\frac{1}{5}\left(2P_3 + 3\overset{P_1}{\cos\theta}\right) = \cos^3\theta$$

$$V_0 = k\left[4\cdot\frac{1}{5}\left(2P_3 + 3P_1\right) - 3P_1\right]$$

$$= k\left[\frac{8}{5}P_3 + \frac{12}{5}P_1 - \frac{15}{5}P_1\right]$$

$$= k\,\frac{1}{5}\left[8P_3 - 3P_1\right]$$

$$\therefore A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi \frac{k}{5}\left(8P_3 - 3P_1\right)P_\ell\sin\theta\,d\theta$$

$$= \frac{(2\ell+1)k}{10R^\ell}\left[8\int_0^\pi P_3P_\ell\sin\theta\,d\theta - 3\int_0^\pi P_1P_\ell\sin\theta\,d\theta\right]$$

$A_\ell = 0 \qquad \ell \neq 1 \text{ and } \ell \neq 3$

$= \frac{3k}{\cancel{10}_{5} R} \cdot \cancel{3} \frac{\cancel{2}}{\cancel{3}} = \frac{3}{5}\frac{k}{R} \qquad \ell = 1$

$= \frac{\cancel{7}k}{\cancel{10}_{5} R^3} \, 8 \, \frac{\cancel{2}}{\cancel{7}} = \frac{8}{5}\frac{k}{R^3} \qquad \ell = 3$

$$\boxed{\therefore \; V(r,\theta) = \frac{3}{5}\frac{k}{R} r\cos\theta + \frac{8}{5}\frac{k}{R^3} r^3 \frac{(5\cos^3\theta - 3\cos\theta)}{2} \qquad r < R}$$

for $r > R \qquad V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi V_0(\theta) P_\ell(\cos\theta)\sin\theta \, d\theta$$

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi \frac{k}{5}(8P_3 - 3P_1) P_\ell \sin\theta \, d\theta$$

$= \frac{\cancel{2\cdot 3+1}}{\cancel{2}} R^4 \frac{k}{5} 8 \frac{\cancel{2}}{\cancel{2\cdot 3+1}} = \frac{8}{5} k R^4 \qquad \ell = 3$

$= \frac{\cancel{2\cdot 1+1}}{\cancel{2}} R^2 \frac{k}{5} 3 \frac{\cancel{2}}{\cancel{2\cdot 1+1}} = \frac{3k}{5} R^2 \qquad \ell = 1$

$= 0 \qquad$ otherwise

$\therefore \; V(r,\theta) = \frac{8}{5} k R^4 \frac{1}{r^4} P_3(\cos\theta) + \frac{3k}{5} R^2 \frac{1}{r^2} P_1(\cos\theta)$

$$\boxed{V(r,\theta) = \frac{8}{5} k \frac{R^4}{r^4} \frac{(5\cos^3\theta - 3\cos\theta)}{2} + \frac{3k}{5}\frac{R^2}{r^2}\cos\theta}$$

$$\sigma_0 = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta)$$

$$= \varepsilon_0 \Big[\frac{3}{5}\frac{k}{R}(2\cdot1+1)R^0 \cos\theta +$$

$$\frac{8}{5}\frac{k}{R^3}(2\cdot3+1)R^2\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right)\Big]$$

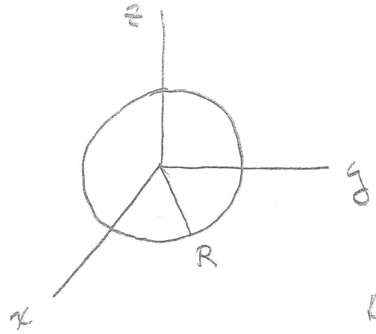
$$= \frac{k\varepsilon_0}{5R}\left[3\cdot3\cos\theta + \frac{\cancel{8}^{4}}{\cancel{R^2}}\frac{7\cancel{R^2}}{\cancel{2}}\left(5\cos^3\theta - 3\cos\theta\right)\right]$$

$$= \frac{k\varepsilon_0}{5R}\left[9\cos\theta + 5\cdot28\cos^3\theta - 3\cdot28\cos\theta\right]$$

$$\boxed{\sigma_0 = \frac{k\varepsilon_0}{5R}\left[140\cos^3\theta - 75\cos\theta\right]}$$

TOPS. 35500

3-26)



$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho\, d\tau'$$

Do the 1st few terms

zeroth order

$$\int (r')^0 P_0 \rho\, d\tau' = \int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{kR}{r'^2}(R-2r')\sin\theta\, r'^2\sin\theta\, dr' d\theta\, d\phi$$

monopole term

$$\int_0^{\pi}\sin^2\theta\, d\theta = \frac{\pi}{2}$$

$$\int_0^{2\pi} d\phi = 2\pi$$

$$= 2\pi\left(\frac{\pi}{2}\right)\left[kR^2 r' - kR r'^2\right]_0^R$$

from Mathematica

$$= 0.$$

1st order

$$\int r'^1 P_1(\cos\theta)\rho\, d\tau' =$$

dipole term

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^R r'\cos\theta' \frac{kR}{r'^2}(R-2r')\sin\theta'\, r'^2\sin\theta'\, dr' d\theta' d\phi'$$

$$= (2\pi)(0)\left(-\frac{kR^4}{6}\right)$$

$$= 0$$

2nd order

$$\int r'^2 P_2(\cos\theta')\rho\, d\tau' =$$

quadrupole term

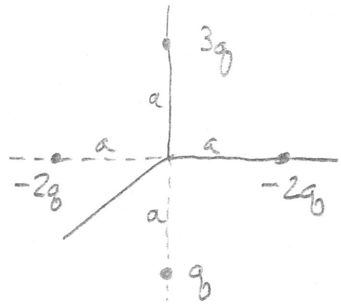
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^R r'^2\left(\frac{3\cos^2\theta' - 1}{2}\right)\frac{kR}{r'^2}(R-2r')\sin^2\theta'\, r'^2\, dr' d\theta' d\phi'$$

$$= 2\pi\left(-\frac{\pi}{16}\right)\left(-\frac{kR^5}{6}\right) = \frac{\pi^2}{48}kR^5$$

$$\therefore \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{\pi^2}{48} kR^5$$

$$\boxed{\therefore \quad V(z) = \frac{1}{4\pi\varepsilon_0} \frac{k\pi^2 R^5}{48 z^2}}$$

3-27)



$$V_{mon} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad Q \equiv \text{total charge} = 0 \text{ here}$$

$$= 0$$

$$V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}\,')\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

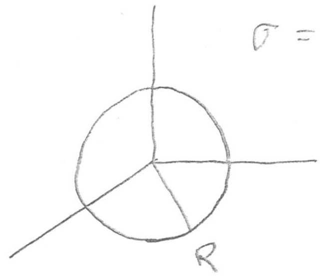
$$\vec{p} = \Sigma q_i \vec{r}_i\,' = -2qa\,\hat{y} + (-2q)(-a)\hat{y} +$$
$$3q(a)\hat{z} + (q)(-a)\hat{z}$$
$$= 2aq\,\hat{z}$$

$$\vec{p}\cdot\hat{r} = 2aq\,\hat{z}\cdot(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$$
$$= 2aqz$$

$$\therefore\ V_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{2aqz}{r^2} \qquad \frac{z}{r} = \cos\theta$$

$$\boxed{V_{dip} = \frac{1}{4\pi\varepsilon}\frac{2aq\cos\theta}{r}}$$

3-28)



$\sigma = k\cos\theta$

$$\vec{p} = \int \vec{r}'\rho(\vec{r}')\,d\tau'$$

$$d\tau' = r'^2 dr'\,d\cos\theta'\,d\phi'$$
$$= R^2 d\cos\theta'\,d\phi'$$

a. $$\vec{p} = \int_0^{2\pi}\int_{-1}^{1} R\,\hat{r}'(k\cos\theta')\,R^2 d\cos\theta'\,d\phi'$$

$$\hat{r}' = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\vec{p} = \int_0^{2\pi}\int_{-1}^{1} kR^3(\sin\theta'\cos\phi'\,\hat{x} + \sin\theta'\sin\phi'\,\hat{y} + \cos\theta'\,\hat{z})\cos\theta'\,d\cos\theta'\,d\phi'$$

$$\int_{-1}^{1}\sin\theta'\cos\theta'\,d\cos\theta' = \int_0^{\pi}\sin^2\theta'\cos\theta'\,d\theta' = 0$$

$$\int_{-1}^{1}\cos^2\theta'\,d\cos\theta' = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

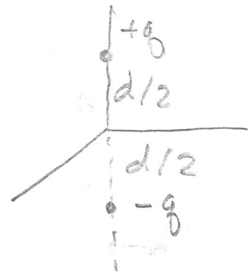
$$\int_0^{2\pi} d\phi' = 2\pi$$

$$\vec{p} = 2\pi\,k\,R^3\,\frac{2}{3}\,\hat{z} = \underline{\frac{4\pi}{3}k\,R^3\,\hat{z}}$$

b. $$V = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\vec{p}\cdot\hat{r} = \frac{1}{\cancel{4\pi}\epsilon_0}\frac{1}{r^2}\frac{\cancel{4\pi}k}{3}R^3\,\hat{z}\cdot\hat{r} = \frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2}$$

The higher multipoles are zero.

3-29



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$\frac{1}{r} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$r' = \frac{d}{2} \quad \theta'_+ = \theta \quad \theta'_- = \pi - \theta$$

$$\frac{1}{r_+} = \frac{1}{r}\left[\left(\frac{d/2}{r}\right)^0 (1) + \left(\frac{d/2}{r}\right)^1 \cos\theta + \left(\frac{d/2}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{d/2}{r}\right)^3\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \dots\right]$$

$$= \frac{1}{r}\left[1 + \frac{1}{2}\frac{d}{r}\cos\theta + \frac{1}{8}\frac{d^2}{r^2}(3\cos^2\theta - 1) + \frac{1}{16}\frac{d^3}{r^3}(5\cos^3\theta - 3\cos\theta) + \dots\right]$$

Now use $\cos(\pi - \theta) = -\cos\theta$

$$\frac{1}{r_-} = \frac{1}{r}\left[\left(\frac{d/2}{r}\right)^0 (1) + \left(\frac{d/2}{r}\right)^1 (-\cos\theta) + \left(\frac{d/2}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{d/2}{r}\right)^3\left(\frac{-5\cos^3\theta + 3\cos\theta}{2}\right) + \dots\right]$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{d}{r}\cos\theta + \frac{1}{8}\frac{d^2}{r^2}(3\cos^2\theta - 1) + - \frac{1}{16}\frac{d^3}{r^3}(5\cos^3\theta - 3\cos\theta) + \dots\right]$$

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{1}{r}\left[0 + \frac{d}{r}\cos\theta + 0 + \frac{1}{8}\left(\frac{d}{r}\right)^3 (5\cos^3\theta - 3\cos\theta) + \cdots\right]$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\Big[\underset{\text{monopole}}{0} + \underset{\text{dipole}}{\frac{d}{r}\cos\theta} + \underset{\text{quadrupole}}{0} + \underset{\text{octopole}}{\frac{1}{8}\left(\frac{d}{r}\right)^3(5\cos^3\theta - 3\cos\theta)}\Big]$$

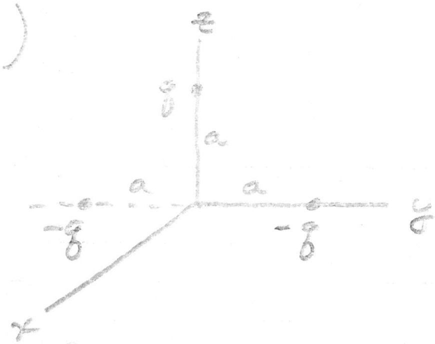
$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^2}P_1(\cos\theta)$$

$$V_{quad} = 0$$

$$V_{oct} = \frac{1}{4\pi\varepsilon_0}\frac{qd^3}{8r^4}(5\cos^3\theta - 3\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qd^3}{4r^4}P_3(\cos\theta)$$

3.32)



$Q = \sum q_i = -q$

$\therefore V_{mono} = \frac{1}{4\pi\varepsilon_0} \frac{-q}{r}$

$\vec{p} = \sum q_i \vec{r}_i'$

$= -qa\hat{y} + (-q)(-a)\hat{y} + qa\hat{z}$

$= qa\hat{z}$

$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \vec{p}\cdot\hat{r}$

$\vec{p}\cdot\hat{r} = qa\hat{z} \cdot (\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})$

$= qa\cos\theta$

$V_{dipole} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} qa\cos\theta$

$V \cong \frac{1}{4\pi\varepsilon_0} q\left(-\frac{1}{r} + \frac{a}{r^2}\cos\theta\right)$

$\therefore \vec{E} = -\nabla V$

$= -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$

$= -\frac{1}{4\pi\varepsilon} q\left[\left(+\frac{1}{r^2} - \frac{2a\cos\theta}{r^3}\right)\hat{r} + \frac{a}{r^3}(-\sin\theta)\hat{\theta}\right]$

$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\left[\left(\frac{2a\cos\theta}{r} - 1\right)\hat{r} + \frac{a\sin\theta}{r}\hat{\theta}\right]$