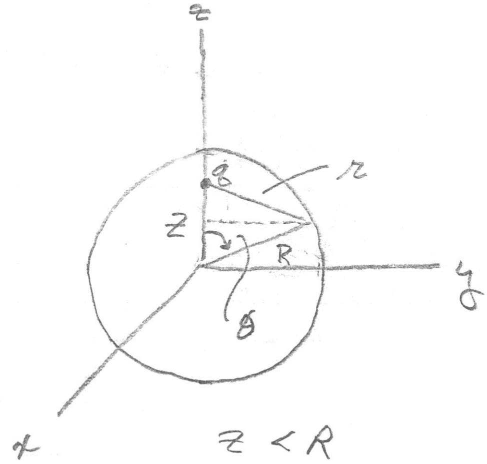
3-1) As in 3.1.4 place the sphere at the origin and the charge on the z axis



$z < R$

$$V_{ave}(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V\,da$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$ for a point on the sphere.

$$da = R^2\, d(\cos\theta)\, d\phi$$

$$r^2 = z^2 + R^2 - 2Rz\cos\theta$$

$$V_{ave} = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_{-1}^{1} \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+R^2-2Rz\cos\theta}}\, R^2\, d(\cos\theta)\, d\phi$$

$$\int_0^{2\pi} d\phi = 2\pi \qquad \text{let } x' = \cos\theta$$

$$= \frac{1}{4\pi \cancel{R^2}}\,\frac{q}{\cancel{4\pi}\epsilon_0}\,\cancel{2\pi}\cancel{R^2}\int_{-1}^{1}\frac{1}{\sqrt{z^2+R^2-2Rzx'}}\,dx'$$

$$= \frac{q}{8\pi\epsilon_0}\left[-\frac{\sqrt{R^2-2Rx'z+z^2}}{Rz}\right]_{-1}^{1}$$

$$= \frac{q}{8\pi\epsilon_0}\left[-\frac{\sqrt{R^2-2R(1)z+z^2}}{Rz} - \left(-\frac{\sqrt{R^2-2R(-1)z+z^2}}{Rz}\right)\right]$$

$$= \frac{q}{8\pi\epsilon_0}\frac{1}{Rz}\left[-\sqrt{R^2+z^2-2Rz} + \sqrt{R^2+z^2+2Rz}\right]$$

$$= \frac{q}{8\pi\epsilon_0}\frac{1}{Rz}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

use $R-z$ (not $z-R$) here since this is a length and we treat them as positive.

$$V_{ave} = \frac{q}{8\pi\varepsilon_0} \frac{1}{Rz} \left(z+R - (R-z) \right)$$

$$= \frac{q}{\cancel{8}\pi\varepsilon_0} \frac{1}{Rz} (2z)$$
4

$$\boxed{V_{ave} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R}}$$

This is for a single charge inside the ~~sphere~~. For many charges inside the ~~sphere~~

$$V_{ave} = \Sigma V_i$$

$$= \Sigma \frac{1}{4\pi\varepsilon_0} \frac{q_i}{R} \longrightarrow Q_{enc.}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{R} \Sigma q_i$$

$$\boxed{V_{ave} = \frac{1}{4\pi\varepsilon_0} \frac{Q_{enc}}{R}}$$

for each external charge q_i

$$V^{(e)}_{ave} = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{z_i}$$

so for many external charges

$$V^{(e)}_{ave} = \frac{1}{4\pi\varepsilon_0} \Sigma \frac{q_i}{z_i} = V_{center}$$

∴ By superposition

$$V^{(total)}_{ave} = V_{center} + \frac{1}{4\pi\varepsilon_0} \frac{Q_{enc}}{R}$$

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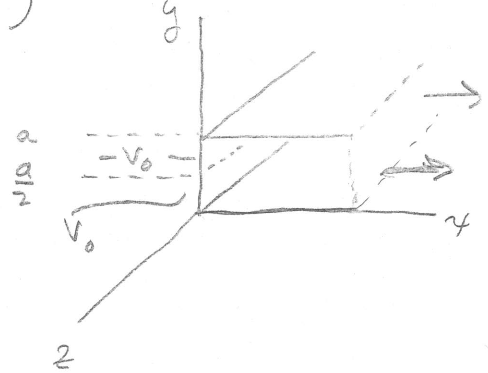
3-2) Laplace's equation requires no local minima in V except at the boundaries so there can't be a stable equilibrium point.

Particles will tend to leak out of the faces because of the repulsion at the corners.

3-12)



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$V = 0$ for $y = 0$

$V = 0$ for $y = a$

$V \to 0$ as $x \to \infty$

$V = V_0$ for $x = 0, y < \frac{a}{2}$

$V = -V_0$ for $x = 0, y > \frac{a}{2}$

let $V = X(x)Y(y)$

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} = 0$$

$$\left[Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0 \right] \frac{1}{XY}$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = k^2 \longrightarrow \text{separation constant}$$

Same equations we saw in class so

$$X = Ae^{kx} + Be^{-kx}$$

$$Y = C \sin ky + D \cos ky$$

For $y = 0$, $V = 0$

$$Y(0) = C \cdot \sin 0 + D \cos 0 = 0$$

$$= D$$

$$= 0 \quad \therefore D = 0$$

For $y = a$, $V = 0$

$$Y(a) = C \sin ka = 0$$

$\therefore$ $C = 0$ dull, dull, dull

or

$\sin ka = 0 \Rightarrow ka = n\pi \quad n = 1, 2, 3, \ldots$

$$\boxed{k = \frac{n\pi}{a}}$$

$$\therefore Y(y) = C \sin \frac{n\pi}{a} y$$

For $x \to \infty$, $V = 0$

$$X = Ae^{kx} + Be^{-kx}$$

$\therefore$ $A = 0$ so $X \to 0$ as $x \to 0$

$$X = Be^{-kx}$$

For $x = 0$ $\quad V = V_0 \quad 0 < y < a/2$

$\qquad\qquad\quad V = -V_0 \quad a/2 < y < a$

$$V(0,y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} y$$

$$\int_0^a V(0,y) \sin \frac{n'\pi}{a} y \, dy = \sum_{n=1}^{\infty} c_n \int_0^a \sin \frac{n\pi}{a} y \sin \frac{n'\pi}{a} y \, dy$$

R.H.S. $= 0 \quad n' \neq n$

$\qquad = \frac{a}{2} \quad n' = n$

$$L.H.S. = \int_0^{\frac{a}{2}} V_0 \sin \frac{n'\pi}{a} y\, dy + \int_{\frac{a}{2}}^{a} (-V_0) \sin \frac{n'\pi}{a} y\, dy$$

$$= \frac{2aV_0 \sin^2\left(\frac{n'\pi}{4}\right)}{n'\pi} +$$

$$- \frac{aV_0\left(\cos \frac{n'\pi}{2} - \cos n'\pi\right)}{n'\pi}$$

from Mathematica

$$= \frac{aV_0}{n'\pi}\left[2\sin^2 \frac{n'\pi}{4} - \cos\frac{n'\pi}{2} + \cos n'\pi\right]$$

See table from Mathematica

$$= \frac{aV_0}{n'\pi}\left[4\right] \qquad n' = 2, 6, 10, 14, 18, \ldots$$

$$= 0 \qquad \text{otherwise}$$

$$\frac{4aV_0}{n\pi} = \frac{a}{2} c_n \qquad n = 2, 6, 10, \ldots$$

$$c_n = \frac{8V_0}{n\pi}$$

$$= 0 \qquad \text{otherwise}$$

Get the sequence of integers

$n = 2, 6, 10, \ldots \rightarrow 4n-2$ for $n = 1, 2, 3, \ldots$

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \frac{8V_0}{(4n-2)\pi} \sin\left[\frac{(4n-2)}{a}\pi y\right] \exp\left[-\frac{(4n-2)}{a}\pi x\right]}$$

3-13) $$V(x,y) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi y}{a}\right) \times \exp\left(-\frac{(2n+1)\pi x}{a}\right)$$

$\sigma(y) = ?$ for $x = 0$

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \frac{\partial V}{\partial x}$$

$$\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{\cancel{2n+1}} \sin\left(\frac{(2n+1)\pi y}{a}\right) \times \left(-\frac{(\cancel{2n+1})\pi}{a}\right) \exp\left(-\frac{(2n+1)\pi x}{a}\right)$$

set $x = 0$ and

$$\sigma = +\frac{4V_0\varepsilon_0}{\cancel{\pi}} \sum_{n=0}^{\infty} \sin\left(\frac{(2n+1)\pi y}{a}\right)\left(+\frac{\cancel{\pi}}{a}\right)(1)$$

$$= \frac{4V_0\varepsilon_0}{a} \sum_{n=0}^{\infty} \sin\left(\frac{(2n+1)\pi y}{a}\right)$$

See plot

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3-17) a. $r < R$ $\quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

$$A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta$$

$r > R$ $\quad V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta$$

for $V(r,\theta) = V_0$

$$A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi V_0 P_\ell(\cos\theta) \sin\theta \, d\theta$$

$$P_0(\cos\theta) = 1$$

$$= \frac{2\ell+1}{2R^\ell} V_0 \int_0^\pi P_0 P_\ell(\cos\theta) \sin\theta \, d\theta$$

$$= \begin{cases} 0 & \ell \neq 0 \\ \frac{1}{2R^0} V_0 \, 2 = V_0 & \ell = 0 \end{cases}$$

$$\boxed{\therefore \text{ for } r < R \quad V(r,\theta) = A_0 r^0 P_0(\cos\theta) = V_0}$$

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi V_0 P_0 P_\ell(\cos\theta) \sin\theta \, d\theta$$

$$= \frac{2\ell+1}{2} R^{\ell+1} V_0 \int_0^\pi P_0 P_\ell \sin\theta \, d\theta$$

$$= \begin{cases} 0 & \ell \neq 0 \\ \frac{1}{2} R V_0 \, 2 = R V_0 & \ell = 0 \end{cases}$$

$$\boxed{\therefore \text{ for } r > R \quad V(r,\theta) = \frac{R V_0}{r}}$$

6. $\sigma(\theta) = \sigma_0$ $\qquad \sigma_0 = \varepsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta)$

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0 P_l(\cos\theta) \sin\theta \, d\theta$$

$$P_0 = 1$$

$$= \frac{1}{2\varepsilon_0 R^{l-1}} \sigma_0 \int_0^{\pi} P_0 P_l \sin\theta \, d\theta$$

$$= \begin{cases} 0 & l \neq 0 \\ \frac{1}{\cancel{2}\varepsilon_0 R^{-1}} \sigma_0 \cancel{2} = \frac{R\sigma_0}{\varepsilon_0} & l = 0 \end{cases}$$

for $r < R$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$= \frac{R\sigma_0}{\varepsilon_0} r^0 P_0$$

$$\boxed{V = \frac{R\sigma_0}{\varepsilon_0}}$$

for $r > R$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

at $r = R$ $\quad V_{inside} = V_{outside}$

$$\frac{R\sigma_0}{\varepsilon_0} = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

for this to be true only the $l=0$ term on the right hand side can be nonzero.

$$\frac{R\sigma_0}{\varepsilon_0} = \frac{B_0}{R}$$

$$B_0 = \frac{\sigma_0 R^2}{\varepsilon_0}$$

$$\boxed{\therefore\ V(r,\theta) = \frac{\sigma_0 R^2}{\varepsilon_0}\frac{1}{r}}$$