Chapter 3

LAGRANGIAN METHOD

The form that Newton's equations of motion take depends on the coordinate system used. For instance, the equations in a polar system are different from those in a cartesian system. The Lagrangian method is a reformulation which makes it simple to write the equations of motion in any coordinate system. In addition, it provides a straightforward and systematic way to handle constraints and to identify conserved quantities. The Lagrangian method allows an attack on many problems whose equations of motion would not otherwise be easy to find. Lagrange's equations (and the related Hamilton's equations) are of fundamental importance in classical mechanics and quantum mechanics.

3.1 Lagrange Equations

For a system consisting of N particles moving in three dimensions a total of $3N$ cartesian coordinates are required. The first particle's coordinates are labeled $\mathbf{r}_1$, the second $\mathbf{r}_2$, and so on up to $\mathbf{r}_N$. There is a Newton's equation for each of these coordinates. As a first step in the Lagrange approach we choose a new set of coordinates $q_1, q_2, \cdots q_{3N}$ called *general coordinates*, collectively denoted by $\{q_j\}$, to describe the configuration of the system. These coordinates do not necessarily have the dimensions of distance; in fact they are often angles. Newton's equations can be expressed in terms of the new coordinates by everywhere substituting for each cartesian coordinate its expression in terms of the new coordinates. These expressions relate the values of the new coordinates to the corresponding cartesian coordinate values which describe the same configuration of the system.

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1(q_1, q_2, \cdots q_{3N};\ t) \\ \mathbf{r}_2 &= \mathbf{r}_2(q_1, q_2, \cdots q_{3N};\ t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \cdots q_{3N};\ t) \end{aligned} \tag{3.1}$$

Note that the expressions (or coordinate transformations) may be different at different times. We use the common physics shorthand notation that the same symbol $\mathbf{r}$ is used for the function $\mathbf{r}(q)$ and its value $\mathbf{r}$. A simple specific example of the transformation (3.1) might be the choice of spherical polar coordinates (r, θ, ϕ) as general coordinates. For one particle, (3.1) becomes

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{3.2}$$

If the particle moves on the surface of a sphere of radius ℓ centered at the origin we may set $r = \ell$ in (3.2) and only two of the general coordinates, θ and ϕ, will vary in time. A relation of this type is called a *constraint.*

The equations of motion which result directly from the substitutions of (3.1) in Newton's equations are usually a mess. A much nicer set of equations, both because they exhibit explicitly the simplifications of symmetries and constraints, and because they are easier to write down, are Lagrange's equations. They are not the same as Newton's but are equivalent; in fact, each Lagrange equation is a linear combination of Newton's equations, and *vice versa.*

3.2 Lagrange's Equations in One Dimension

We will first derive the Lagrange equation for one particle moving in one dimension. With this as a guide we can then extend the derivation to a system with an arbitrary number of degrees of freedom. The derivation is purely mathematical and involves formal manipulations with partial and total derivatives.

We introduce a general coordinate $q(t)$ expressed in terms of x by

$$q(t) = q[x(t),\, t] \tag{3.3}$$

or inversely, as in (3.1),

$$x(t) = x[q(t),\, t] \tag{3.4}$$

An explicit dependence on t in the transformation allows for the possibility that the q- and x-coordinates are related differently at different times. The velocity $\dot{x} = dx/dt$ can be expressed in terms of the general velocity

$\dot{q}$ by chain differentiation

$$\dot{x} = \frac{\partial x}{\partial q}\dot{q} + \frac{\partial x}{\partial t} \tag{3.5}$$

where $\partial x/\partial q$ and $\partial x/\partial t$ are by (3.4) functions of q and t.

The momentum $p = m\dot{x}$ of the particle can be written in terms of the kinetic energy $K(\dot{x}) = \frac{1}{2}m\dot{x}^2$ as

$$p = \frac{dK}{d\dot{x}} \tag{3.6}$$

We introduce a new momentum $\mathfrak{p}(t)$ called the *general momentum* by a formula analogous to (3.6)

$$\mathfrak{p}(t) = \frac{\partial K}{\partial \dot{q}}(\dot{q}, q, t) \tag{3.7}$$

where $K(q, \dot{q}, t)$ means $\frac{1}{2}m\dot{x}^2$ with $\dot{x}$ expressed by (3.5). By chain differentiation the general momentum $\mathfrak{p}$ is related to the ordinary momentum p

$$\mathfrak{p} = \frac{dK}{d\dot{x}}\frac{\partial \dot{x}}{\partial \dot{q}} = p\frac{\partial \dot{x}}{\partial \dot{q}} \tag{3.8}$$

By use of (3.4) and (3.5), the partial derivative $\partial \dot{x}/\partial \dot{q}$ (q held fixed) simplifies to

$$\frac{\partial \dot{x}}{\partial \dot{q}} = \frac{\partial x}{\partial q} \tag{3.9}$$

Therefore we have

$$\mathfrak{p} = p\frac{\partial x}{\partial q} \tag{3.10}$$

Newton's equation of motion is

$$\dot{p} = F(x, \dot{x}, t) \tag{3.11}$$

The corresponding Lagrange equation of motion has $\dot{\mathfrak{p}}$ instead of $\dot{p}$ on the left-hand side; it is derived by differentiating both sides of (3.10) with respect to t

$$\dot{\mathfrak{p}} = \dot{p}\frac{\partial x}{\partial q} + p\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) \tag{3.12}$$

To simplify the second term, we interchange the order of differentiation,

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = \frac{\partial \dot{x}}{\partial q} \tag{3.13}$$

We briefly digress to justify the result in (3.13). Proceeding just as in Eq. (3.5), the total time derivative of $\partial x/\partial q$ is

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial x}{\partial q}\right)\dot{q} + \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial q}\right) \tag{3.14}$$

An important point must now be addressed. In the Lagrangian formalism we regard q and $\dot{q}$ as independent variables in the sense that $\partial \dot{q}/\partial q \equiv 0$. Other quantities such as the general momentum defined in (3.7) are *derived* since they ultimately depend on $q, \dot{q}$ and t. Differentiating (3.5) with respect to q (and treating $\dot{q}$ as an independent variable) we obtain

$$\frac{\partial \dot{x}}{\partial q} = \frac{\partial}{\partial q}\left(\frac{\partial x}{\partial q}\right)\dot{q} + \frac{\partial}{\partial q}\left(\frac{\partial x}{\partial t}\right) \tag{3.15}$$

Since the right-hand sides of (3.14) and (3.15) are identical, (3.13) follows.

Returning to the derivation, we multiply (3.13) by p and replace p on the right-hand side by the expression (3.6) to obtain

$$p\frac{d}{dt}\left(\frac{\partial x}{\partial q}\right) = \frac{dK}{d\dot{x}}\frac{\partial \dot{x}}{\partial q} = \frac{\partial K}{\partial q} \tag{3.16}$$

The substitution of (3.11) and (3.16) into (3.12) yields the following equation of motion in the q-coordinate system

$$\dot{\mathfrak{p}} = F\frac{\partial x}{\partial q} + \frac{\partial K}{\partial q} \tag{3.17}$$

The first term on the right-hand side is called the *general force*

$$\mathcal{Q}(\dot{q}, q, t) = F\frac{\partial x}{\partial q} \tag{3.18}$$

Then the equation of motion

$$\dot{\mathfrak{p}} = \mathcal{Q} + \frac{\partial K}{\partial q} \tag{3.19}$$

is of universal form for an arbitrary choice of coordinate q. The term $\partial K/\partial q$ in this equation represents a "fictitious" force which appears whenever the coefficients $\partial x/\partial q$ or $\partial x/\partial t$ in (3.5) vary with q.

If the force F is separated into a part $-dV(x)/dx$ which is derived from a potential energy, and a part F' which cannot be expressed (or which we do not choose to express) in terms of a potential energy, the general force can be separated into corresponding parts.

$$Q = -\frac{dV(x)}{dx}\frac{\partial x}{\partial q} + F'\frac{\partial x}{\partial q} = -\frac{\partial V(q)}{\partial q} + Q' \tag{3.20}$$

What we mean by $V(q)$ is the quantity at each point that is the same as $V(x)$, that is, $V[x(q)]$. For simplicity, we use the notation $V(q)$, although it is not the same *function* as $V(x)$. Notice that the potential (conservative) part of the general force has the same form, $-\partial V/\partial q$, as the conservative cartesian force $-\partial V/\partial x$.

If (3.20) for Q is substituted into (3.19) the terms $\partial K/\partial q$ and $-\partial V/\partial q$ can be combined, giving the *Lagrange equation of motion*

$$\dot{p} = \frac{\partial L}{\partial q} + Q' \tag{3.21}$$

where

$$L(q,\, \dot{q},\, t) \equiv K(q,\, \dot{q},\, t) - V(q) \tag{3.22}$$

is the *Lagrangian* function. Since

$$p = \frac{\partial K}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} \tag{3.23}$$

follows from (3.7) and $\partial V(q)/\partial \dot{q} = 0$, the Lagrange equation of motion (3.21) can also be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q' \tag{3.24}$$

The general force Q' must include all forces F' on the particle which are not included in the potential energy.

3.3 Lagrange's Equations in Several Dimensions

The derivation of the Lagrange equation in § 3.2 was for the motion of one particle in one dimension. The generalization to the motion of N particles in three dimensions is made by repeating, step by step, the derivation in the one-dimenensional case.

There are now $3N$ cartesian components x_k and likewise $3N$ general coordinates q_j. In analogy to (3.6) the k^{th} cartesian momentum is $p_k = \partial K/\partial \dot{x}_k$, and from (3.8) and (3.10) the general momentum can be written as

$$p_j = \frac{\partial K}{\partial \dot{q}_j} = \frac{\partial K}{\partial \dot{x}_k}\frac{\partial \dot{x}_k}{\partial \dot{q}_j} = p_k\frac{\partial x_k}{\partial q_j} \tag{3.25}$$

where the generalization of (3.9), the identity $\partial \dot{x}_k/\partial \dot{q}_j = \partial x_k/\partial q_j$, has been used. In (3.25) and subsequent equations a summation over repeated indices (in this case k) is implied. As in (3.12) the time derivative of the general momentum is

$$\dot{p}_j = \dot{p}_k\frac{\partial x_k}{\partial q_j} + p_k\frac{\partial \dot{x}_k}{\partial q_j} \tag{3.26}$$

In parallel to the derivation in one dimension we find

$$\begin{aligned}\dot{p}_j &= \left(-\frac{\partial V}{\partial x_k} + F_k'\right)\frac{\partial x_k}{\partial q_j} + \frac{\partial K}{\partial \dot{x}_k}\frac{\partial \dot{x}_k}{\partial q_j}\\ &= -\frac{\partial V}{\partial q_j} + Q_j' + \frac{\partial K}{\partial q_j}\\ &= \frac{\partial}{\partial q_j}(K - V) + Q_j'\\ &= \frac{\partial}{\partial q_j}L + Q_j'\end{aligned} \tag{3.27}$$

where L is the Lagrangian

$$L\left(\{q\},\{\dot{q}\};t\right) = K\left(\{q\},\{\dot{q}\};t\right) - V\left(\{q\}\right) \tag{3.28}$$

The general forces derived from a potential are

$$Q_j^{\text{pot}} = F_i\frac{\partial x_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{3.29}$$

and

$$Q'_j = F'_i \frac{\partial x_i}{\partial q_j} \tag{3.30}$$

are the other general forces. It follows from (3.25) and $\partial V/\partial \dot{q}_j = 0$ that the general momentum can be written as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{3.31}$$

and so (3.27) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q'_j \tag{3.32}$$

As an elementary application of Lagrangian techniques, we determine the r and θ equations of motion for a particle moving in a plane under the influence of a central potential energy $V(r)$. As general coordinates we take

$$q_1 = r\,, \qquad q_2 = \theta \tag{3.33}$$

in terms of which the cartesian coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{3.34}$$

The kinetic energy

$$K = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{3.35}$$

is easily expressed in polar coordinates by taking the time derivative of (3.34) to get the cartesian velocities

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \tag{3.36}$$

and therefore

$$K = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{3.37}$$

This result for K also follows from (2.126) with $K = \frac{1}{2}m(v_r^2 + v_\theta^2)$. We note that K is a function of q_1, $\dot{q}_1$, and $\dot{q}_2$ but not of q_2. The Lagrangian is

$$L = K - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{3.38}$$

In this case there are no constraint forces or non-conservative forces so that $Q'_r = Q'_\theta = 0$. Using this in the Lagrange equations of motion (3.32), with $Q'_j = 0$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \end{aligned} \tag{3.39}$$

we find

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 &= -\frac{\partial V}{\partial r} \\ \frac{d}{dt}(mr^2\dot{\theta}) &= r(mr\ddot{\theta} + 2m\dot{r}\dot{\theta}) = 0 \end{aligned} \tag{3.40}$$

These correspond to (2.130), obtained from direct application of Newton's Laws, with $F_r = -\partial V/\partial r$ and $F_\theta = 0$.

Since L does not depend on θ, $\dot{p}_\theta = 0$ from (3.39); hence the general momentum p_θ is constant,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \tag{3.41}$$

This conserved quantity is the angular momentum L_z.

This conservation law is an example of a general principle that can be deduced from the Lagrange equation (3.32): if a general coordinate q_j does not appear in the Lagrangian and $Q'_j = 0$, the corresponding general momentum $p_j = \partial L/\partial \dot{q}_j$ is constant in time — it is a constant of the motion (a conserved quantity).

3.4 Constraints

As a simple example of a constrained system we return to the simple pendulum. A mass m moves in a vertical plane as illustrated in Fig. 3-1, subject to gravitational force and to the tension force of an attached string of length ℓ which constrains the mass to always be at a distance ℓ from the other end of the string. To begin with, we shall suppose that this other end of the string is held at a fixed position; later it will be allowed to move arbitrarily. The essence of the problem is that we are to find the motion of the mass according to Newton's equations, but we are not given all the forces; instead, we are given partial information about the motion, namely the constraint(s). The unknown force, in this case the tension, is called the constraint force, and is whatever it has to be for the given motion to obey the constraint(s). Note that the number of the unknown components of constraint force must be the same as the number of constraints on the motion, otherwise the motion will be over- or under-determined.

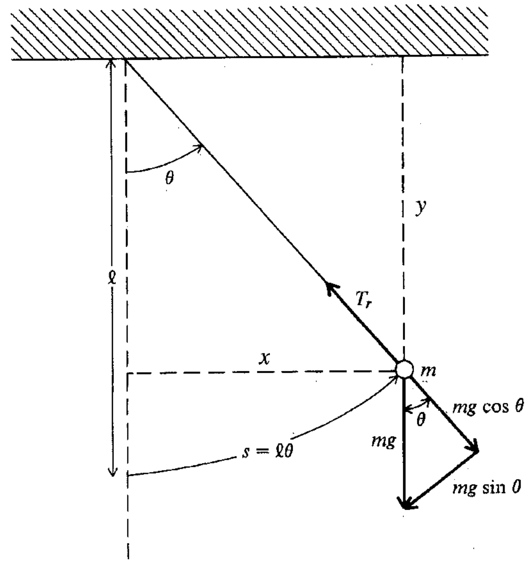


FIGURE 3-1. Simple pendulum

How can we systematically treat a mechanical system with constraints? The first step is to find combinations of the Newton equations in which the constraint forces are absent. As we shall more-or-less demonstrate below, these are precisely Lagrange equations, for an appropriate choice of coordinates. One has enough information to solve these equations (while assuming the constraints to hold), and the solutions to these determine the motion. The remaining equations, into which one substitutes the solution to the motion, then determine the constraint forces; if only the solution for the motion were desired, this step would be unnecessary. The above procedure was carried out already for the pendulum in § 2.7. By expressing Newton's laws in polar coordinates we found (2.133) that the θ equation did not contain the string tension. We solved this equation for $\dot{\theta}$ as a function of θ and then substituted back into the radial equation (2.132) to find the tension, the constraint force.

We now proceed with the simple pendulum using the Lagrangian method. As in Fig. 3-1, let x (horizontal) and y (vertically downward) be cartesian coordinates of the mass and let the origin be at the other (fixed) end of the string, so that the constraint is $\ell \equiv r = \sqrt{x^2+y^2}$. We first do the calculation in an awkward but informative way. Consider the following Lagrangian in polar coordinates

$$L' = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - V^{\text{constraint}}(r) \tag{3.42}$$

where $V(r,\theta) = -mgr\cos\theta$ is the gravitational potential energy $-mgy$ and $V^{\text{constraint}}(r)$ is a potential energy that will enforce the constraint $r = \ell$ by having a deep and narrow minimum at $r = \ell$. It is understood here that only motions with low energies are being considered so that vibrations about the constraint (here $r = \ell$) are of negligible amplitude. In the real world a little friction rapidly damps these high frequency motions.

The radial Lagrange equation for the pendulum is

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{r}}\right) - \frac{\partial L'}{\partial r} = 0 \tag{3.43}$$

or

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \frac{dV^{\text{constraint}}}{dr} = 0 \tag{3.44}$$

Due to the deep and narrow minimum of $V^{\text{constraint}}(r)$ only $r = \ell$ is allowed. Thus the constraint force necessary to keep $r = \ell$ is

$$Q_r^{\text{constraint}} \equiv -\frac{dV^{\text{constraint}}}{dr} = -m\ell\dot{\theta}^2 - mg\cos\theta \tag{3.45}$$

This constraint force is directed inward if the string is taut and is the negative of the string tension T_r. One sees from (3.43) and (3.44) that the condition for r = constant is $\frac{\partial L'}{\partial r} = 0$, i.e. $\frac{\partial L}{\partial r} - \frac{\partial V^{\text{constraint}}}{\partial r} = 0$ or

$$Q_r^{\text{constraint}} \equiv -\left.\frac{\partial L}{\partial r}\right|_{r=\ell} \tag{3.46}$$

The angular Lagrange equation for the pendulum is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(r^2\dot\theta\right) + gr\sin\theta &= 0 \end{aligned} \tag{3.47}$$

Imposing the constraint $r = \ell$ here leads to the usual pendulum equation

$$\ddot\theta + \frac{g}{\ell}\sin\theta = 0 \tag{3.48}$$

The result of the above excercise is that:
1. We can impose constraints directly in the Lagrangian and determine the correct equations of motion without ever explicitly referring to the constraint forces.
2. If we wish to find the force required to enforce a constraint, we choose an additional general coordinate (in this case r) so that when it is held to be a particular constant ($r = \ell$ here) the constraint is maintained. The constraint force then follows as in (3.46).

We can now describe the general case: let the system, with $3N$ degrees of freedom, have C constraints, that is, the motion $x_k(t)$, $k = 1, 2, \ldots, 3N$ is to satisfy

$$f_j(\{x_k(t)\}, t) = 0, \qquad j = 1, 2, \ldots, C \tag{3.49}$$

Choose general coordinates so that C of them are

$$q_j = f_j(\{x_k(t)\}, t) \qquad j = 1, 2, \ldots, C \tag{3.50}$$

so that the constraint conditions read

$$q_j = 0 \qquad j = 1, 2, \ldots, C \tag{3.51}$$

In our pendulum case this would mean $q_1 = r - \ell$. The constraint forces can be imagined to be from a potential $V^{\text{constraint}}(q_1, q_2, \ldots, q_C; t)$ which

has a deep and narrow minimum at $q_1 = q_2 = \cdots = q_C = 0$; the Lagrangian is

$$L' = K - V - V^{\text{constraint}} = L - V^{\text{constraint}} \tag{3.52}$$

with $L = K - V$. Then the combinations of Newton's equations in which no constraint forces appear, and which therefore determine the motion, are the Lagrange equations for the "non-constraint" coordinates,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) = \frac{\partial L}{\partial q_j} \qquad j = C+1,\ C+2, \ldots, 3N \tag{3.53}$$

where $L = K - V$ and the "constraint" coordinates q_j are taken to obey (3.51). Consequently to determine the motion K and V need only be known for constrained configurations of the system. As in the pendulum example, the constraint forces are given by $0 = \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q_j}\right) = \frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} - \frac{\partial V^{\text{constraint}}}{\partial q_j}$, i.e.

$$Q_j^{\text{constraint}} = -\frac{\partial L}{\partial q_j} \quad j = 1, 2, \ldots, C \tag{3.54}$$

As in (3.53) $L = K - V$ and the q_j are all zero. (Note that $Q_j^{\text{constraint}}$ is a general force, so for example it will be a torque if q_j is an angle.)

The type of constraint $f_j(\{x_k\}, t) = 0$ is called a *holonomic* constraint. An important holonomic constraint is the rigid body constraint, in which the distances between every point in the body remain constant. The rigid body constraint can be expressed as

$$|\boldsymbol{r}_i - \boldsymbol{r}_j| = d_{ij} \tag{3.55}$$

where d_{ij} is the constant distance between particles i and j. As we will discuss in Chapter 6, the result of the rigid body constraints is that the configuration of a rigid body is described by six general coordinates — three angles and the three coordinates of the center of mass.

For completeness we should mention that some mechanical systems have constraints which *cannot* be expressed as relations among the coordinates; these are called *non-holonomic* constraints. An important class of these are expressed as constraints on the velocities

$$a_{ij}\,\dot{q}_j + b_i = 0 \tag{3.56}$$

(the a_{ij} and b_i may depend on the q_i's) where the equations cannot be integrated. If such a set of relations could be integrated, the result would be relations between coordinates and the constraints would actually be holonomic. Consider the example of a ball rolling without slipping on a surface. There are two types of constraints here. One is that the ball touches the surface; this is holonomic and can be expressed by saying that the distance of the center of the ball above the surface is always equal to the radius of the ball. The other is the "rolling without slipping" constraint which can be expressed by saying that the ball at the point of contact must be at rest relative to the plane. This constraint is non-holonomic because it cannot be integrated to a relation among the coordinates. This is evident from the fact that the ball can be rolled to any position and orientation (note that the ball can rotate *around* the point of contact).

An example of another class of non-holonomic constraints is given by a pendulum bob on a flexible string; the distance r between the bob and the other end of the string cannot exceed the length ℓ of the string. For some initial conditions the string may not remain taut and the bob will then fall inward (*i.e.*, $r < \ell$). When this happens the string no longer exerts any force on the bob and the bob moves as a projectile until the string becomes taut once more. The precise transition from constrained to unconstrained motion and back to constrained motion requires the solution of the equations of motion at each step and cannot be cast in the usual holonomic form.

3.5 Pendulum With Oscillating Support

You may have wondered if the Lagrangian method is actually advantageous, since the examples we have solved are just as easy to do by Newton's second law. To illustrate the merits of the Lagrangian approach, we shall treat the motion of a pendulum with an oscillating support. This example also provides a simple demonstration of the forced harmonic oscillator.

The point of suspension (x_s, y_s) of a simple pendulum is moved as a specified function of time, as shown in Fig. 3-2. We take as coordinates x, y the relative coordinates of the bob to the point of suspension; thus

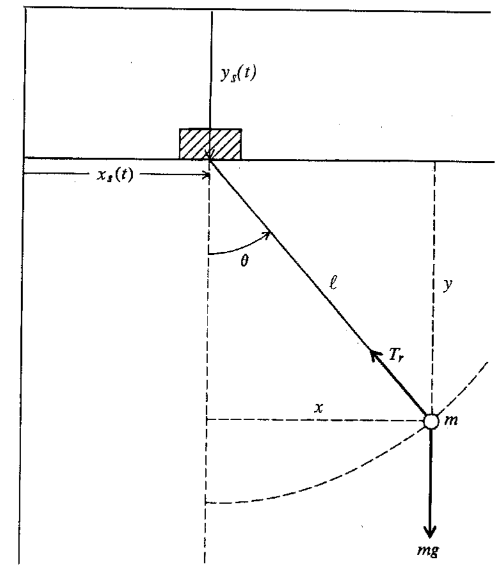


FIGURE 3-2. Simple pendulum with a moving support. The direction of positive y_s and y is downwards.

the bob has position $(x+x_s, y+y_s)$ with respect to a fixed system, where $x_s(t)$ and $y_s(t)$ are given functions of time representing the horizontal and vertically downward coordinates of the support and

$$\begin{aligned} x &= \ell \sin\theta \\ y &= \ell\cos\theta \end{aligned} \tag{3.57}$$

The bob's kinetic energy is

$$K = \tfrac{1}{2}m\left[(\dot{x}+\dot{x}_s)^2 + (\dot{y}+\dot{y}_s)^2\right] = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+2\dot{x}\dot{x}_s+2\dot{y}\dot{y}_s+\dot{x}_s^2+\dot{y}_s^2) \tag{3.58}$$

and its potential energy is

$$V = -mg\ell\cos\theta - mgy_s \tag{3.59}$$

Thus the Lagrangian $L = K - V$ is

$$L = \tfrac{1}{2}m(\ell^2\dot{\theta}^2+2\ell\dot{\theta}\dot{x}_s\cos\theta+2\ell\dot{\theta}\dot{x}_s\sin\theta+\dot{x}_s^2+\dot{y}_s^2)+mg\ell\cos\theta+mgy_s \tag{3.60}$$

For the general coordinate θ, the derivatives appearing in the Lagrange equation are

$$\frac{\partial L}{\partial \dot\theta} = m\ell^2\dot\theta + m\ell\dot x_s\cos\theta + m\ell\dot y_s\sin\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m\ell^2\ddot\theta + m\ell\ddot x_s\cos\theta + m\ell\ddot y_s\sin\theta + m\ell\dot\theta\dot y_s\cos\theta - m\ell\dot\theta\dot x_s\sin\theta$$
$$\frac{\partial L}{\partial \theta} = m\ell\dot\theta\dot y_s\cos\theta - m\ell\dot\theta\dot x_s\sin\theta - mg\ell\sin\theta \tag{3.61}$$

The resulting equation of motion is

$$\ddot\theta + \left(\frac{g}{\ell} + \frac{\ddot y_s}{\ell}\right)\sin\theta = -\frac{\ddot x_s}{\ell}\cos\theta \tag{3.62}$$

(Note the cancellation of $\dot\theta\dot x_s$ and $\dot\theta\dot y_s$ terms.) For small angular displacements ($\theta \ll 1$) and a horizontal sinusoidal motion of the support,

$$\begin{aligned} x_s &= x_0\cos\omega t \qquad y_s = 0 \\ \ddot x_s &= -\omega^2 x_0\cos\omega t \end{aligned} \tag{3.63}$$

the equation of motion (3.62) becomes

$$\ddot\theta + \omega_0^2\theta = \frac{x_0}{\ell}\omega^2\cos\omega t \tag{3.64}$$

where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency. This equation is mathematically identical with that of a forced harmonic oscillator [see (1.115)]. Because of this similarity a pendulum with a horizontally oscillating support can be used to demonstrate the properties of a driven harmonic oscillator.

We stress that the advantage of using Lagrangian methods is the methodical and straightforward procedure. Once a Lagrangian function is constructed from the kinetic and potential energies, the task of obtaining the equations of motion is simply a matter of differentiation. In complex problems there is less chance of error using this method.

3.6 Hamilton's Principle and Lagrange's Equations

An elegant method known as *Hamilton's Principle* provides an instructive demonstration that Lagrange's equations are equivalent to Newton's equations. Given the Lagrangian $L(\{q\}, \{\dot q\}, t)$, one defines for a given motion $q_j(t)$ of the system between the times t_1 and t_2 the quantity S called the *action* of the motion

$$S = \int_{t_1}^{t_2} L(\{q\}, \{\dot q\}, t)\,dt \tag{3.65}$$

We will be interested in how S changes when the motion is changed to another motion. More specifically, we will be interested only in a slightly restricted class of motions, namely those which have a specified initial point, $q_j(t_1) = q_j^{\text{initial}}$, and likewise a specified final point, $q_j(t_2) = q_j^{\text{final}}$, as illustrated in Fig. 3-3. As the motion is varied, subject to the fixed-end-point conditions just stated, the value of S varies (in general). We can now state Hamilton's principle as the following theorem: If any small variation (satisfying the fixed-end-point conditions) of a motion produces no variation of S (to first order) then the motion satisfies Lagrange's equations and vice versa.

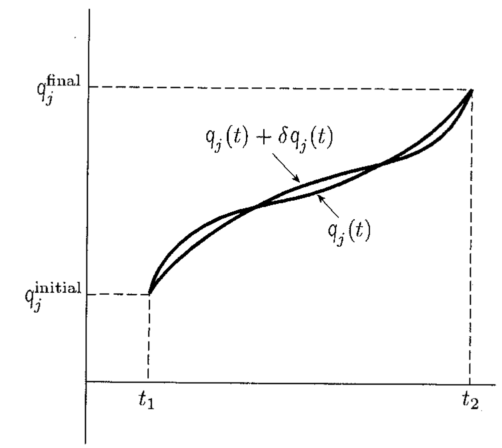


FIGURE 3-3. Two nearby trajectories having the same initial and final values, for the case of one coordinate.

In other words, the action S is *stationary* at just those motions which satisfy Lagrange's equations. [Often, a stationary value of S is a minimum value, that is, all variations of the motion raise the value of S. For this reason, Hamilton's principle is often (misleadingly) called the Least Action Principle.]

The demonstration is as follows. We consider a particular motion $q_j(t)$ and a slightly different motion $\tilde{q}_j(t) = q_j(t) + \delta q_j(t)$, where the difference of motions, $\delta q_j(t)$, is small. To satisfy the fixed-end-point conditions, δq_j must vanish at t_1 and t_2. The variation of the action when the motion is varied from $q_j(t)$ to $\tilde{q}_j(t)$ is

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt L\left(\{\tilde{q}\}, \{\dot{\tilde{q}}\}, t\right) - \int_{t_1}^{t_2} dt L\left(\{q\}, \{\dot{q}\}, t\right) = \int_{t_1}^{t_2} dt\, \delta L \\ &= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right)\end{aligned} \tag{3.66}$$

where sums over j are implied. The final step used chain differentiation, *i.e.* $\delta f(x, y, \ldots) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \cdots$. Clearly, for δS to vanish it is sufficient that $\partial L/\partial q_j = 0$ and $\partial L/\partial \dot{q}_j = 0$; however, this is not a necessary condition because the functions δq_j and $\delta \dot{q}_j$ are not independent. This is dealt with as follows. Note that

$$\delta \dot{q}_j = \frac{\delta q_j(t + dt) - \delta q_j(t)}{dt} = \frac{d}{dt}(\delta q_j) \tag{3.67}$$

Thus the terms in $\delta \dot{q}_j$ can be integrated by parts

$$\int dt \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j = \int dt \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\delta q_j = \frac{\partial L}{\partial \dot{q}_j}\delta q_j - \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j \tag{3.68}$$

and (3.66) becomes

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j + \left.\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right|_{t_1}^{t_2} \tag{3.69}$$

The last term vanishes by the fixed-end-point condition, $\delta q_j = 0$ at $t = t_2$ and $t = t_1$. Thus δS vanishes for arbitrary small $\delta q_j(t)$ if and only if the square-bracketed expression vanishes for all t between t_1 and t_2 and for all values of the coordinate index j. This vanishing is Lagrange's equations (3.32),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{3.70}$$

and so Hamilton's Principle has been shown. We remark that Hamilton's Principle is an example of a *variational principle*, and our treatment of it is an example of the *calculus of variations.*

It is now easy to see that Lagrange's equations have the same form in any coordinate system. According to Hamilton's Principle, the statement that a motion $q(t)$ satisfies Lagrange's equations (3.32) is equivalent to the statement that the action of the motion is stationary. The latter statement is independent of the choice of coordinates, and so the former statement must be as well. To be more explicit, if we change coordinates from q to $\bar{q}$ [*cf.* Eq. (3.1)] then the reexpression of the action in terms of the new description $\bar{q}(t)$ of the orbit goes as follows:

$$\begin{aligned}S &= \int_{t_1}^{t_2} dt L(\{q(t)\}, \{\dot{q}(t)\}, t) \\ &= \int_{t_1}^{t_2} dt L(\{q(\bar{q}(t))\}, \{\dot{q}(\bar{q}(t), \dot{\bar{q}}(t), t)\}, t) \\ &= \int_{t_1}^{t_2} dt \bar{L}(\{\bar{q}(t)\}, \{\dot{\bar{q}}(t)\}, t)\end{aligned} \tag{3.71}$$

where the last step defines $\bar{L}$ to be the function of the $\bar{q}_j, \dot{\bar{q}}_j$ and t which results from substituting into $L(\{q\}, \{\dot{q}\}, t)$ the expressions of the old coordinates and velocities q, $\dot{q}$ in terms of the new,

$$\begin{aligned}q_j &= q_j(\{\bar{q}\}, t) \\ \dot{q}_j &= \frac{\partial q_j}{\partial \bar{q}_i}\dot{\bar{q}}_i + \frac{\partial q_j}{\partial t}\end{aligned} \tag{3.72}$$

Thus Hamilton's Principle tells us that if the motion $q(t)$ satisfies Lagrange's equations with Lagrangian $L(\{q\}, \{\dot{q}\}, t)$ then its description in terms of the new coordinates, $\bar{q}(t)$, satisfies Lagrange's equations with Lagrangian $\bar{L}(\{\bar{q}\}, \{\dot{\bar{q}}\}, t)$.

Now that we have shown that Lagrange's equations have the same form in any coordinates, it follows that they are equivalent to Newton's equations if $L = K - V$ because if the $\{q\}$ and $\{\dot{q}\}$ are chosen to be cartesian coordinates, then $K = \frac{1}{2}\sum_k m_k \dot{x}_k^2$ and Lagrange's equations (3.70) become

$$\dot{p}_k = \frac{\partial(K-V)}{\partial x_k} = -\frac{\partial V}{\partial x_k} = F_k$$
$$p_k = \frac{\partial L}{\partial \dot{x}_k} = \frac{\partial K}{\partial \dot{x}_k} = m\dot{x}_k \tag{3.73}$$

A final remark is that if the motion is to satisfy holonomic constraints, the equations which determine the motion of the system result from using in Hamilton's Principle only motions which satisfy the constraints.

3.7 Hamilton's Equations

The Lagrange equations of motion, which are equivalent to Newton's equations, are a set of second-order differential equations. An alternative formulation of Newton's law consists of twice as many first-order differential equations known as *Hamilton's equations.*

In Lagrangian mechanics the independent variables are $q_j, \dot{q}_j$, and t, and the general momentum p_j is given in terms of these by (3.31).

$$p_j = \frac{\partial K}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} \tag{3.74}$$

In Hamiltonian mechanics, q_j, p_j, and t are chosen as independent variables and $\dot{q}_j$ is a dependent quantity.

$$\dot{q}_j = \dot{q}_j(q_1, q_2, \ldots, q_n;\ p_1, p_2, \ldots, p_n;\ t) \tag{3.75}$$

The Hamiltonian function H is defined as

$$H(q_1, q_2, \ldots, q_n;\ p_1, p_2, \ldots, p_n;\ t) \equiv p_j\dot{q}_j - L \tag{3.76}$$

where a summation over the repeated index j on the right-hand side is implied. In this definition the variables $\dot{q}_j$ are understood to be functions of the general coordinates, momenta, and time as in (3.75). From (3.76) the total differential of H is

$$\begin{aligned} dH &= \dot{q}_j\,dp_j + p_j\,d\dot{q}_j - \frac{\partial L}{\partial q_j}dq_j - \frac{\partial L}{\partial \dot{q}_j}d\dot{q}_j - \frac{\partial L}{\partial t}dt \\ &= \dot{q}_j\,dp_j - \frac{\partial L}{\partial q_j}dq_j - \frac{\partial L}{\partial t}dt \end{aligned} \tag{3.77}$$

where in the second line we have used $p_j = \partial L/\partial \dot{q}_j$. Since the independent differentials are now dp_j, dq_j and dt, we see that the replacement of the $\dot{q}_j$ by the p_j as the fundamental variables is achieved by the definition $H = p_j\dot{q}_j - L$. This cancellation of the coefficient of $d\dot{q}$ is an example of a *Legendre transformation.* Such variable changes are encountered frequently in the study of thermodynamics.

To derive Hamilton's equations of motion we compare (3.77) with the total differential

$$dH = \frac{\partial H}{\partial p_j}dp_j + \frac{\partial H}{\partial q_j}dq_j + \frac{\partial H}{\partial t}dt \tag{3.78}$$

and use (3.32) to find

$$\begin{aligned} \frac{\partial H}{\partial p_j} &= \dot{q}_j \\ \frac{\partial H}{\partial q_j} &= -\frac{\partial L}{\partial q_j} = -\dot{p}_j + Q_j' \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned} \tag{3.79}$$

To establish the physical significance of the Hamiltonian we relate the quantities on the right-hand side of (3.76) to the kinetic and potential energies of the system. In cartesian coordinates the kinetic energy is

$$K = \sum_k \tfrac{1}{2}m_k\dot{x}_k^2 = \tfrac{1}{2}p_k\dot{x}_k \tag{3.80}$$

By use of the chain rule expression for $\dot{x}_k$ the kinetic energy can be rewritten in the form

$$K = \frac{1}{2} p_k \left(\frac{\partial x_k}{\partial \mathrm{q}_j} \dot{\mathrm{q}}_j + \frac{\partial x_k}{\partial t} \right) = \frac{1}{2} \mathrm{p}_j \dot{\mathrm{q}}_j + \frac{1}{2} p_k \frac{\partial x_k}{\partial t} \tag{3.81}$$

Solving for $\mathrm{p}_j \dot{\mathrm{q}}_j$ from (3.81),

$$\mathrm{p}_j \dot{\mathrm{q}}_j = 2K - p_k \frac{\partial x_k}{\partial t} \tag{3.82}$$

and substituting the result into (3.76), we obtain

$$\begin{aligned} H &= 2K - p_k \frac{\partial x_k}{\partial t} - L \\ &= K + V - p_k \frac{\partial x_k}{\partial t} \end{aligned} \tag{3.83}$$

Hence if the transformation between the cartesian and general coordinates in (3.1) has no explicit time dependence,

$$\frac{\partial x_k}{\partial t} = 0 \tag{3.84}$$

the Hamiltonian is the total energy of the system,

$$H = K + V \tag{3.85}$$

Of course this equation holds only in cases where the potential energy exists.

To find the conditions under which the Hamiltonian is a conserved quantity, we compute the total time derivative.

$$\frac{dH}{dt} = \frac{\partial H}{\partial \mathrm{q}_j} \dot{\mathrm{q}}_j + \frac{\partial H}{\partial \mathrm{p}_j} \dot{\mathrm{p}}_j + \frac{\partial H}{\partial t} \tag{3.86}$$

Upon use of (3.79), this reduces to

$$\frac{dH}{dt} = \mathrm{Q}'_j \dot{\mathrm{q}}_j + \frac{\partial H}{dt} \tag{3.87}$$

Thus, if the forces are derivable from a potential energy ($\mathrm{Q}'_j = 0$) and H has no explicit time dependence ($\partial H / \partial t = 0$), the Hamiltonian is a constant of the motion. This constant is the total energy of the system if (3.84) holds.

As an elementary example of the Hamiltonian method, we consider the one-dimensional harmonic oscillator, for which

$$\begin{aligned} K &= \tfrac{1}{2} m \dot{x}^2 \\ V &= \tfrac{1}{2} k x^2 \\ L &= K - V = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 \end{aligned} \tag{3.88}$$

In this case $\mathrm{q} = x$ and $\mathrm{Q}' = 0$. The momentum is found by differentiation according to (3.74),

$$p = \frac{\partial K}{\partial \dot{x}} = m \dot{x} \tag{3.89}$$

The Hamiltonian from (3.76) is

$$\begin{aligned} H &= p \dot{x} - \left(\tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 \right) \\ &= \frac{p^2}{2m} + \tfrac{1}{2} k x^2 \end{aligned} \tag{3.90}$$

where (3.89) has been used to eliminate $\dot{x}$ in favor of p. This Hamiltonian is immediately recognizable as the total energy of the oscillator.

Hamilton's equations of motion from (3.79),

$$\frac{\partial H}{\partial p} = \dot{x} \qquad \frac{\partial H}{\partial x} = -\dot{p} \tag{3.91}$$

yield

$$\frac{p}{m} = \dot{x} \qquad kx = -\dot{p} \tag{3.92}$$

When p is eliminated between these two first-order equations, we obtain the usual second-order differential form of Newton's second law:

$$m \frac{d^2 x}{dt^2} = -kx \tag{3.93}$$

Because of the similar role that coordinate and momentum play in Hamilton's equations they provide the jumping off point for the formulation of abstract mechanics, celestial mechanics, and quantum mechanics. In the latter the generalization to subatomic mechanics begins with the classical Hamiltonian. The coordinates and momenta are now operators; for example, in coordinate space $x_{\mathrm{op}} = x$ and $p_{x\,\mathrm{op}} = -i\hbar \frac{\partial}{\partial x}$, where $\hbar$ is the reduced Planck's constant $\hbar = h/2\pi$. In quantum mechanics all information on a physical system resides in the wavefunction $\psi(\mathbf{r}, t)$ which satisfies the Schrödinger equation

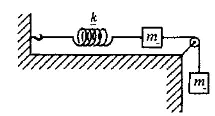
$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{3.94}$$

Since $\mathbf{r}$ and $\mathbf{p}$ have operator forms this amounts to a partial differential equation for the wavefunction. The introduction of quantum mechanics, when combined with special relativity, has extended the range of experimental validity of Hamiltonian mechanics down to at least 10^{-18} m.

PROBLEMS

3-1. Two equal masses $m_1 = m_2 = m$ with coordinates x_1 and x_2 in one dimension are connected by a spring of spring constant k. Use Lagrangian methods to find the equations of motion. What is the angular frequency of simple harmonic motion for the relative displacement $x_1 - x_2$ of the two masses?

3-2. Two equal masses are constrained by the spring-and-pulley system shown in the accompanying sketch. Assume a massless pulley and a frictionless surface. Let x be the extension of the spring from its relaxed length. Derive the equations of motion by Lagrangian methods. Solve for x as a function of time with the boundary conditions $x = 0$, $\dot{x} = 0$ at $t = 0$.



3-3. Use Lagrangian methods to find the equations of motion for Problem 2-21.

3-4. Use Lagrangian methods to find the equations of motion for Problem 2-22.

3-5. Two masses m_1 and m_2 are connected by a spring of rest length ℓ and spring constant k. The system slides without friction on a horizontal surface in the direction of the spring's length.

a) Set up the Lagrangian for the motion.

b) Find the normal modes of this system and the corresponding frequencies.

c) Give general solutions to the equations of motion. Note that an equation of motion with a zero angular frequency is not simple harmonic.

d) For the initial conditions $x_1(0) = 0$, $\dot{x}_1(0) = v_0$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$, find the subsequent motion.

e) Using the solution from part d), evaluate the center of mass coordinate $x_{CM} = [m_1 x_1 + m_2(x_2 + \ell)]/(m_1 + m_2)$ and the relative coordinate $x_2 - x_1$ as a function of time.

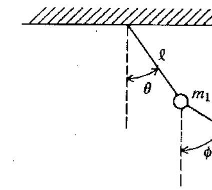
3-6. A bead of mass m is constrained to move without friction on a hoop of radius R. The hoop rotates with constant angular velocity ω around a vertical diameter of the hoop. Use a polar angle θ and an azimuthal angle ϕ to describe the position of the bead on the hoop, with $\omega = \dot{\phi}$. Take $\theta = 0$ at the bottom of the hoop.

a) Set up the Lagrangian and obtain the equation of motion on the bead.

b) Find the critical angular velocity $\omega = \Omega$ below which the bottom of the hoop provides a stable equilibrium position for the bead.

c) Find the stable equilibrium position for $\omega > \Omega$.

3-7. A double pendulum consists of two weightless rods connected to each other and a point of support, as illustrated. The masses m_1 and m_2 are not equal, but the lengths of the rods are. The pendulums are free to swing only in one vertical plane.

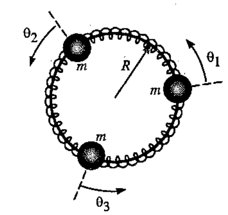


a) Set up the Lagrangian of the system for arbitrary displacements and derive the equations of motion from it.

b) Find the normal-mode frequencies of the system when both angles of oscillation are small.

c) Show that the frequencies become approximately equal if $m_1 \gg m_2$; interpret this. For $m_2 \gg m_1$ interpret the normal-mode frequencies and describe the motion of each mass.

3-8. A model of a ring molecule consists of three equal masses m which slide without friction on a fixed circular wire of radius R. The masses are connected by identical springs of spring constant $m\omega_0^2$. The angular positions of the three masses, θ_1, θ_2 and θ_3, are measured from a rest position.



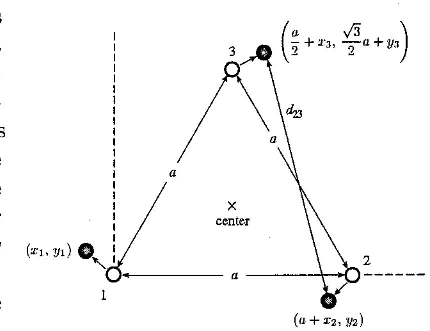
a) Write down the Lagrangian and show that the equations of motion are

$$\ddot{\theta}_1 + \omega_0^2(2\theta_1 - \theta_2 - \theta_3) = 0$$
$$\ddot{\theta}_2 + \omega_0^2(2\theta_2 - \theta_1 - \theta_3) = 0$$
$$\ddot{\theta}_3 + \omega_0^2(2\theta_3 - \theta_1 - \theta_2) = 0$$

b) Show that the mode in which $\theta_1 = \theta_2 = \theta_3$ corresponds to constant total angular momentum $L = \sum_{i=1}^{3} p_{\theta_i}$.

c) Assume the total angular momentum is zero and that $\theta_1+\theta_2+\theta_3 = 0$. Find two degenerate oscillatory modes and their frequency.

3-9. A triangular molecule has three identical atoms with rest separations a as shown. The molecule is represented as a mechanical system of masses and springs with the springs representing the chemical bonds. For small motions in the x, y plane about the equilibrium configuration, the kinetic and potential energies are



$$K = \frac{1}{2}m\sum_{i=1}^{3}(\dot{x}_i^2 + \dot{y}_i^2)$$

$$V = \frac{1}{2}k\left[(x_2 - x_1)^2 + \frac{1}{4}\left(\sqrt{3}y_3 - \sqrt{3}y_1 + x_3 - x_1\right)^2 + \frac{1}{4}\left(\sqrt{3}y_3 - \sqrt{3}y_2 + x_2 - x_3\right)^2\right]$$

Use Lagrange's equations to find the equations of motion.

3.4 Constraints

3-10. A bead slides without friction on a parabolic wire of shape $y = ax^2$ with the force of gravity in the negative y direction. Write down the Lagrangian in terms of x and y coordinates. Then use the constraint equation to express the Lagrangian solely in terms of x. Find the equation of motion and then simplify it for the case of small oscillations.

3.6 Hamilton's Principle and Lagrange's Equations

3-11. A particle of mass m falls vertically in a constant gravity field g. Assume that the position as a function of time is

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

where y increases upward and the $\{c_k\}$ are constant coefficients to be determined.

a) Evaluate the action S between $t = 0$, where $y(0) = 0$, and $t = T$ where $y(T) = \ell$. As a function of c_2, c_3, T and ℓ show that

$$\frac{S}{mT^3} = \frac{\ell}{2T^4}(\ell - gT^2) + \left(\frac{g}{6}\right)c_2 + \left(\frac{1}{6}\right)c_2^2 + \left(\frac{T}{2}\right)c_2c_3 + \left(\frac{gT}{4}\right)c_3 + \left(\frac{2T^2}{5}\right)c_3^2$$

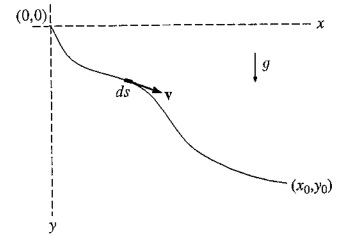
b) For fixed T and ℓ show that the action is an extremum for $c_2 = -g/2$ and $c_3 = 0$.

c) What kind of extremum does S have at this point?

3-12. Using the methods of the calculus of variations show that the curve of shortest length connecting the two points (x_1, y_1) and (x_2, y_2) in the x, y plane is a straight line.

Hint: the length is $s = \int_{x_1}^{x_2} L\left(y, \frac{dy}{dx}, x\right)dx$ *where* $L = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

3-13. A bead slides without friction on a wire in the vertical x, y plane as shown. The elapsed time for the trip between the origin (0,0) and the point (x_0, y_0) is $t = \int_{(0,0)}^{(x_0,y_0)} \frac{ds}{v}$, where $ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ is the element of arc length and $\mathbf{v}$ is the velocity ($v = \sqrt{2gy}$ from energy conservation). Assuming that the bead is released at rest, the shape of the wire $y(x)$ is to be found for which the elapsed time is minimum. This famous *brachistochrone* problem (or curve of quickest descent), first proposed and solved by John Bernoulli in 1696, led to the development of the calculus of variations.



a) Show that the differential equation defining the wire shape is

$$2y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

b) Demonstrate that the solution is a *cycloid*

$$x = a(\phi - \sin\phi)$$
$$y = a(1 - \cos\phi)$$

How are the parameter a and the values of ϕ at the endpoints determined?

3.7 Hamilton's Equations

3-14. For a particle moving in a plane under the influence of a central potential energy $V(r)$, find the Hamiltonian as a function of r, θ, p_r, and p_θ. Find the four Hamilton equations of motion. Show that the results are equivalent to Eqs. (3.40).

3-15. In a $2N$-dimensional *phase space* with coordinates (q_j, p_j) show that the "flow velocity" $(\dot{q}_j, \dot{p}_j)$ in this space satisfies $\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = 0$, assuming that the general forces appear only in H. This indicates that the "flow" in this space is incompressible. This result is fundamental to statistical mechanics.

Chapter 4

MOMENTUM CONSERVATION

The conservation of linear momentum is a universal law for all of physics. In classical mechanics this conservation law is a direct consequence of Newton's laws. In the absence of external forces, the equation of motion

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = 0 \tag{4.1}$$

implies that $\mathbf{p}$ is independent of time. In other words, a particle with definite mass moves with constant velocity $\mathbf{v}$ in a force-free region. The most interesting ramifications of momentum conservation concern systems of more than one particle.

For a two-particle system, internal forces $\mathbf{F}^{\text{int}}$ between the particles and external forces $\mathbf{F}^{\text{ext}}$ on the particles can be present. The laws of motion for particles 1 and 2 are

$$\begin{aligned} \frac{d\mathbf{p}_1}{dt} &= \mathbf{F}_1^{\text{int}} + \mathbf{F}_1^{\text{ext}} \\ \frac{d\mathbf{p}_2}{dt} &= \mathbf{F}_2^{\text{int}} + \mathbf{F}_2^{\text{ext}} \end{aligned} \tag{4.2}$$

The total momentum of the system

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{4.3}$$

obeys an equation given by the sum of (4.2)

$$\frac{d\mathbf{P}}{dt} = (\mathbf{F}_1^{\text{int}} + \mathbf{F}_2^{\text{int}}) + (\mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}}) \tag{4.4}$$

If the total external force is zero,

$$\mathbf{F}^{\text{ext}} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} = 0 \tag{4.5}$$

and the internal forces cancel,

$$\mathbf{F}_1^{\text{int}} = -\mathbf{F}_2^{\text{int}} \tag{4.6}$$

as implied by Newton's third law, then the momentum is conserved