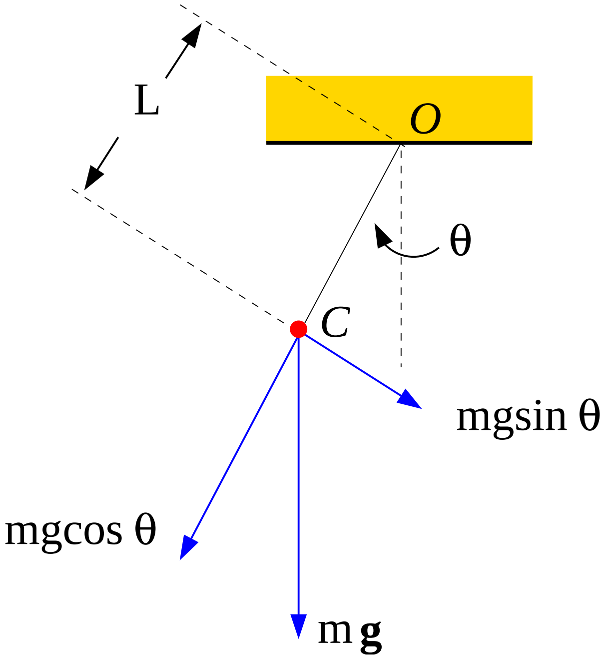
The Pendulum - Stating the Problem

The physics of the pendulum evokes a wide range of applications from circuits to ecology. We start with a simple pendulum consisting of a point mass on a negligibly light string.

1. Show the simple pendulum obey's Hooke's Law $F_s = -kx$ for small angles where F_s is the force exerted by a spring (the restoring force) and x is the displacement from the equilibrium point.
2. What differential equation does x satisfy?
3. What is the solution?
4. How would you prove the solution is correct?
5. What is the physical meaning of the constants in the solution?

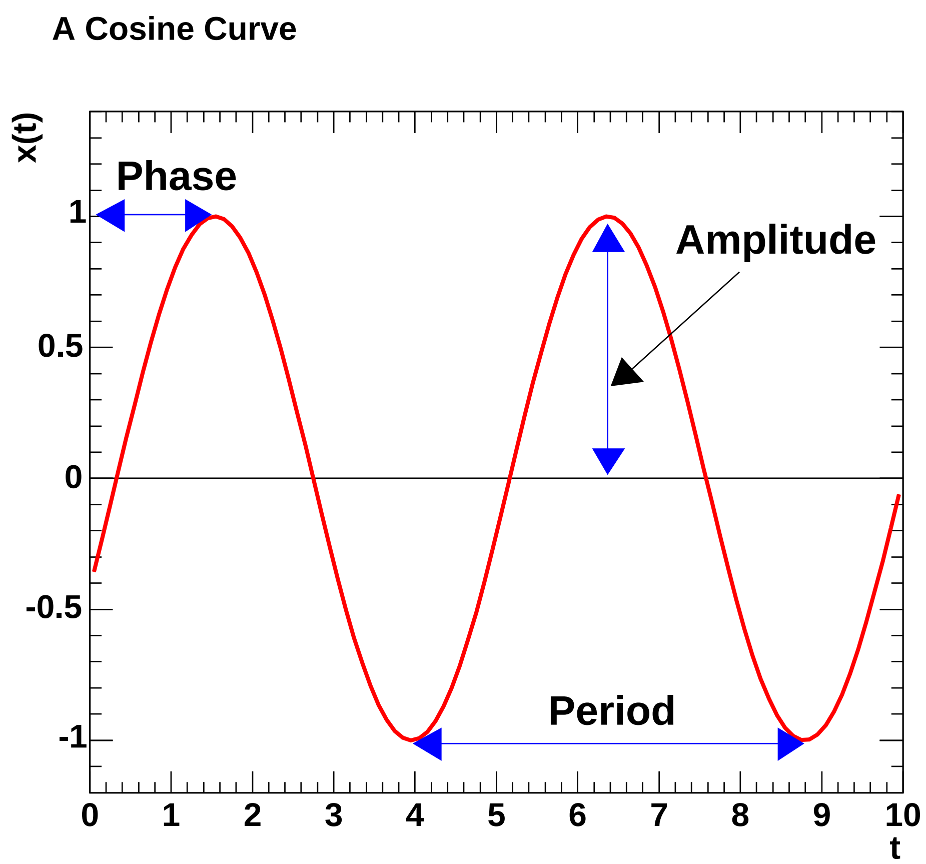


The Simple Pendulum - The Solution

The solution for Hooke's Law is

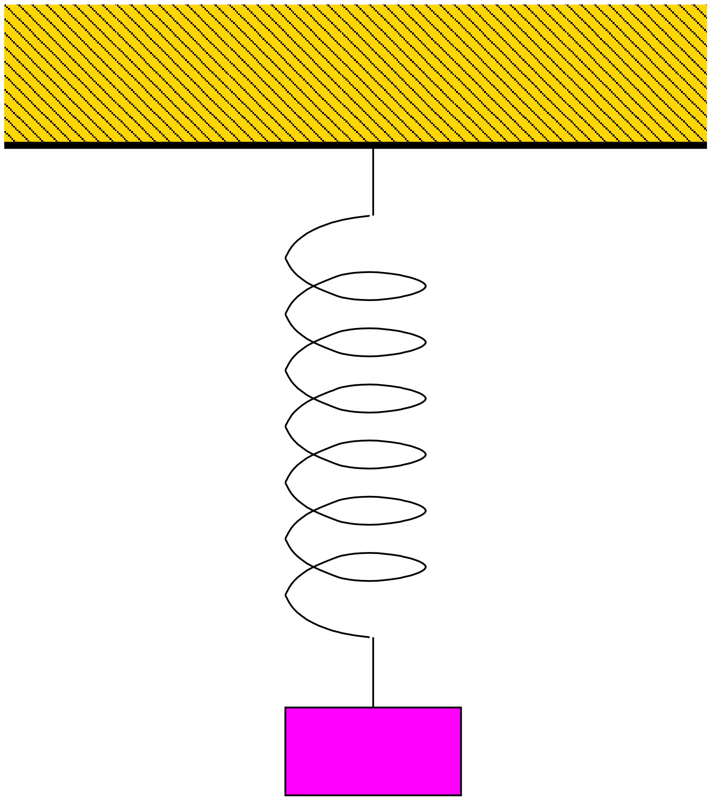
$$x(t) = A\cos(\omega t + \phi)$$

where $x(t)$ is the displacement from equilibrium.



The Simple Pendulum - An Example

A harmonic oscillator consists of a block of mass $m = 0.33\ kg$ attached to a spring with spring constant $k = 400\ N/m$. See the figure below. At time $t = 0.0\ s$ the block's displacement from equilibrium and its velocity are $x = 0.100\ m$ and $v = -13.6\ m/s$. Find the particular solution for this oscillator. Use a centered derivative formula to generate an algorithm for solving the equation of motion.



The Simple Pendulum with Friction

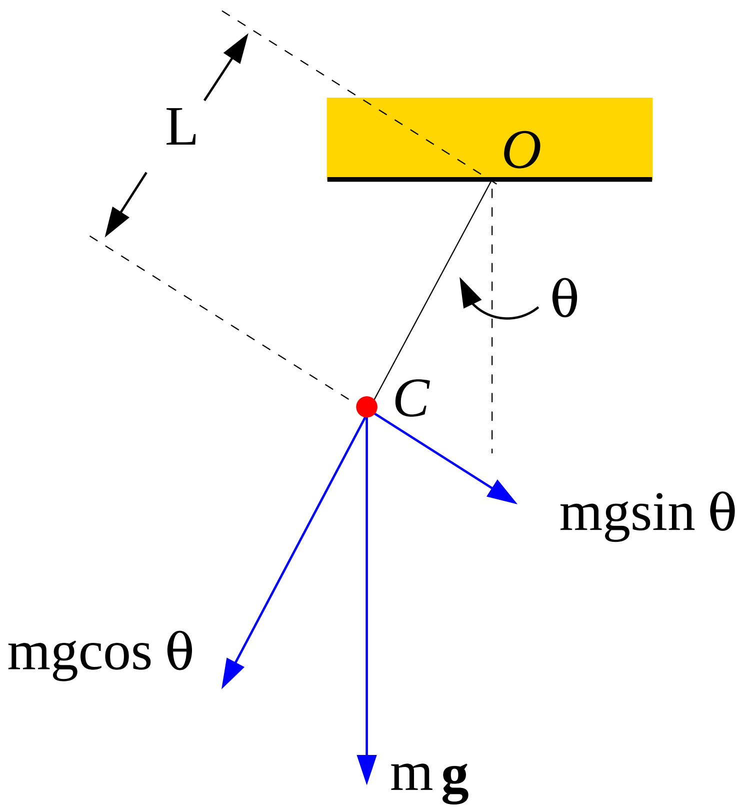
Consider the simple pendulum shown here. What is the differential equation describing the motion when the following forces are included in addition to gravity? For friction use

$$F_{friction} = -\frac{q}{L} v$$

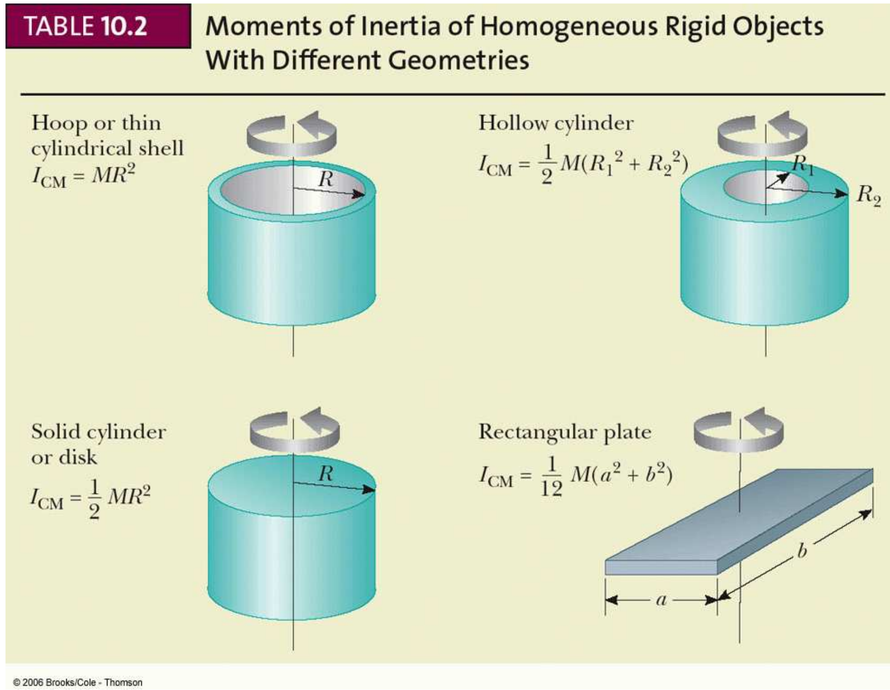
where q is a constant specific to a particular body. For the driving force use

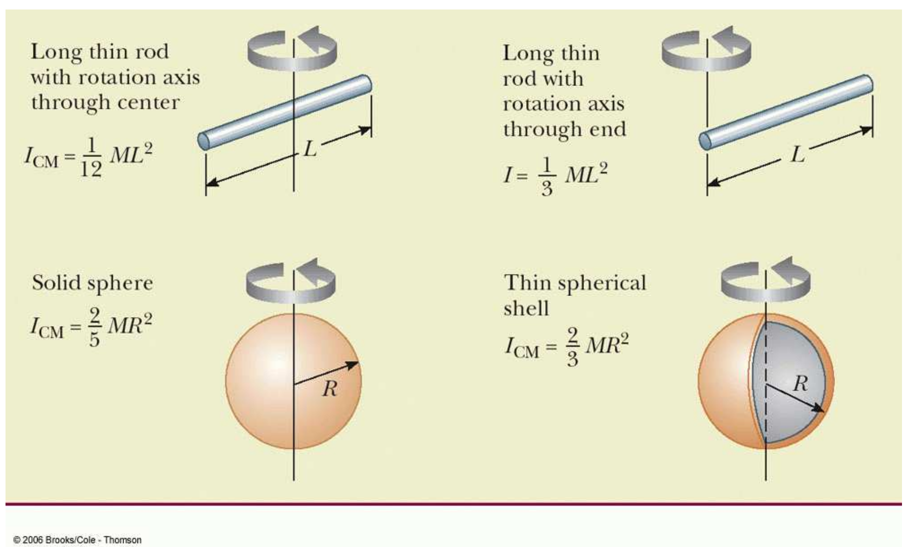
$$F_{driving} = F_D \sin(\Omega t)$$

where F_D is the magnitude of the driving force and Ω is its angular frequency.



Moments of Inertia





The Physical Pendulum

Consider the rod rotating about an end point in the figure. Starting from the definition of the torque

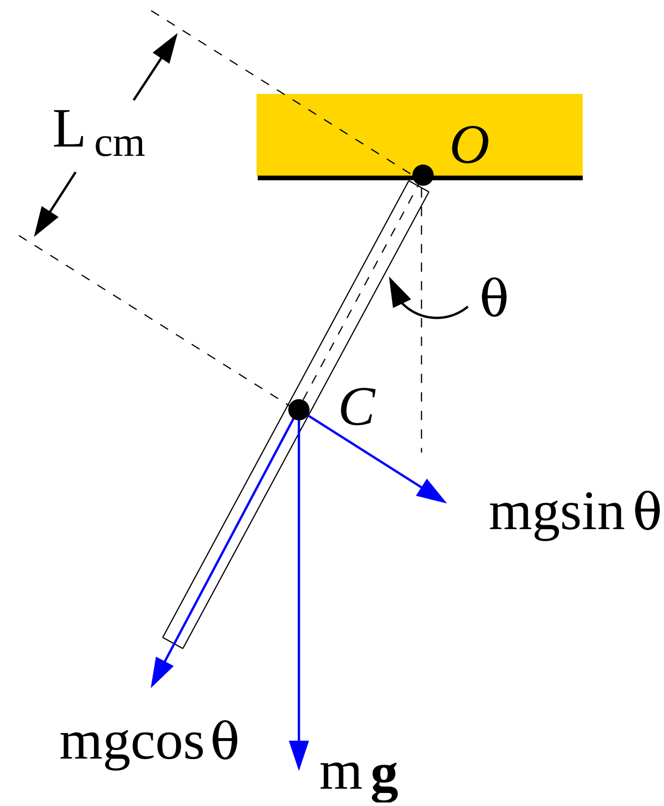
$$\vec{\tau} = \vec{r} \times \vec{F}$$

(1) derive the differential equation the angular position θ must satisfy.

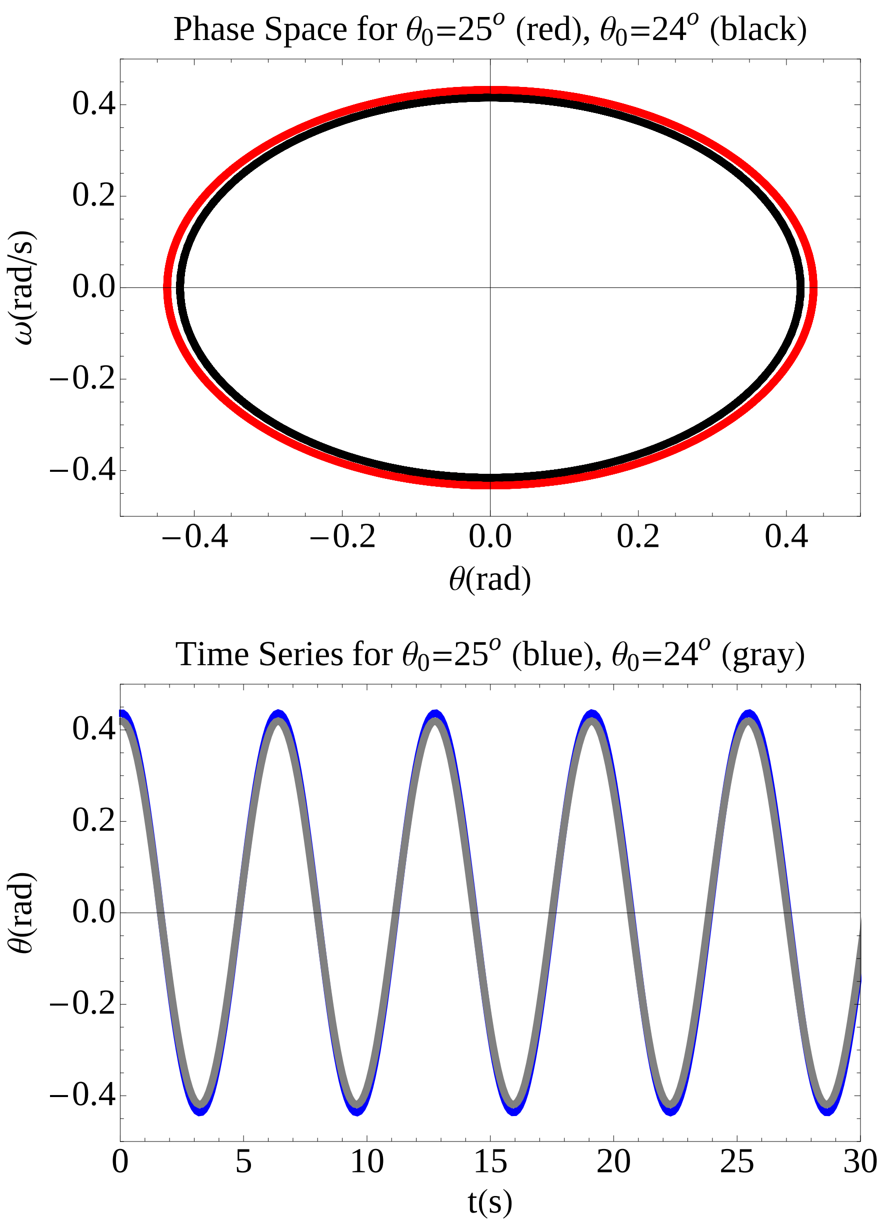
(2) Derive a new differential equation if the pendulum is damped by a friction force $\vec{F}_f = -D\vec{v}$ where b is some constant describing the the pendulum.

(3) Derive a final differential equation if the pendulum is now also driven by a force $\vec{F}_{drive} = F_D \sin(\Omega t)\hat{\theta}$.

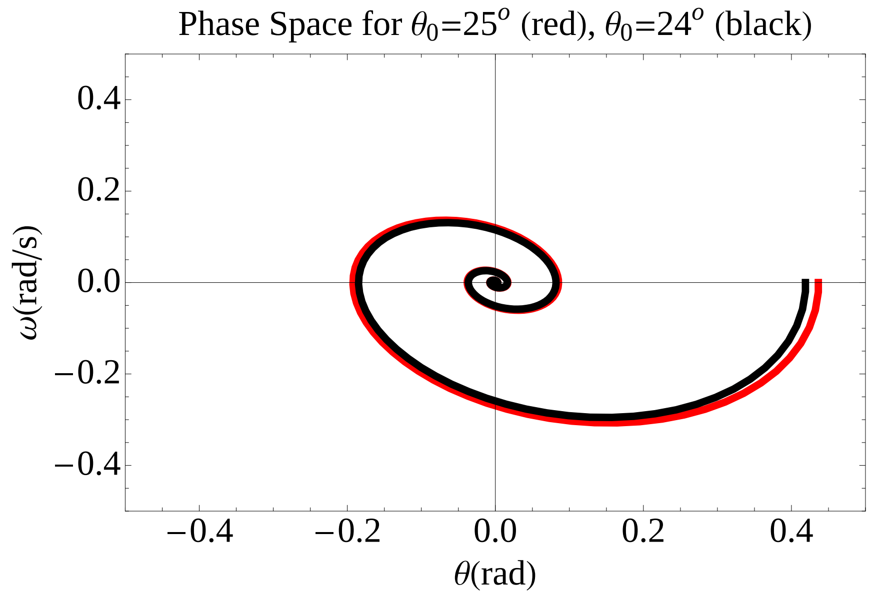
(4) What does the phase space look like for each set of conditions if the initial conditions are $\theta_0 = 25^\circ$ and $\omega_0 = 0\, rad/s$?

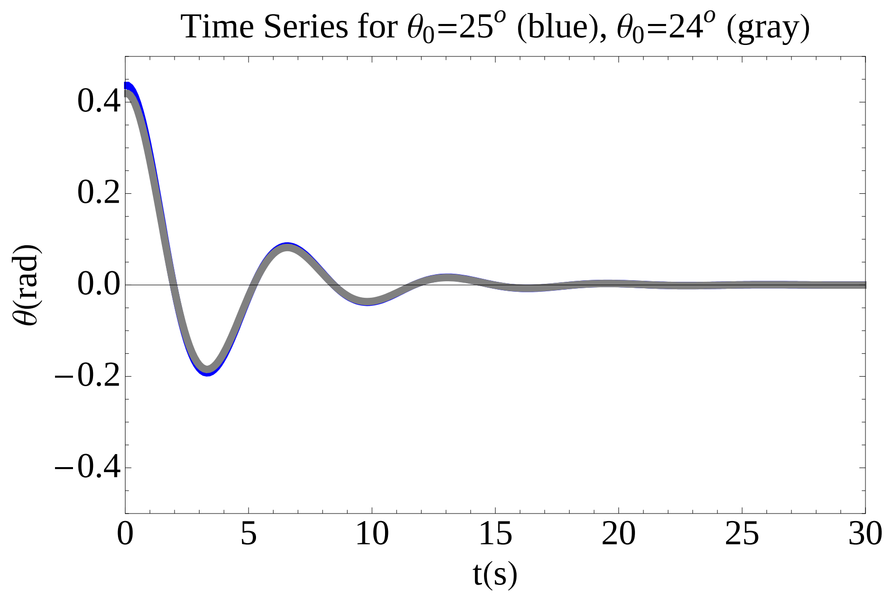


Nonlinear, Physical Pendulum Phase Space and Time Series

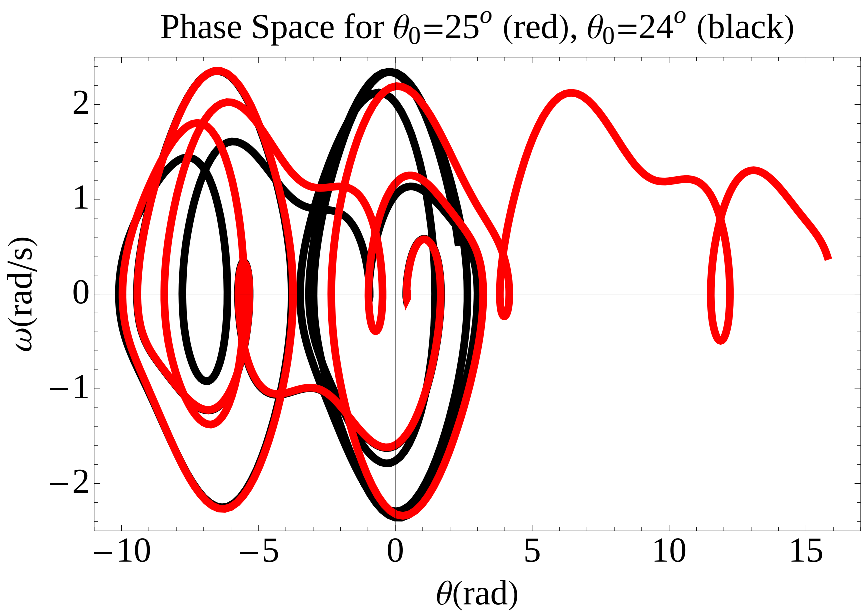


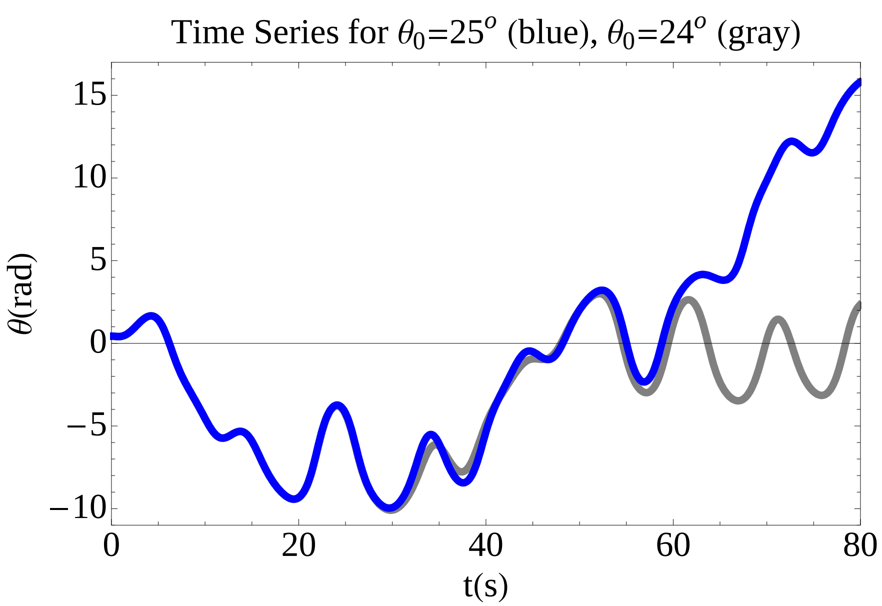
Nonlinear, Damped, Physical Pendulum Phase Space and Time Series





Nonlinear, Damped, Driven, Physical Pendulum Phase Space and Time Series



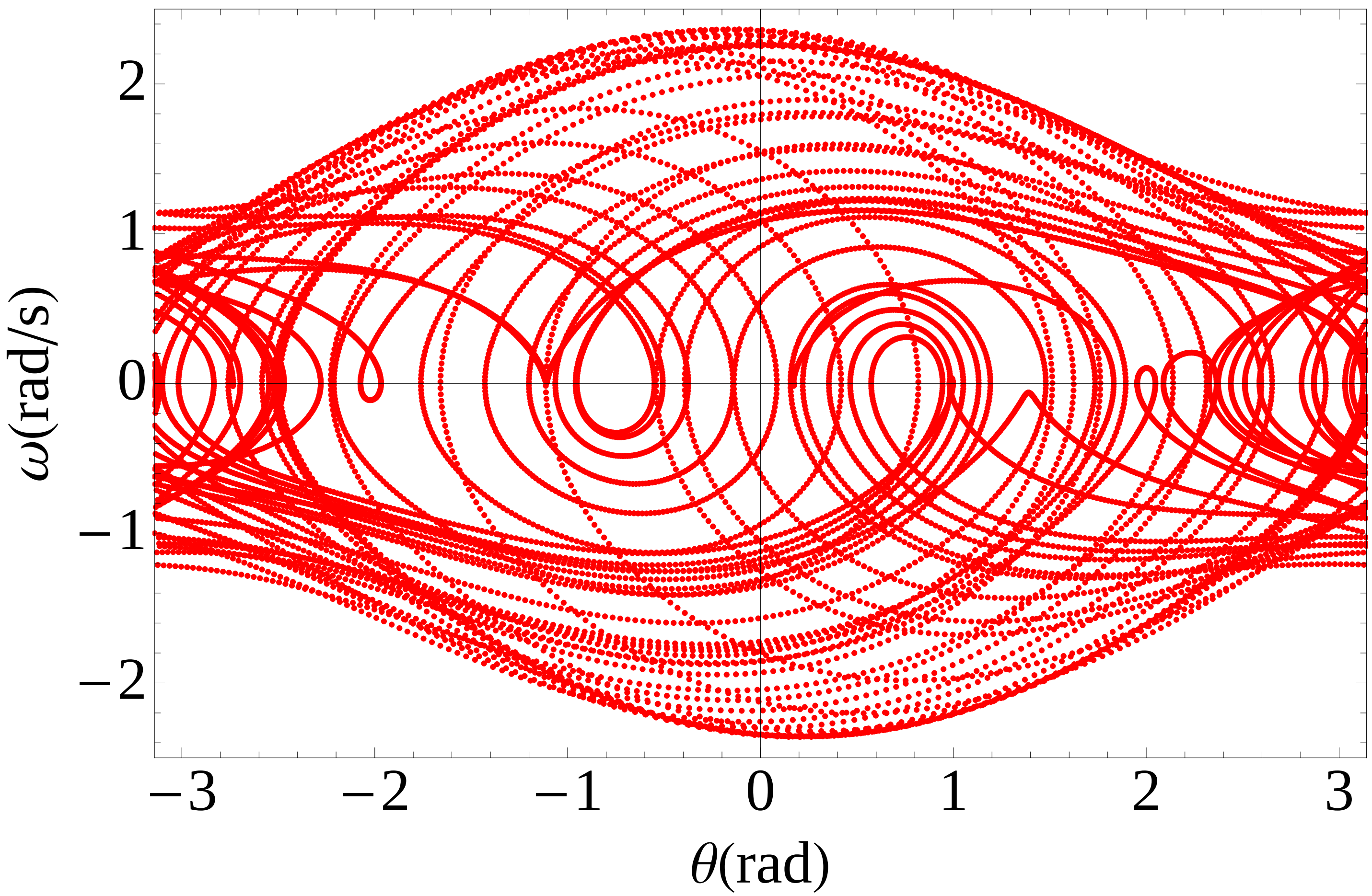


Code for Nonlinear, Damped, Driven, Physical Pendulum

```
(* Initial conditions and parameters *)
th0 =  25.0*Pi/180; (* initial position in meters *)
w0 = 0.0; (* initial velocity in m/s *)
t0 = 0.0; (* initial time in seconds *)
grav = 9.8; (* acceleration of gravity *)
length = 14.7; (* length of pendulum *)
mass = 0.245; (* mass of pendulum *)

(* driving force amplitude and friction force. See below for more *)
qDrag = 0.6; (* drag coefficient *)
DriveForce = 11.8; (* DriveForce = 11.8; cool plot value *)
DriveFreq = 0.67; (* driving force angular frequency *)
DrivePeriod = 2*Pi/DriveFreq; (* period of the driving force *)

(* step size *)
step = 0.10;

(* limits of the iterations. since we already have theta(t=0) and we \
have calculated theta(t=step) then the first value in the table will \
be for t=2*step. *)
tmin = 2*step;
tmax = 80.0;

(* condense the constants into coefficients for the appropriate terms. *)
f1 = 1 + (3*qDrag*step/(2*mass*length));
f2 = 3*DriveForce*(step^2)/(2*length);
f3 = -3*grav*(step^2)/(2*length);
f4 = -1 + (3*qDrag*step/(2*mass*length));
```

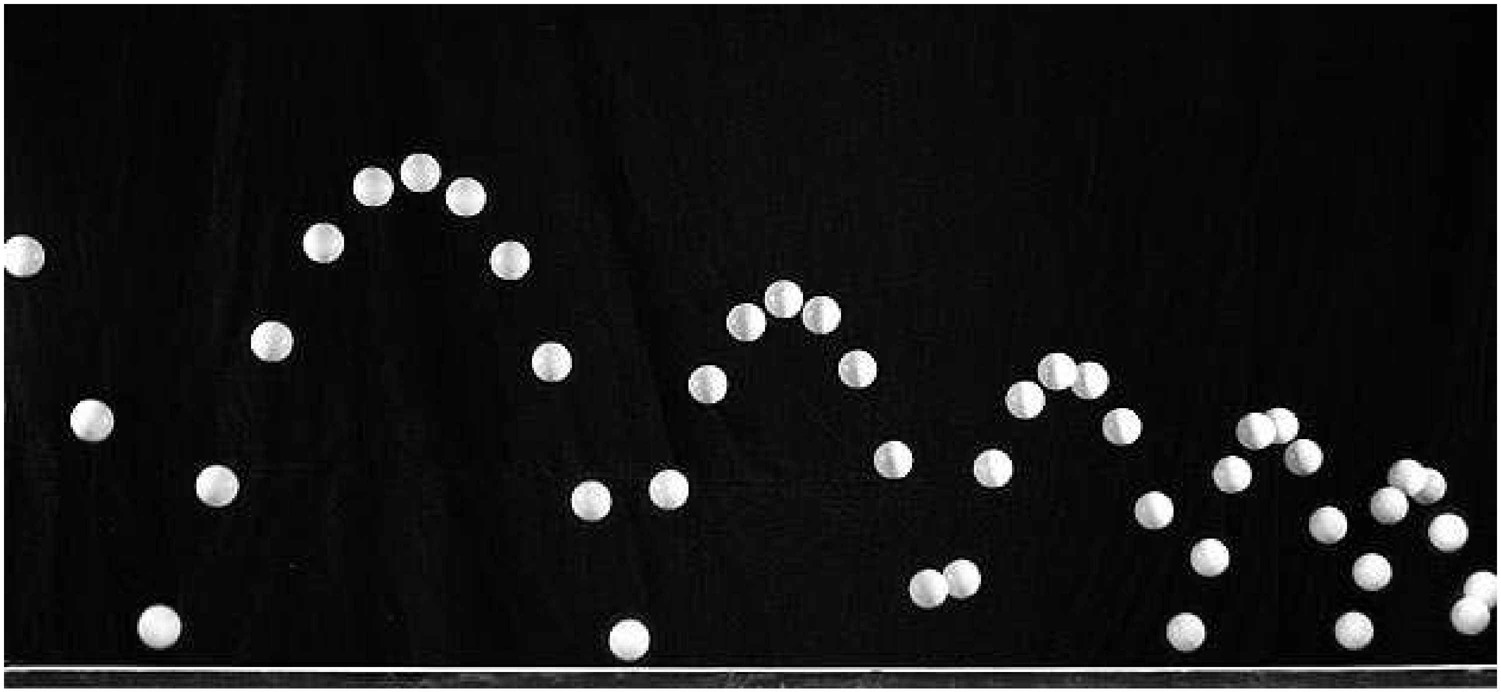
```
(* set up the first two points. *)
t1   = t0 + step;
th1 = th0 + w0*step;
(* get rid of the previous results for the table and proceed *)

Clear[pdispl]
Clear[tdispl]

(* A centered second derivative formula is used to generate a
iterative solution for the mass on a spring.
   first load the starting point values for the algorithm. *)
thmid  = th0;  (*starting value of theta *)
thplus = th1;   (* second value \
of theta *)
tmid = t0;

(* create a table of ordered (theta,w). for each component the next
value is calculated first and then the variables are incremented in
preparation for the next interation. *)
pdispl = {{th0, w0}};
tdispl = {{t0, th0}};
Do[thminus = thmid;
   thmid      = thplus;
  tmid = tmid + step;
  thplus   =  (f2*Sin[DriveFreq*t] +  2*thmid + f3*Sin[thmid] +
      f4*thminus)/f1; wmid = (thplus - thminus)/(2*step);
  pdispl = Append[pdispl, {thmid, wmid}] ;
  tdispl = Append[tdispl, {tmid, thmid}] ,
  {t, tmin, tmax, step}
  ];
```

Visualizing Chaos - The Phase Space Trajectory

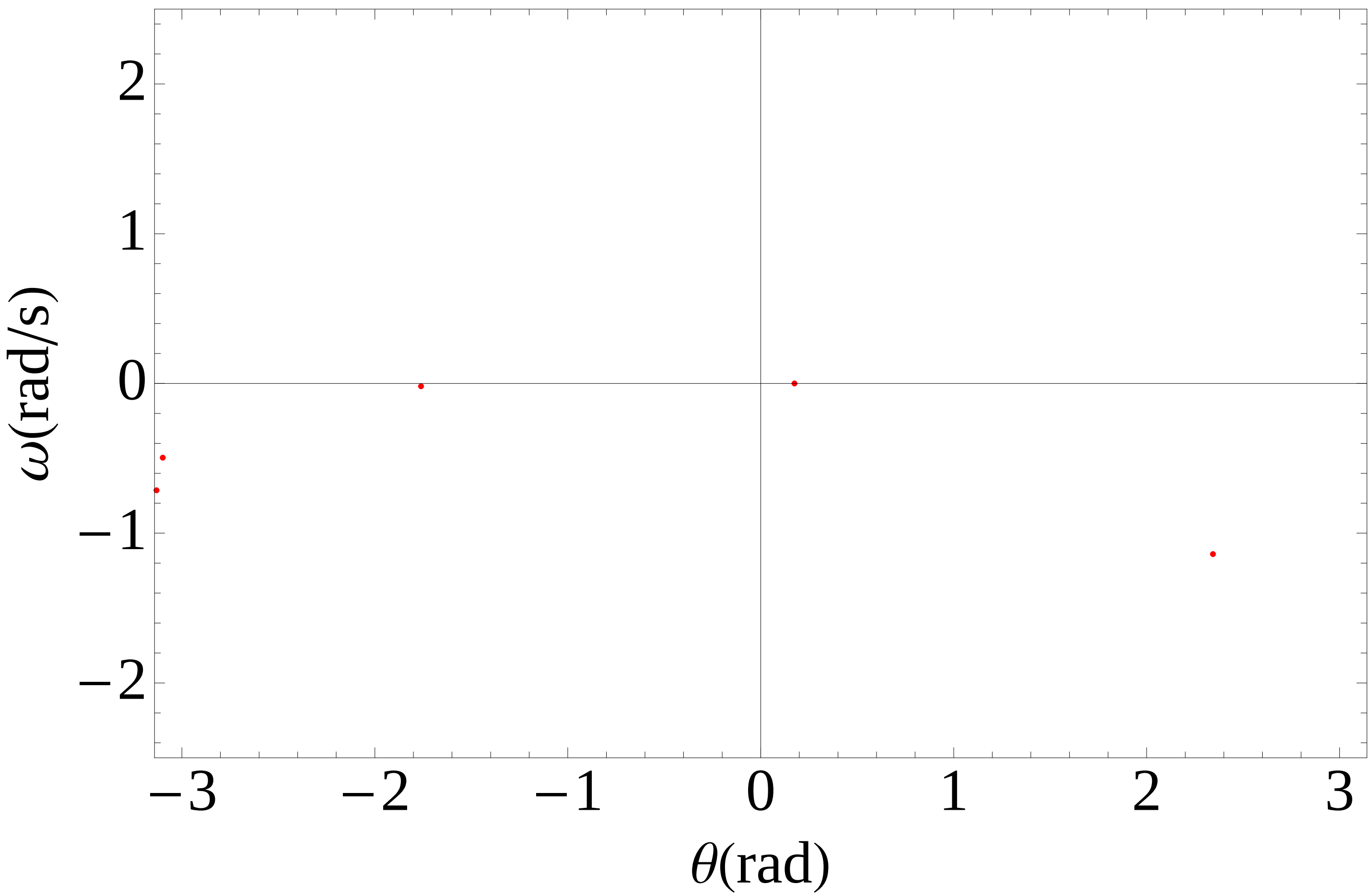


$$\theta_0 = 10^\circ$$

Visualizing Chaos - Stroboscopic Pictures

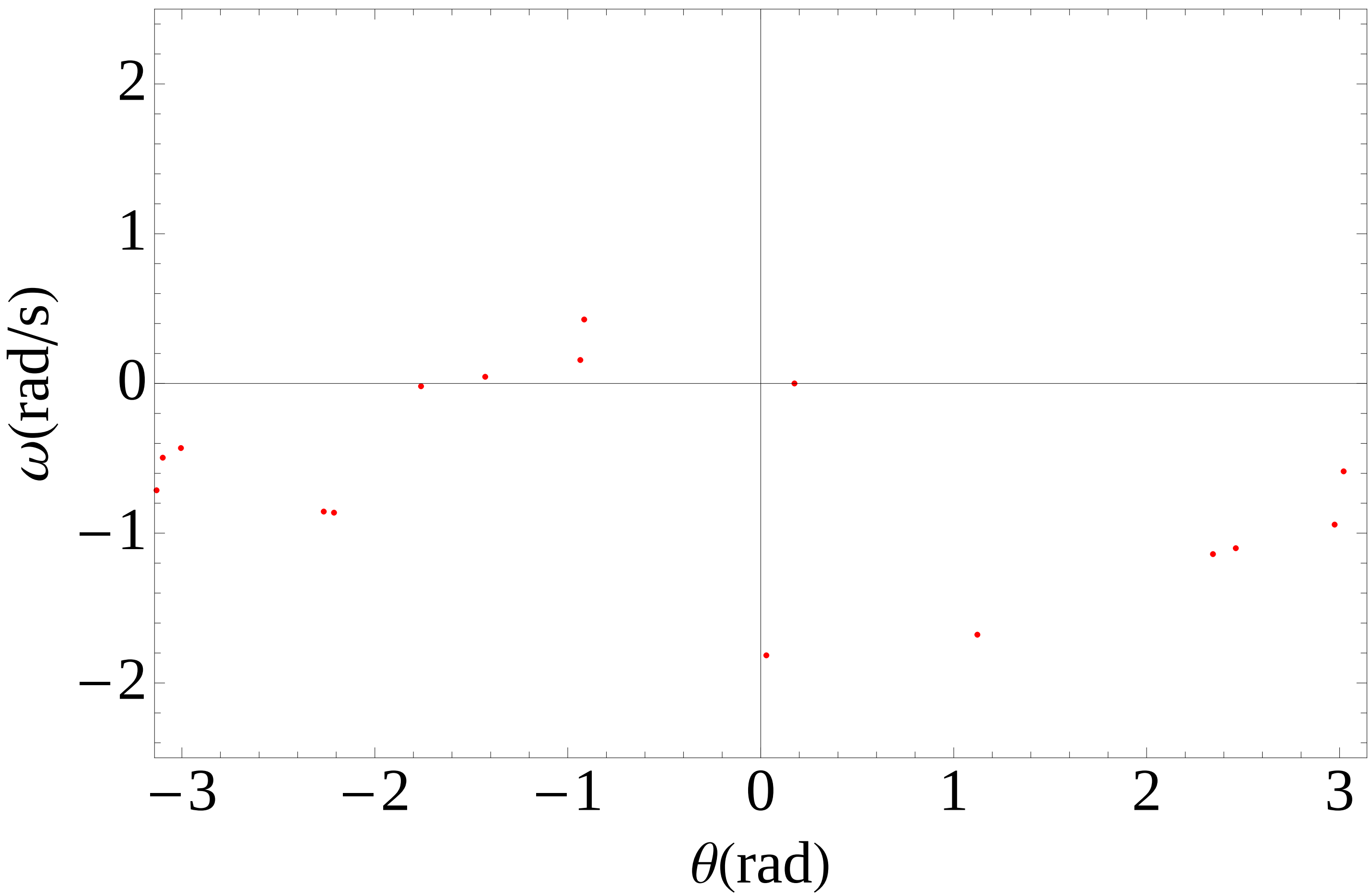


Visualizing Chaos - The Poincare Section



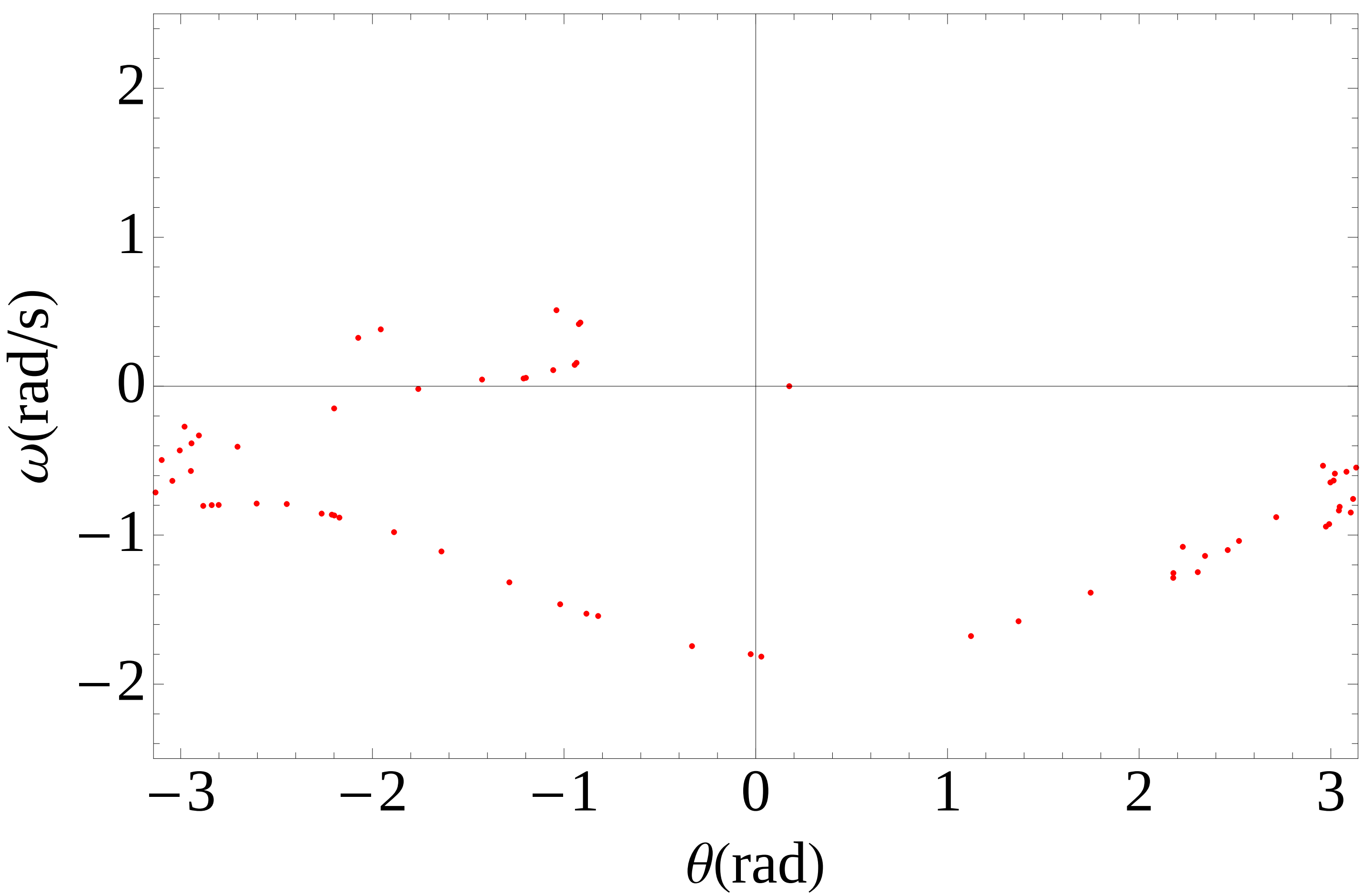
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



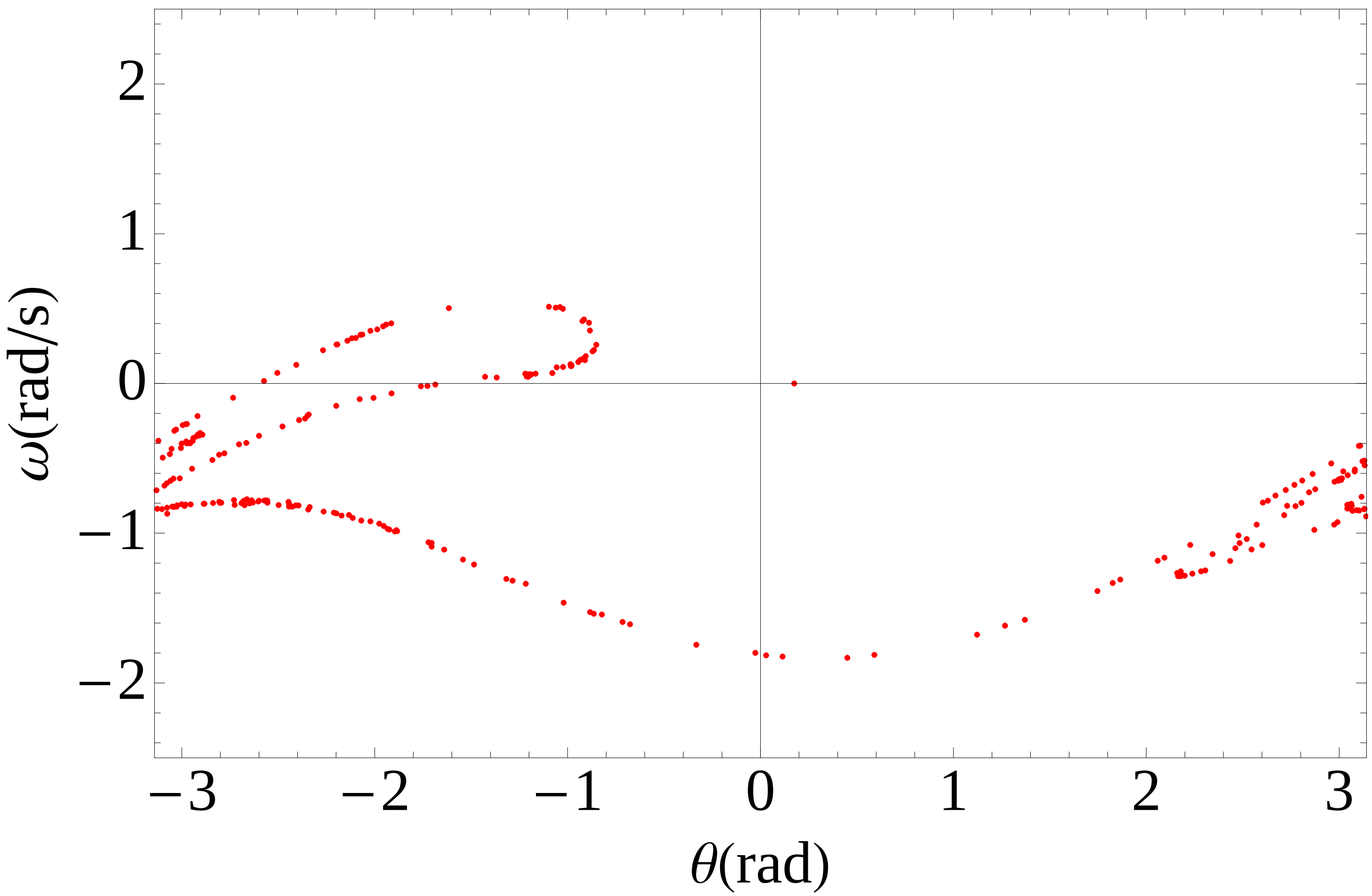
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



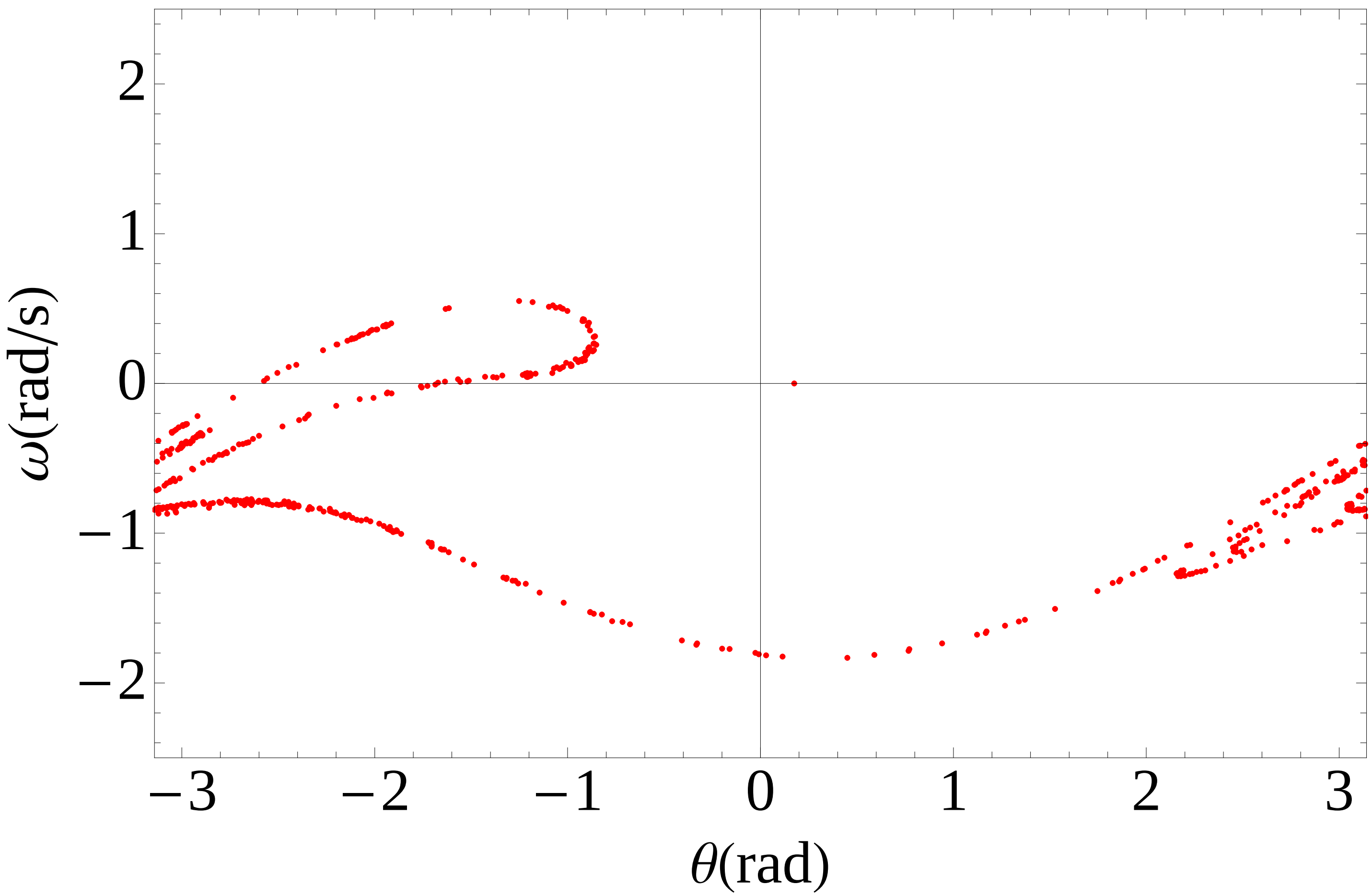
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



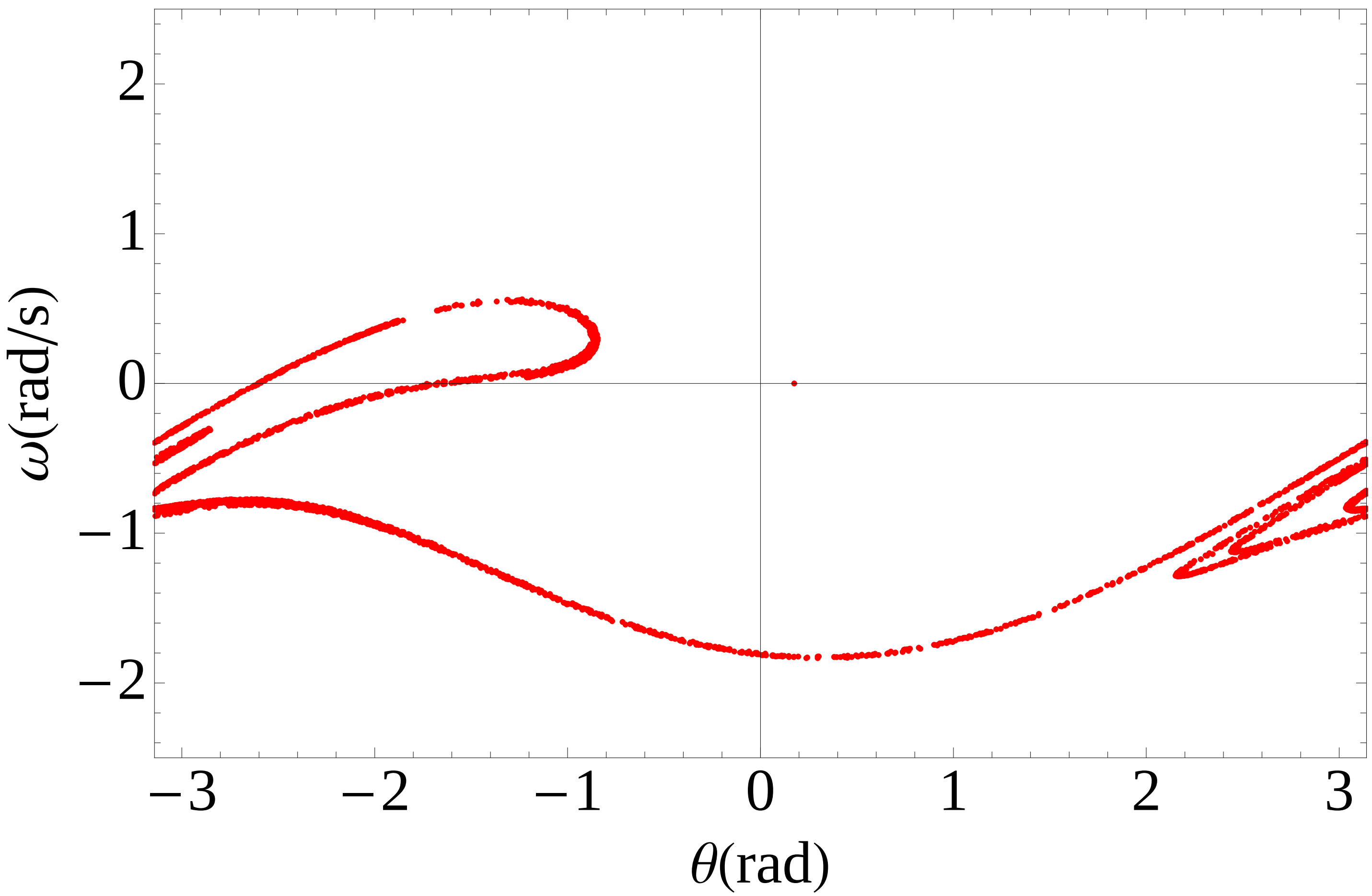
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



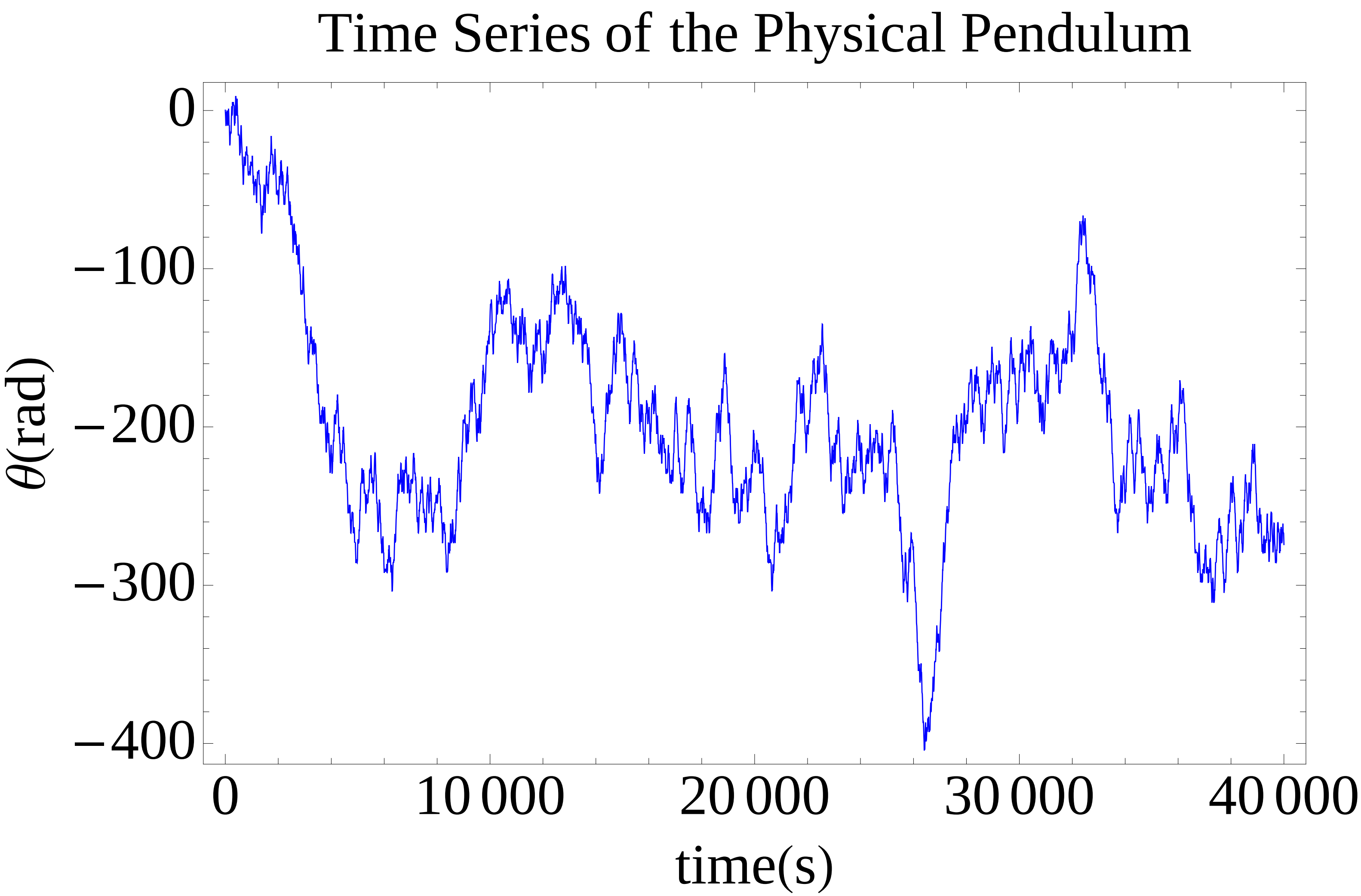
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Time Series



Calculating Chaos - The Poincare Series - 1

```
(* initial conditions and parameters *)
t0 = 0.0;
x0 = 1.0;
v0 = 0.2;
step = 0.01;

(* get the second and third points on the curve *)

t1 = t0 + step;
x1 = x0 + step*v0;
x2 = 2*x1 - x0 - (step*step*x1);
v1 = (x2 - x0)/(2*step);

(* put the first point in the table *)
MyTable = {{x0, v0}, {x1, v1}};

(* Use a Do loop and store the points when t = n\[Pi]. A centered formula is used to
approximate the second derivative. Set parameters needed to test when to store the data. *)
TimeTest = Pi;
PeriodCounter = 1;

(* first point of the algorithm *)
xminus = x0;
xmid = x1;
xplus = x2;
tmin = t1 + step;
tmax = 50.0;
```

Calculating Chaos - The Poincare Section - 2

```
Do[xminus = xmid;
       xmid = xplus;
       xplus = 2*xmid - xminus - (step*step*xmid);
       vmid = (xplus - xminus)/(2*step);
    If[t > TimeTest,
            MyTable = Append[MyTable, {xmid, vmid}];
            PeriodCounter = PeriodCounter + 1;
            TimeTest = PeriodCounter*2*Pi
          ],
       {t, tmin, tmax, step}
   ]
```