Physics 303 Final

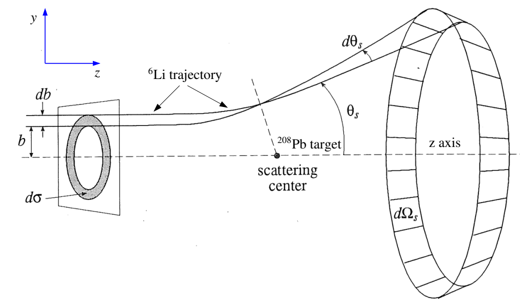
I pledge that I have neither given nor received unauthorized assistance during the completion of this work.

Signature ________________________________

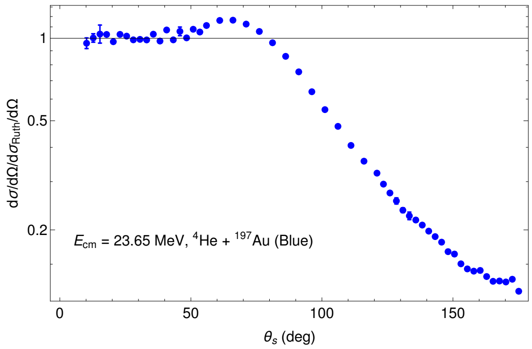
Questions (3 pts. apiece) Answer questions 1-8 in complete, well-written sentences WITHIN the spaces provided.

1. Can the method we used to discover extra-solar planets be applied to all solar systems? Explain.

2. In constructing the differential cross section we made the following definition $d\sigma = 2\pi b|db|$ where b is the impact parameter of the collision (see the figure below). Why?



3. The figure shows the ratio of the measured cross section to the Rutherford cross section as a function of scattering angle θ_s. What is the grazing angle θ_g? Explain your reasoning.

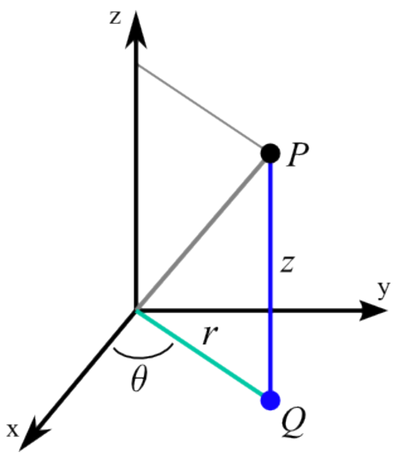


4. What is the evidence that an asteroid struck the Earth and killed all the dinosaurs?

5. Consider the Lagrangian for a two-dimensional harmonic oscillator which can be written in cylindrical coordinates r, θ, and z (see the figure) as the following

$$\mathscr{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + kr^2$$

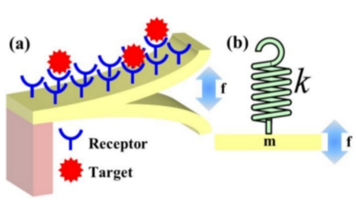
where k is the spring constant and m is the mass. From the structure of the Lagrangian what quantities would you expect to be conserved? Why?



6. What is the difference between the motion of a 'plucked' , overdamped oscillator and a 'plucked', underdamped oscillator?

7. An astronaut in the International Space Station is handed two balls with identical outward appearances (same size, material, color, *etc*). However, one is hollow while the other is filled with lead. How can the astronaut determine which is which? Cutting or altering the balls is not allowed nor is probing them with any form of radiation.

8. You have designed a cantilever-based biosensor to detect plague spores for first responders. See the figure. Your prototype has passed all the tests for operation in air when the buyer asks if it will work just as well in water. What do you say and why?



Problems (1-6). Clearly show all reasoning for full credit. Use a separate sheet to show your work.

1. 8 pts. Starting from the energy integral for central force motion

$$\int \frac{dr}{\sqrt{r^2(\frac{2\mu E}{l^2}r^2 + \frac{2\mu\alpha}{l^2}r - 1)}} = \theta + C$$

make the change of variable $u = 1/r$ to obtain a modified integral.

2. 12 pts. What are the eigenvalues and eigenvectors for the following matrix?

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$$

3. 12 pts. Jumping on the Earth an astronaut can reach a height $y_0 = 0.4\ m$. If she jumps on an asteroid of mass $m = 6.7 \times 10^{13}\ kg$ and radius $r = 2\ km$ with the same initial velocity, will she return?

4. 12 pts. The two atoms in a diatomic molecule with masses m_1 and m_2 interact through a potential

$$V(r) = \frac{a}{r^4} - \frac{b}{r^3}$$

where r is the inter-atomic separation. How much energy must be added to the molecule to break it apart when it is at the equilibrium separation?

5. 14 pts. Apply a direct integration to the following equation

$$v_y(t) = -v_t \tanh\left(\frac{gt}{v_t}\right)$$

where v_t is the terminal velocity of a falling object to get the general solution for $y(t)$ and then use the initial conditions that at $t = 0$, $y = y_0$ to find the particular solution.

6. 18 pts. The Nobel prizewinner Hideki Yukawa built a successful model of the strong force that binds nucleons (protons and neutrons) together into nuclei using the attractive potential energy

$$V(r) = -\frac{Ke^{-\alpha r}}{r} \qquad K > 0,\ \alpha > 0$$

where r is the nucleon separation.

   a. What is the force?

   b. What is the angular momentum l and total energy E for motion on a circle of radius a in terms of a, the nucleon mass m, and the parameters K and α?

   c. What is the period of circular motion in terms of the same quantities?

Physics 303 Final Equation Sheet

$$\vec{F} = m\vec{a} = \dot{\vec{p}} = -\frac{dV}{dr}\hat{r} \quad \vec{F}_G = -\frac{Gm_1m_2}{r^2}\hat{r} \quad \vec{F}_C = \frac{kq_1q_2}{r^2}\hat{r} \quad \vec{F}_g = -mg\hat{y} \quad \vec{F}_s = -kr\hat{r}$$

$$|\vec{F}_{cent}| = m\frac{v^2}{r} \quad \vec{F}_f = -bv\hat{v} \quad \int \frac{df}{dx} = \int df \quad \ddot{y} + A\dot{y} + By = 0 \Rightarrow y = Ce^{\lambda t}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \ddot{y} + \omega_0^2 y = 0 \Rightarrow y = A\sin(\omega_0 t + \phi) = \alpha_1 e^{i\omega_0 t} + \alpha_2 e^{-i\omega_0 t}$$

$$\ddot{y} + \omega_0^2 y = \omega_0^2 l \Rightarrow y = C + A\sin(\omega_0 t + \phi) \quad \omega_0^2 = \frac{k}{m} \quad \omega_0^2 = \frac{g}{l} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$V = -\int_{x_s}^{x} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' \quad E = K + V \quad V_s = \frac{kx^2}{2} \quad V_g = mgy \quad V_G = -\frac{Gm_1m_2}{r} \quad V_C = \frac{kq_1q_2}{r}$$

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \mathscr{L} = K - V \quad \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0 \quad (\mathbf{A} - \lambda\mathbf{1})\vec{C} = 0$$

$$\vec{L} = \vec{r} \times \vec{p} \quad \vec{N} = \frac{d\vec{L}}{dt} \quad l = \mu r^2\dot{\theta} \quad V_{cent} = \frac{L^2}{2\mu r^2} \quad \vec{p} = m\vec{v} \quad \vec{p}_i = \vec{p}_f \quad K_i = K_f \quad e = \frac{|\vec{v}_{2f} - \vec{v}_{1f}|}{|\vec{v}_{2i} - \vec{v}_{1i}|}$$

$$\frac{1}{r} = \frac{\mu\alpha}{l^2}\left(1 + \epsilon\cos(\theta - \theta_0)\right) \quad \epsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha^2}} \quad \sin\left(\frac{\theta_s}{2}\right) = \frac{1}{\epsilon} \quad \frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E_{cm}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta_s}{2}\right)}$$

$$\alpha = Gm_1m_2 \text{ or } -e^2 Z_1 Z_2 \qquad e^2 = \frac{\hbar c}{137} = 1.44\ MeV - fm \quad \vec{F}(r) = -\frac{GmM(r)}{r^2}\hat{r}$$

$$V_{eff}(r) = \frac{l^2}{2\mu r^2} + V(r) \quad \vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \vec{0} \quad v_T = r\dot{\theta} \quad r = \left[\frac{GT^2M_s}{4\pi^2}\right]^{1/3}$$

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_s} \quad r(\theta) = \frac{\lambda(1+\epsilon)}{(1 + \epsilon\cos(\theta - \theta_0))} \quad \lambda = \frac{l^2}{m\alpha}\frac{1}{1+\epsilon} \quad a = \frac{\lambda}{1-\epsilon} \quad b = \lambda\sqrt{\frac{1+\epsilon}{1-\epsilon}}$$

$$\sin A = \cos(A - \pi/2) \quad \cos A = -\sin(A - \pi/2) \quad \arccos(-x) = \pi - \arccos(x)$$

More Equations, Conversions, and Constants

Speed of light (c)	$3.0 \times 10^8 \ m/s$	g	$9.8 \ m/s^2$
Gravitation constant (G)	$6.67 \times 10^{-11} \ N - m^2/kg^2$	Earth's radius	$6.37 \times 10^6 \ m$
Coulomb constant (k_e)	$8.99 \times 10^9 \frac{N-m^2}{C^2}$	Earth's mass	$5.97 \times 10^{24} \ kg$
Elementary charge (e)	$1.60 \times 10^{-19} \ C$	Proton/Neutron mass	$1.67 \times 10^{-27} \ kg$
Planck's constant (h)	$6.626 \times 10^{-34} \ J - s$	Proton/Neutron mass	$932 \times 10^6 \ eV/c^2$
Permittivity constant (ϵ_0)	$8.85 \times 10^{-12} \frac{kg^2}{N-m^2}$	Electron mass	$9.11 \times 10^{-31} \ kg$
Permeability constant (μ_0)	$4\pi \times 10^{-7} N/A^2$	Electron mass	$0.55 \times 10^6 \ MeV/c^2$
1 MeV	$10^6 \ eV$	1.0 eV	$1.6 \times 10^{-19} \ J$
1 kg	$931.5 \ MeV/c^2$	1 u	$1.67 \times 10^{-27} \ kg$

$$\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) \qquad \int \tanh x dx = \ln\left[\cosh x\right] \qquad \int \coth x dx = \ln[\sinh x]$$

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \qquad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \frac{x}{a} \qquad \frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \sec x = \sec x \tan x \qquad \frac{d}{dx} \ln ax = \frac{1}{x} \qquad \frac{d}{dx} \cot x = -\frac{1}{\sin^2 x}$$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \qquad \int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2} \ln\left[x + \sqrt{x^2 - a^2}\right]$$

$$\int \tanh^2(x) dx = x - \tanh x \qquad \int \tanh^3(x) dx = \ln\left[\cosh x\right] + \frac{\operatorname{sech}^2(x)}{2}$$

$$\int \sqrt{\tanh x} dx = -\tan^{-1}\left[\sqrt{\tanh x}\right] - \frac{1}{2} \ln\left[1 - \sqrt{\tanh x}\right] + \frac{1}{2} \ln\left[1 + \sqrt{\tanh x}\right]$$

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx} \coth x = -\operatorname{csch}^2 x \qquad \frac{d}{dx} \sinh x = \cosh x$$

$$\int \frac{1}{\sqrt{ar^4 + br^3 - r^2}} dr = \frac{r\sqrt{-1 + br + ar^2} \arctan\left(\frac{-2+br}{2\sqrt{-1+br+ar^2}}\right)}{\sqrt{r^2(-1 + br + ar^2)}}$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = -\frac{1}{\sqrt{-a}} \arcsin\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right)$$

$$\int \frac{1}{ax^2 + bx + c} dx = \frac{2}{\sqrt{4ac - b^2}} \arctan\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Pythagorean identities :

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Reciprocal identities :

$\csc x = \frac{1}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x}$

Co - function identities :

$\cos\left(\frac{\pi}{2} - x\right) = \sin x$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x$

$\cot\left(\frac{\pi}{2} - x\right) = \tan x$

$\csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\sec\left(\frac{\pi}{2} - x\right) = \csc x$

Periodicity identities :

$\sin(x \pm 2\pi) = \sin x$

$\cos(x \pm 2\pi) = \cos x$

$\tan(x \pm \pi) = \tan x$

$\cot(x \pm \pi) = \cot x$

$\sec(x \pm 2\pi) = \sec x$

$\csc(x \pm 2\pi) = \csc x$

Even - odd identities :

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

Sum and difference formulas :

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Product to sum formulas :

$\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

$\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$

$\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$

Half - angle formulas :

$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan\left(\frac{x}{2}\right) = \frac{(1-\cos x)}{\sin x}$

Sum to product :

$\sin x \pm \sin y = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Law of sines :

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines :

$a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$

Double - angle formulas :

$\sin 2\theta = 2 \cdot \sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Area of triangle :

$\frac{1}{2}ab\sin C$

$\sqrt{s(s-a)(s-b)(s-c)},$

where $s = \frac{1}{2}(a+b+c)$

- Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = $$
$$= \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

- Hyperbolic cotangent: $x \neq 0$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = $$
$$= \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}$$

- Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = $$
$$= \frac{2e^x}{e^{2x} + 1} = \frac{2e^{-x}}{1 + e^{-2x}}$$

- Hyperbolic cosecant: $x \neq 0$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = $$
$$= \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$
$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right); x \geq 1$$
$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); |x| < 1$$
$$\operatorname{arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right); |x| > 1$$

Odd and even functions:

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$

Hence:

$$\tanh(-x) = -\tanh x$$
$$\coth(-x) = -\coth x$$
$$\operatorname{sech}(-x) = \operatorname{sech} x$$
$$\operatorname{csch}(-x) = -\operatorname{csch} x$$