rest at the origin ($x = 0$) at time $t = 0$. Find the solution of the equation of motion which satisfies the specified initial condition.

1-25. Find the average power dissipated per driving period by the frictional force of a sinusoidally driven harmonic oscillator in steady state. (Recall that power = force × velocity.) Show that maximum dissipation occurs at $\omega = \omega_0$ and evaluate this maximum.

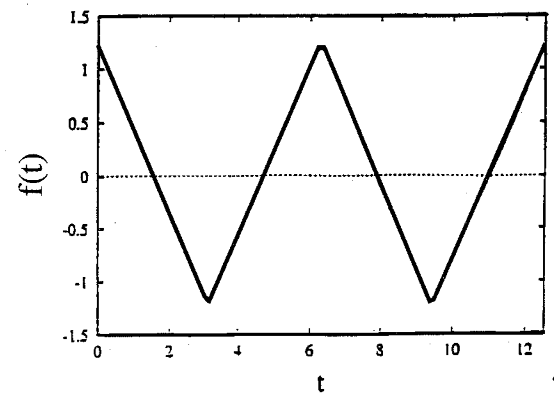
1-26. A sawtooth wave (see accompanying figure) can be decomposed into an infinite sum of cosines

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(\omega_n t) ,$$

where $\omega_n = (2n+1)\omega$ and ω is the "angular frequency" of the sawtooth. Find the steady-state motion of an oscillator driven by this force per unit mass

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) .$$

Hint: find the solution for a given ω_n and use the principle of superposition.



Chapter 2

ENERGY CONSERVATION

There are three important conservation laws of mechanics: energy, momentum, and angular momentum. The three laws can be derived from Newtonian theory. However, their range of validity is much broader, extending even to the domain of relativistic elementary particles, although slightly changed in form. In their ramifications in all branches of science, these conservation laws have exceptionally far-reaching consequences. In this chapter we discuss energy conservation and then in later chapters we take up in turn momentum and angular momentum conservation.

2.1 Potential Energy

To derive the energy-conservation law in the case of one-dimensional motion, we start with the second law of motion for a body of mass m

$$\frac{d}{dt}(mv) = F(x, v, t) \tag{2.1}$$

and multiply by v. Since $v\,dv/dt = \frac{1}{2}d(v^2)/dt$ we obtain the equation

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = F(x, v, t)v \tag{2.2}$$

Substituting $v = dx/dt$ on the right-hand side and integrating with respect to t gives

$$\tfrac{1}{2}mv^2(t_2) - \tfrac{1}{2}mv^2(t_1) = \int_{t_1}^{t_2} F(x(t), v(t), t)\frac{dx}{dt}dt = \int_{x_1}^{x_2} F\big(x, v(x), t(x)\big)dx \tag{2.3}$$

The left-hand side is the difference at two times of the familiar expression for the kinetic energy

$$K = \tfrac{1}{2}mv^2 \tag{2.4}$$

Equation (2.3) is the *Work-Energy theorem*

$$K_2 - K_1 = \Delta K = \text{Work} = \int_{x_1}^{x_2} F\big(x, v(x), t(x)\big)\,dx \tag{2.5}$$

This theorem states that the work done by the force acting on m equals the change in kinetic energy.

Some forces depend only on position, and then the integrand of (2.5) is a function only of x and the integral does not depend on the particular motion $x(t)$. In such cases it is valuable to define the potential energy

$$V(x) \equiv -\int_{x_s}^{x} F(x')dx' \tag{2.6}$$

where x_s is an arbitrary but fixed reference point. The right-hand side of (2.5) can be expressed in terms of V as

$$\int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_s} F(x)dx + \int_{x_s}^{x_2} F(x)dx = V(x_1) - V(x_2) \tag{2.7}$$

for any x_s, even one which is outside the range x_1 to x_2. Using this in (2.5) yields

$$K_2 + V(x_2) = K_1 + V(x_1) \equiv E \tag{2.8}$$

The quantity E is known as the total energy of the body. Since this E is independent of coordinate, the energy is constant in time; *i.e.*, the energy is conserved. If F has an explicit dependence on either v or t, there is no conserved energy of the form (2.8). This does not mean that the energy of the universe is not conserved, but only that energy is transferred between the mechanical form (2.8) and other forms such as thermal energy (microscopic motion of molecules).

The expression for the energy in (2.8) can be simply written as

$$E = K + V(x) = \tfrac{1}{2}mv^2(x) + V(x) \tag{2.9}$$

for any value of the coordinate x. The term *potential energy* means that V is a form of energy which potentially may appear as kinetic energy.

By differentiating (2.6), we can solve for the force in terms of the potential energy:

$$F(x) = -\frac{dV(x)}{dx} \tag{2.10}$$

A force that is derivable in this way from a potential energy is called a *conservative force*. In one-dimensional motion, any force which is a function only of position is a conservative force. The effect of changing from a reference point x_s to a new reference point x'_s is just to change $V(x)$ in (2.6) by a constant, independent of x. The force is unaffected by a change in reference point since it is calculated from the derivative of the potential energy. Because the motion of the particle is determined by the force, all measurable quantities are independent of x_s; hence x_s can be chosen arbitrarily.

Since the energy is a constant of the motion for a conservative force, we can use (2.9) to determine v as a function of x, given E. If the energy is known for a conservative system, it can be used as one of the initial conditions. The conservative nature of spring and gravitational forces allows the use of energy conservation. The potential energy of the spring force is

$$V(x) = -\int_0^x (-kx')dx' = \tfrac{1}{2}kx^2 \tag{2.11}$$

where we have chosen $x_s = 0$. Equation (1.47) of the archery example is a special case of (2.11) with this potential energy for $x < 0$. The total energy of an oscillator can be calculated from (2.11) using the solutions for x and v from (1.64) and (1.66). We find

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = m[-\omega a \sin(\omega t + \alpha)]^2 + \tfrac{1}{2}k[a \cos(\omega t + \alpha)]^2 \tag{2.12}$$

which simplifies to

$$E = \tfrac{1}{2}ka^2 \tag{2.13}$$

where we have used $\omega^2 = k/m$. The energy is proportional to the square of the maximum displacement a, which is called the amplitude. At the turning points of the motion, $x = \pm a$, the energy is entirely potential energy. At $x = 0$, the kinetic energy is greatest; see Fig. 2-1.

2.2 Gravitational Escape

The gravitational potential energy due to the earth's attraction on a mass m at a distance $x \geq R_E$ from the center of the earth is

$$V(x) = -\int_{\infty}^{x} \left(-\frac{GmM_E}{x'^2}\right) dx' = -\frac{GmM_E}{x} \tag{2.14}$$

We have chosen x_s so that the potential energy vanishes at infinite distance. We may express the gravitational constant G in terms of the

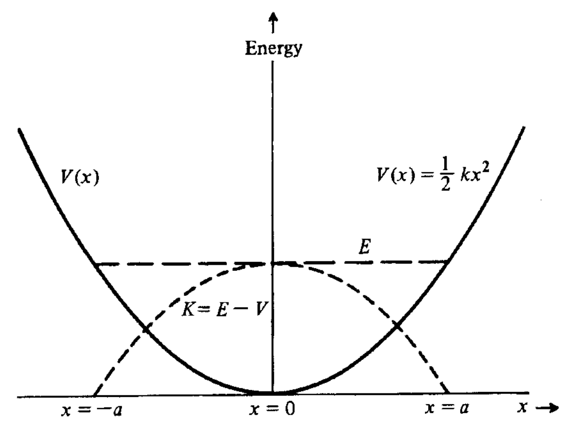


FIGURE 2-1. Potential and kinetic energies for motion under the spring force.

gravitational acceleration on the surface of the earth using (1.7),

$$GM_E = gR_E^2 \tag{2.15}$$

to obtain

$$V(x) = -\frac{mgR_E^2}{x}, \qquad x \geq R_E \tag{2.16}$$

We can use (2.16) in (2.9) to calculate the minimum velocity needed by a rocket at the earth's surface to go to $x = \infty$, that is, to "to escape from the earth's gravitational attraction" (see Fig. 2-2). From (2.9) the velocity at some position x is

$$v(x) = \pm\sqrt{\frac{2}{m}\left(E + \frac{mgR_E^2}{x}\right)} \tag{2.17}$$

For the velocity to be always real as x increases to ∞, $E \geq 0$ is required. The minimum velocity for escape from the earth's surface is consequently obtained by putting $E = 0$, $x = R_E$ in (2.17), yielding

$$\begin{aligned} v_{\text{esc}} &= \sqrt{2gR_E} \\ &= \sqrt{2(9.8)(6.371 \times 10^6)}\ \text{m/s} \\ &= 11.2\ \text{km/s} \quad (40{,}200\ \text{km/h}) \end{aligned} \tag{2.18}$$

The escape velocity is independent of the mass of the rocket. To get to the moon, a spacecraft launched from the earth needs a velocity nearly equal to the escape velocity.

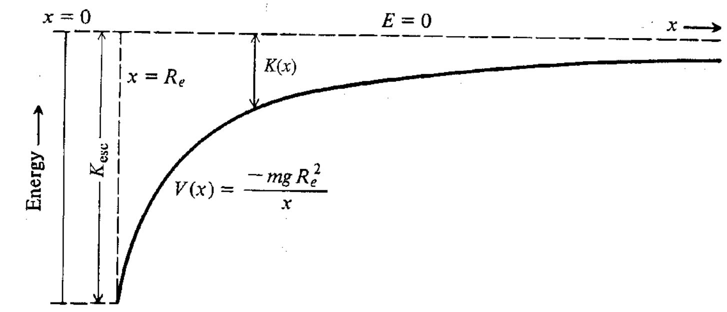


FIGURE 2-2. Gravitational potential energy due to the earth and the minimum kinetic energy K_{esc} needed for escape from the earth's gravitational attraction.

2.3 Small Oscillations

For a general potential energy the velocity can be calculated from (2.9) to be

$$v(x) = \pm\sqrt{\frac{2}{m}[E - V(x)]} \tag{2.19}$$

This expression determines only the magnitude of the velocity. The sign depends on the previous history of the motion. Since the velocity must be real, the accessible region is

$$V(x) \leq E \tag{2.20}$$

The positions at which $V(x) = E$ are turning points, where the velocity goes through zero and changes sign, *i.e.*, the particle comes to rest and reverses its direction of motion. The qualitative nature of the motion of a particle can be described using (2.19); see Fig. 2-3.

For the potential energy sketched in Fig. 2-4, at the total energy indicated by the dashed horizontal line there are three turning points,

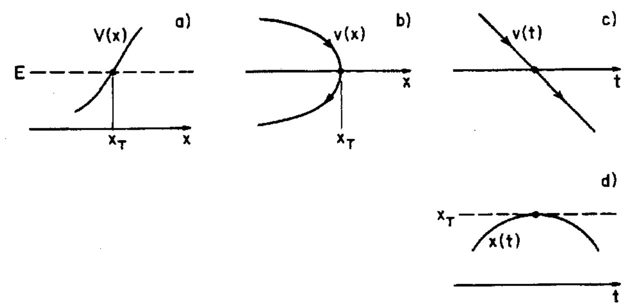


FIGURE 2-3. Behavior of a) the potential energy $V(x)$, b) the velocity $v(x)$, c) the velocity $v(t)$, and d) the position $x(t)$, near a turning point x_T of the motion. We illustrate here the case of a particle initially moving with a velocity in the positive x direction.

x_1, x_2, x_3. The regions $0 \le x < x_1$ and $x_2 < x < x_3$ are forbidden by (2.20). The motion $x(t)$ of a particle in the region $x_1 \le x < x_2$ will be oscillatory, *i.e.*, the particle will move back and forth between x_1 and x_2. The sign of the velocity in this region changes at the turning points as in (2.19). Finally, a particle approaching x_3 from infinity will slow down, reverse its motion at $x = x_3$, and go back out toward infinity.

The motion of a particle in the potential valley, $x_1 \le x \le x_2$, is particularly simple if the maximum displacements from the minimum potential energy at $x = x_e$ are small. In such a case, we can approximate the potential by a few terms of a series expansion about $x = x_e$:

$$V(x) = V(x_e) + (x - x_e)\left[\frac{dV(x)}{dx}\right]_{x=x_e} + \tfrac{1}{2}(x - x_e)^2\left[\frac{d^2V(x)}{dx^2}\right]_{x=x_e} + \cdots \tag{2.21}$$

The derivative dV/dx vanishes at a minimum. Since the second derivative of $V(x)$ is positive at a minimum of $V(x)$, a particle at $x = x_e$ is in stable equilibrium, so for small displacements the potential energy can be approximated by

$$V(x) \cong V(x_e) + \tfrac{1}{2}k(x - x_e)^2 \tag{2.22}$$

where

$$k \equiv \left[\frac{d^2V(x)}{dx^2}\right]_{x=x_e} \ge 0 \tag{2.23}$$

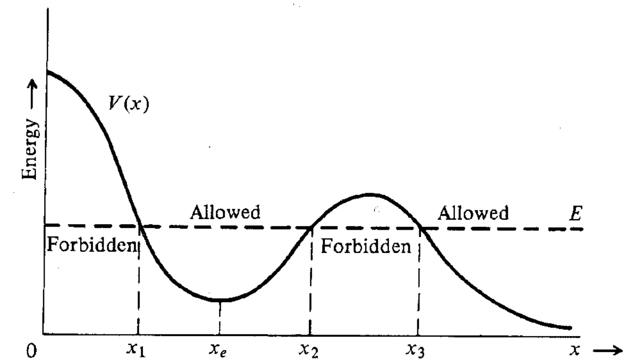


FIGURE 2-4. Allowed and forbidden regions for motion of a particle with energy E for a potential energy $V(x)$.

The constant term $V(x_e)$ can be dropped since it has no consequences for the physical motion. If we make a change of variable to the displacement from equilibrium, we see that the potential energy in (2.22) is that of a simple harmonic oscillator, (2.11), with x replaced by $x - x_e$. Small oscillations in any system can be approximately treated in terms of simple harmonic motion. The expansion about a potential-energy minimum as in (2.22) also provides justification for Hooke's law on the springlike elastic deformation in solids.

In the case discussed above, the effective spring constant k was positive and x_e was a *stable equilibrium point.* If instead, $V(x_e)$ were a local maximum, $F(x_e)$ would still vanish since $\left.\frac{dV}{dx}\right|_{x_e} = 0$, but then $\left.\frac{d^2V}{dx^2}\right|_{x_e}$ would be negative. Since the effective spring constant k would be negative under a small displacement from x_e, the force would be directed away from x_e. In this instance x_e is called an *unstable equilibrium point.*

As an illustration, we find an approximate solution for the motion of a particle of mass m in the potential energy

$$V(x) = \frac{-g^2}{x} + \frac{h^2}{x^2} \tag{2.24}$$

At the equilibrium position,

$$\left[\frac{dV(x)}{dx}\right]_{x=x_e} = \frac{g^2}{{x_e}^2} - \frac{2h^2}{{x_e}^3} = 0 \tag{2.25}$$

which gives

$$x_e = \frac{2h^2}{g^2} \tag{2.26}$$

From (2.23) the spring constant for small oscillations about x_e is

$$k = \left[\frac{d^2V(x)}{dx^2}\right]_{x=x_e} = \frac{-2g^2}{{x_e}^3} + \frac{6h^2}{{x_e}^4} \tag{2.27}$$

or upon substitution from (2.26),

$$k = \frac{g^8}{8h^6} \tag{2.28}$$

which is positive so that x_e is a point of stable equilibrium. The approximate solution to the equation of motion from (2.28), (1.62) and (1.64) is

$$x(t) - \frac{2h^2}{g^2} = a\cos\left[\left(\frac{g^4}{h^3\sqrt{8m}}\right)t - \alpha\right] \tag{2.29}$$

where a and α are arbitrary constants to be determined from the initial conditions.

2.4 Three-Dimensional Motion: Vector Notation

In three dimensions the position of a particle of mass m can be specified by its cartesian coordinates (x, y, z). Newton's second law can then be stated as the three equations

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned} \tag{2.30}$$

where (F_x, F_y, F_z) are called the x, y, z components of the force of the particle. If one chooses to use a different cartesian coordinate system, which is translated and rotated with respect to the original system, the equations of motion must have the same form. In the new coordinate system the equations of motion are

$$\begin{aligned} m\ddot{x}' &= F_{x'} \\ m\ddot{y}' &= F_{y'} \\ m\ddot{z}' &= F_{z'} \end{aligned} \tag{2.31}$$

where (x', y', z') are the coordinates of the particle in the new coordinate frame. Each of (2.31) is a linear combination of (2.30). As an example, suppose the new coordinate system has the same origin as the original system but is rotated by an angle ϕ around the z axis, as illustrated in Fig. 2-5. The coordinates of the particle in the two frames are related by

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \\ z' &= z \end{aligned} \tag{2.32}$$

By time-differentiating, we see that analogous relations hold for velocities and accelerations; *e.g.*,

$$\begin{aligned} \ddot{x}' &= \ddot{x}\cos\phi + \ddot{y}\sin\phi \\ \ddot{y}' &= -\ddot{x}\sin\phi + \ddot{y}\cos\phi \\ \ddot{z}' &= \ddot{z} \end{aligned} \tag{2.33}$$

Substituting (2.30) into (2.33), we obtain

$$\begin{aligned} m\ddot{x}' &= F_x\cos\phi + F_y\sin\phi \\ m\ddot{y}' &= -F_x\sin\phi + F_y\cos\phi \\ m\ddot{z}' &= F_z \end{aligned} \tag{2.34}$$

When we identify

$$\begin{aligned} F_{x'} &= F_x\cos\phi + F_y\sin\phi \\ F_{y'} &= -F_x\sin\phi + F_y\cos\phi \\ F_{z'} &= F_z \end{aligned} \tag{2.35}$$

the set of all three new equations is equivalent to the old set; the new equations are just linear combinations of the old equations. For instance, $m\ddot{x}' = F_{x'}$ is just $\cos\phi$ times the equation $m\ddot{x} = F_x$ plus $\sin\phi$ times the equation $m\ddot{y} = F_y$. Notice that $F_{x'}, F_{y'}, F_{z'}$ are related to F_x, F_y, F_z in the same way as x', y', z' are related to x, y, z [(2.32)].

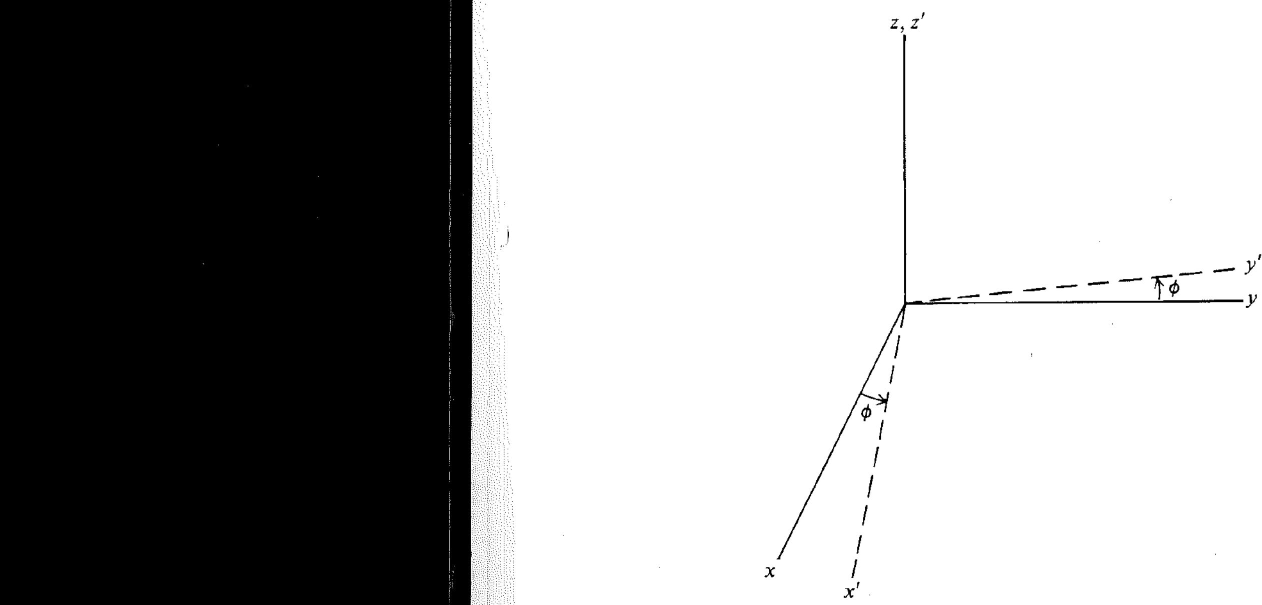


FIGURE 2-5. Two coordinate systems related by a rotation by an angle ϕ about the z axis.

To symbolize the above state of affairs and at the same time realize a great simplification in notation, we introduce the concept of a vector. A *vector* is a set of three quantities (in a three-dimensional coordinate system) whose components in differently oriented (*i.e.*, rotated) coordinate systems are related in the same way as the set of coordinates (x, y, z). Symbolically, we denote a vector with components (a_x, a_y, a_z) by $\mathbf{a}$. Examples of vectors which we have already encountered are the position vector $\mathbf{r} \equiv (x, y, z)$, the velocity vector $\mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$, the acceleration vector $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$, and the force vector $\mathbf{F} \equiv (F_x, F_y, F_z)$. The basic idea of a vector is that it is a quantity with components that change in a specific way when the coordinate system is changed [*e.g.*, (2.32)].

In vector notation Newton's second law can be written as a single equation

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{2.36}$$

This is shorthand for the set of (2.30) or (2.31). An advantage of vector notation is that no specific reference frame is necessary in this statement of the laws of motion. Vector notation is also often useful in manipulation and solving the equations of motion.

The distance of a point (x, y, z) from the origin of the coordinate system is $\sqrt{x^2 + y^2 + z^2}$. This distance is independent of the rotational orientation of the coordinate system,

$$\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2} \tag{2.37}$$

as can easily be checked for the transformation in (2.32). The above quantity is called the length, or magnitude, of $\mathbf{r}$ and is denoted by

$$|\mathbf{r}| \equiv r \equiv \sqrt{x^2 + y^2 + z^2} \tag{2.38}$$

For a general vector $\mathbf{a} = (a_x, a_y, a_z)$, the length, or magnitude, is similarly defined as

$$|\mathbf{a}| \equiv a \equiv \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \tag{2.39}$$

Since by the definition of a vector given above the components of $\mathbf{a}$ transform under rotations of the coordinate system in the same way as the components of $\mathbf{r}$, the length of $\mathbf{a}$ is independent of the orientation of the coordinate frame. A quantity, such as $|\mathbf{a}|$, that is independent of frame orientation is called a *scalar*, to distinguish it from a quantity such as F_x, which is the component of a vector, and therefore is different in different cartesian coordinate systems [see (2.35)].

If vectors are multiplied by scalars and added together by the rule

$$\alpha\mathbf{a} + \beta\mathbf{b} = (\alpha a_x + \beta b_x,\ \alpha a_y + \beta b_y,\ \alpha a_z + \beta b_z) \tag{2.40}$$

the resulting quantity is again a vector because its components transform under coordinate-system rotations according to the definition of a vector. Since any linear combination of vectors is a vector, many new vectors can be generated from the position vector $\mathbf{r}$. For instance, the relative coordinate of two particles $\mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ is a vector, as is the change of coordinate of a particle between two times:

$$\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t) = \left[x(t+\Delta t) - x(t),\ y(t+\Delta t) - y(t),\ z(t+\Delta t) - z(t)\right] \tag{2.41}$$

It follows that the velocity,

$$\mathbf{v} = \dot{\mathbf{r}} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} \tag{2.42}$$

and the acceleration,

$$\mathbf{a} = \ddot{\mathbf{r}} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t} \tag{2.43}$$

are vectors. We note that all vectors which are constructed from the difference of two position vectors (such as $\mathbf{r}_2 - \mathbf{r}_1$, $\Delta\mathbf{r}$, $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$) are unchanged by a shift in origin of the coordinate frame. Under a change in origin, all position vectors $\mathbf{r}$ are replaced by $\mathbf{r}' = \mathbf{r} + \mathbf{s}$, where $\mathbf{s}$ is a constant vector. It follows that the vectors formed from differences of two position vectors are independent of $\mathbf{s}$. Since the acceleration is unchanged by a shift in origin, the force vector must also share this property in order for (2.36) to hold in translated frames. The position vector $\mathbf{r}$ is the only vector which depends upon the origin of the coordinate system, and therefore is sometimes said not to be a true vector.

The geometrical representation of a vector as a directed line segment, or "arrow," is a powerful intuitive tool. We represent the position vector $\mathbf{r} = (x,\ y,\ z)$ by an arrow drawn from the origin to the point $(x,\ y,\ z)$, as illustrated in Fig. 2-6. The length of $\mathbf{r}$ is then just the length of the arrow. The components of $\mathbf{r}$ are the coordinates of the orthogonal projections of the arrow's point onto the coordinate axes. We can also represent an arbitrary vector $\mathbf{a}$ by an arrow, since under rotations of the coordinate frame the components of $\mathbf{a}$ transform the same way as the components of $\mathbf{r}$. The length of the arrow is proportional to the magnitude of the vector, and the projections of the arrow on the coordinate axes are proportional to the components of the vector, as illustrated in Fig. 2-7. The location of the arrow is arbitrary (so long as the arrow represents a "true" vector, not the position vector) and may be chosen for convenience. For instance, arrows representing the velocity, acceleration, or force on a particle may be attached to the point representing the position of the particle. The addition of vectors is represented by the head-to-tail construction illustrated in Fig. 2-8.

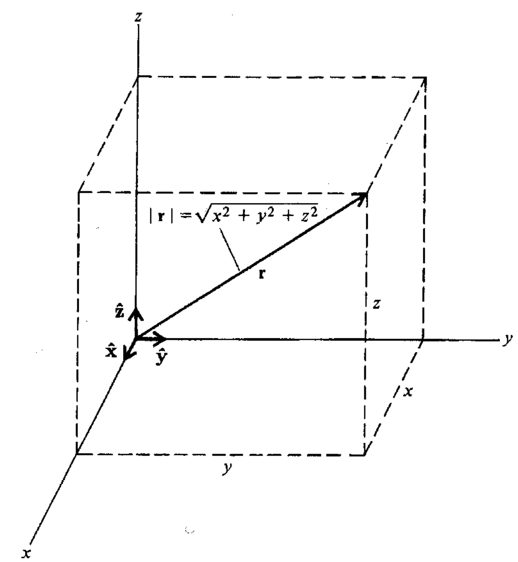


FIGURE 2-6. Position vector $\mathbf{r}$ and coordinate-system unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$.

The *dot product* of two vectors is defined as

$$\mathbf{a} \cdot \mathbf{b} \equiv a_x b_x + a_y b_y + a_z b_z \tag{2.44}$$

The dot product is a *scalar* (*i.e.*, independent of the frame orientation), as we can readily demonstrate from the identity

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z &= \tfrac{1}{2}\Big[(a_x + b_x)^2 - a_x^2 - b_x^2 + (a_y + b_y)^2 \\ &\quad - a_y^2 - b_y^2 + (a_z + b_z)^2 - a_z^2 - b_z^2\Big] \\ &= \tfrac{1}{2}\left(|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2\right)\end{aligned} \tag{2.45}$$

Since the vector magnitudes $|\mathbf{a}|$, $|\mathbf{b}|$, $|\mathbf{a} + \mathbf{b}|$ are scalars, it follows that the dot product is a scalar. From the defining (2.44), we further observe that

$$\begin{aligned}\mathbf{a} \cdot \mathbf{a} &= |\mathbf{a}|^2 \equiv a^2 \\ \mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\ (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}\end{aligned} \tag{2.46}$$

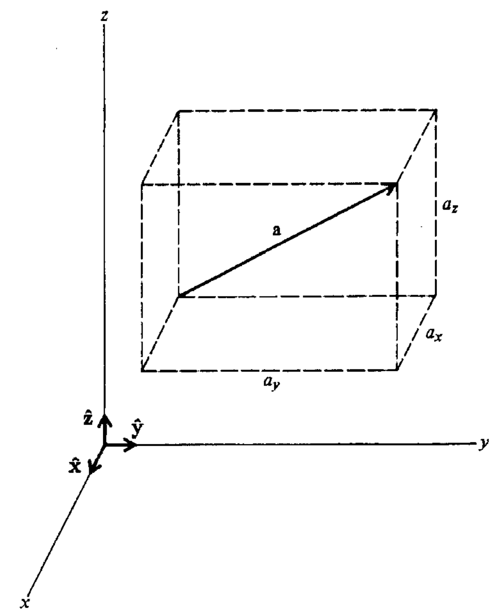


FIGURE 2-7. Arrow representation of an arbitrary vector $\mathbf{a}$.

The magnitude of the vector $\mathbf{a}+\mathbf{b}$ is given in terms of the dot product $\mathbf{a}\cdot\mathbf{b}$ by

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} \tag{2.47}$$

Furthermore, inasmuch as the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ form a triangle as illustrated in Fig. 2-8, the opposite side $|\mathbf{a}+\mathbf{b}|$ of the triangle is related by trigonometry to the adjacent sides $|\mathbf{a}|$ and $|\mathbf{b}|$ by

$$|\mathbf{a}+\mathbf{b}|^2 = a^2 + b^2 + 2ab\cos\theta \tag{2.48}$$

where θ is the angle between the arrows representing $\mathbf{a}$ and $\mathbf{b}$. Equating the above two formulas for $|\mathbf{a}+\mathbf{b}|^2$, we deduce the following result for the dot product:

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta \tag{2.49}$$

Thus the dot product represents the product of the length of one vector times the orthogonal projection of the other vector on it, as indicated in

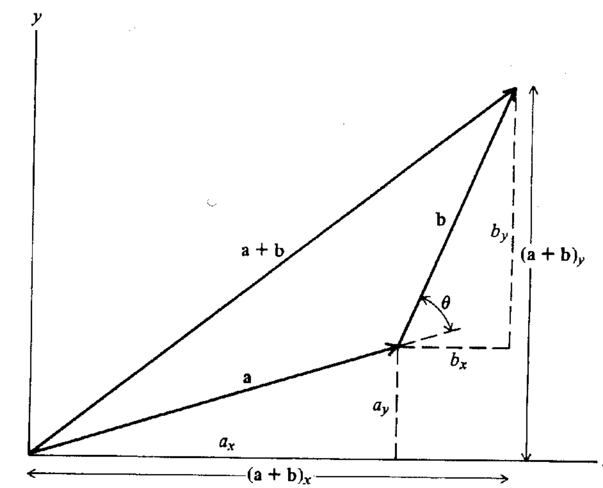


FIGURE 2-8. Head-to-tail construction of the addition of two vectors $\mathbf{a}$ and $\mathbf{b}$. (For convenience of illustration the x, y-coordinate axes are taken to lie in the plane defined by $\mathbf{a}$ and $\mathbf{b}$.)

Fig. 2-9. If $\mathbf{a}\cdot\mathbf{b} = 0$, even though $a \neq 0$ and $b \neq 0$, the angle between the vectors is 90° and the vectors are said to be *orthogonal*.

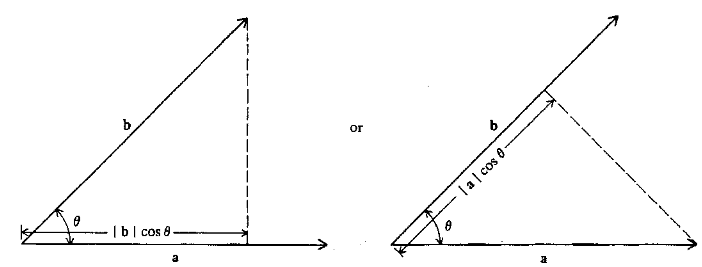


FIGURE 2-9. Geometrical illustration of the dot product.

It is useful to define a set of coordinate-axis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ of unit length $|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = |\hat{\mathbf{z}}| = 1$ which are directed along the x, y, z axes of the coordinate system, as in Fig. 2-6. The components of these orthogonal unit vectors are

$$\begin{aligned}\hat{\mathbf{x}} &= (1,0,0)\\ \hat{\mathbf{y}} &= (0,1,0)\\ \hat{\mathbf{z}} &= (0,0,1)\end{aligned} \tag{2.50}$$

From this definition the dot products of unit vectors with each other are

$$\begin{aligned}\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1\\ \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0\end{aligned} \tag{2.51}$$

In a given frame a general vector $\mathbf{a}$ can be represented in terms of the unit vectors of the frame as

$$\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \tag{2.52}$$

The sum of two vectors can be expressed as

$$\mathbf{a}+\mathbf{b} = (a_x+b_x)\hat{\mathbf{x}} + (a_y+b_y)\hat{\mathbf{y}} + (a_z+b_z)\hat{\mathbf{z}} \tag{2.53}$$

Another type of product of two vectors of considerable importance is the *cross product*, written $\mathbf{a}\times\mathbf{b}$. The cross product has three components, defined by

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})_x &= a_yb_z - a_zb_y\\ (\mathbf{a}\times\mathbf{b})_y &= a_zb_x - a_xb_z\\ (\mathbf{a}\times\mathbf{b})_z &= a_xb_y - a_yb_x\end{aligned} \tag{2.54}$$

Thus, in terms of the unit vectors of the coordinate system, the cross product is

$$\mathbf{a}\times\mathbf{b} = (a_yb_z - a_zb_y)\hat{\mathbf{x}} + (a_zb_x - a_xb_z)\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\hat{\mathbf{z}} \tag{2.55}$$

Alternatively, the definition can be symbolically written as the determinant

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{pmatrix} \tag{2.56}$$

From the symmetry properties of the determinant or directly from (2.55),

we note that

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{2.57}$$

The cross product of a vector with itself vanishes:

$$\mathbf{a}\times\mathbf{a} = 0 \tag{2.58}$$

The cross product transforms like an ordinary vector under rotations of the coordinate system. For instance, consider the transformation equation (2.32). The vectors $\mathbf{a}$ and $\mathbf{b}$ transform in the same way as the position vector $\mathbf{r}$; that is,

$$\begin{aligned}a_{x'} &= a_x\cos\phi + a_y\sin\phi & b_{x'} &= b_x\cos\phi + b_y\sin\phi\\ a_{y'} &= -a_x\sin\phi + a_y\cos\phi & b_{y'} &= -b_x\sin\phi + b_y\cos\phi\\ a_{z'} &= a_z & b_{z'} &= b_z\end{aligned} \tag{2.59}$$

The components of $\mathbf{a}\times\mathbf{b}$ in the rotated frame are then found to be

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})_{x'} &= (a_{y'}b_{z'} - a_{z'}b_{y'})\\ &= (a_yb_z - a_zb_y)\cos\phi + (a_zb_x - a_xb_z)\sin\phi\\ &= (\mathbf{a}\times\mathbf{b})_x\cos\phi + (\mathbf{a}\times\mathbf{b})_y\sin\phi\\ (\mathbf{a}\times\mathbf{b})_{y'} &= -(\mathbf{a}\times\mathbf{b})_x\sin\phi + (\mathbf{a}\times\mathbf{b})_y\cos\phi\\ (\mathbf{a}\times\mathbf{b})_{z'} &= (\mathbf{a}\times\mathbf{b})_z\end{aligned} \tag{2.60}$$

which corresponds to the transformation of (x, y, z) in (2.32). For this reason the cross product is sometimes called the vector product. However, the cross product behaves differently from ordinary vectors under *inversion* of the coordinate axes (that is, $x' = -x$, $y' = -y$, $z' = -z$). We have

$$\mathbf{r}' = -\mathbf{r} \qquad \mathbf{a}' = -\mathbf{a} \qquad (\mathbf{a}\times\mathbf{b})' = (\mathbf{a}\times\mathbf{b}) \tag{2.61}$$

A three-component quantity such as $(\mathbf{a}\times\mathbf{b})$, which behaves like a vector under rotation of the coordinate axes but does not change sign under inversion, is called an *axial vector*.

The dot product of the vector $\mathbf{a}$ with $\mathbf{a} \times \mathbf{b}$ is zero, as we show by use of (2.44) and (2.54),

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_x(a_y b_z - a_z b_y) + a_y(a_z b_x - a_x b_z) + a_z(a_x b_y - a_y b_x) = 0 \tag{2.62}$$

Equivalently

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0 \tag{2.63}$$

since $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors, and we can rename $\mathbf{a} \leftrightarrow \mathbf{b}$. Thus the cross-product vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to the vectors $\mathbf{a}$ and $\mathbf{b}$. The arrow representing $\mathbf{a} \times \mathbf{b}$ must therefore be perpendicular to the plane defined by the arrows of $\mathbf{a}$ and $\mathbf{b}$. By the definition in (2.55), the direction of $\mathbf{a} \times \mathbf{b}$ is the direction in which a right-hand screw moves when it turns from $\mathbf{a}$ toward $\mathbf{b}$, as indicated in Fig. 2-10. The square of the magnitude of the cross product

$$|\mathbf{a} \times \mathbf{b}|^2 = (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2 \tag{2.64}$$

can be rewritten

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) - (a_x b_x + a_y b_y + a_z b_z)^2 \\ &= a^2 b^2 - |\mathbf{a} \cdot \mathbf{b}|^2 \end{aligned} \tag{2.65}$$

Since a, b and $\mathbf{a} \cdot \mathbf{b}$ are scalars under rotations, the length of $\mathbf{a} \times \mathbf{b}$ is also a scalar. By substitution of (2.49), we obtain

$$|\mathbf{a} \times \mathbf{b}|^2 = a^2 b^2 (1 - \cos^2 \theta) \tag{2.66}$$

and so

$$|\mathbf{a} \times \mathbf{b}| = ab|\sin \theta| \tag{2.67}$$

where θ is the angle between the arrows representing $\mathbf{a}$ and $\mathbf{b}$. The length of $\mathbf{a} \times \mathbf{b}$ is just the area of the parallelogram, with the arrows $\mathbf{a}$ and $\mathbf{b}$ as sides.

The cross products of the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ of (2.50) are found [from (2.55)] to be

$$\begin{array}{lll} \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} & \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}} & \hat{\mathbf{x}} \times \hat{\mathbf{x}} = 0 \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} & \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}} & \hat{\mathbf{y}} \times \hat{\mathbf{y}} = 0 \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} & \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}} & \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \end{array} \tag{2.68}$$

A new kind of scalar can be formed by taking the dot product of a vector $\mathbf{a}$ with an axial vector $(\mathbf{b} \times \mathbf{c})$. This scalar, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the

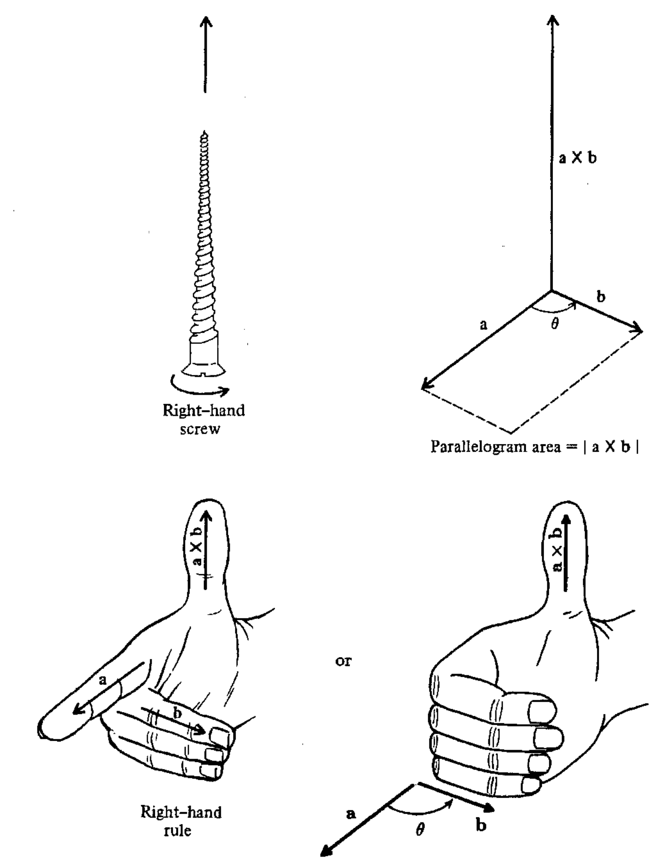


FIGURE 2-10. Geometrical illustration of the cross product.

triple product. From (2.44) and (2.56) the triple product can be written as a determinant of the vector components:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \tag{2.69}$$

The symmetry properties of the determinant under interchange of rows imply that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \tag{2.70}$$

This interchangeability of the dot and cross products,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \tag{2.71}$$

is a useful property of vector algebra. Although the triple product is a scalar under rotations it is called a *pseudoscalar* because it changes sign under coordinate inversion.

The repeated cross product of three vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ can be worked out to

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \tag{2.72}$$

When the cross products are carried out in a different order, the result is

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{2.73}$$

A useful formula for the dot product of two cross products can be derived from (2.73). We take the dot product of (2.73) with a vector $\mathbf{d}$,

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\cdot\mathbf{d} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{2.74}$$

then interchange the dot and cross products on the left-hand side to obtain

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{2.75}$$

The components of a vector are often labeled $\mathbf{a} = (a_1, a_2, a_3)$, the subscripts 1, 2, 3 denoting the x, y, z components, respectively. In this notation the dot product of two vectors can be written as

$$\mathbf{a}\cdot\mathbf{b} = \sum_i a_i b_i \tag{2.76}$$

where the summation is over $i = 1, 2, 3$. As a convenient shorthand notation, we shall often omit the $\sum_i$ symbol and simply write

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i \tag{2.77}$$

where a summation over the repeated vector component index i is implied. This is known as the *summation convention.*

From the components a_i and b_i of two vectors $\mathbf{a}$ and $\mathbf{b}$, we can form $3\times 3 = 9$ products $a_i b_j$. We denote these nine components by the symbol T_{ij}:

$$\mathrm{T}_{ij} = a_i b_j \tag{2.78}$$

In vector notation we regard the nine quantities as components of

$$\mathbb{T} = \mathbf{ab} \tag{2.79}$$

with no dot or cross between the vectors $\mathbf{a}$ and $\mathbf{b}$. This is sometimes called the *direct* or *outer product* of the vectors $\mathbf{a}$ and $\mathbf{b}$. Any such quantity whose nine elements in one coordinate system transform to those in a rotated coordinate system in the same way as a product of vector components transform is called a *tensor* (more precisely, a *tensor of second rank*). Any linear combination of tensors is also a tensor. A general tensor can always be written as a linear combination of outer products. The sum of the diagonal elements $(i = j)$ of the tensor $\mathbb{T} = \mathbf{ab}$,

$$\mathrm{T}_{11} + \mathrm{T}_{22} + \mathrm{T}_{33} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2.80}$$

is just the dot product $\mathbf{a}\cdot\mathbf{b}$. The components of the cross product $\mathbf{a}\times\mathbf{b}$ are constructed from the off-diagonal elements $(i \neq j)$ of this tensor.

If we make a dot product of the tensor $(\mathbf{ab})$ with a vector $\mathbf{c}$, we get a vector

$$\begin{aligned}(\mathbf{ab})\cdot\mathbf{c} &= \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\\ \mathbf{c}\cdot(\mathbf{ab}) &= \mathbf{b}(\mathbf{c}\cdot\mathbf{a})\end{aligned} \tag{2.81}$$

because $(\mathbf{b}\cdot\mathbf{c})$ and $(\mathbf{c}\cdot\mathbf{a})$ are scalars, and a vector multiplied by a scalar is a vector. Hence, for a general tensor $\mathbb{T}$, the dot products $\mathbb{T}\cdot\mathbf{c}$ and $\mathbf{c}\cdot\mathbb{T}$ are vectors. In terms of components,

$$\begin{aligned}(\mathbb{T}\cdot\mathbf{c})_i &= \mathrm{T}_{ij}\mathrm{c}_j\\ (\mathbf{c}\cdot\mathbb{T})_i &= \mathrm{c}_j\mathrm{T}_{ji}\end{aligned} \tag{2.82}$$

with a summation over the index j implied. The most important use of a tensor is to relate two vectors in this way. The unit tensor $\mathbb{I}$, with the property that

$$\mathbf{a}\cdot\mathbb{I} = \mathbb{I}\cdot\mathbf{a} = \mathbf{a} \tag{2.83}$$

for any vector $\mathbf{a}$, is given in terms of unit vectors by

$$\mathbb{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}} \tag{2.84}$$

The components of a second-rank tensor are often written in a 3×3 matrix array as

$$\mathbb{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{2.85}$$

and a vector $\mathbf{c}$ is represented by a column vector

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{2.86}$$

or a row vector

$$\mathbf{c} = (c_1, c_2, c_3) \tag{2.87}$$

The dot product $\mathbb{T}\cdot\mathbf{c}$ can then be worked out by matrix multiplication:

$$\mathbb{T}\cdot\mathbf{c} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} T_{11}c_1 + T_{12}c_2 + T_{13}c_3 \\ T_{21}c_1 + T_{22}c_2 + T_{22}c_3 \\ T_{31}c_1 + T_{32}c_2 + T_{33}c_3 \end{pmatrix} \tag{2.88}$$

Similarly, to evaluate $\mathbf{c}\cdot\mathbb{T}$ we use the row vector form of $\mathbf{c}$. Tensor methods are important in the treatment of rigid body dynamics, as discussed in Chapter 6.

2.5 Conservative Forces in Three Dimensions

We want to find the conditions on the force $\mathbf{F}$ for which energy conservation methods apply in three dimensions. With vector notation, Newton's laws of motion can compactly be expressed as

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}(\mathbf{r}, \mathbf{v}, t) \tag{2.89}$$

The appearance of the vectors $\mathbf{r}$ and $\mathbf{v}$ in the argument of $\mathbf{F}$ indicates that each component of $\mathbf{F}$ can depend on all the components of $\mathbf{r}$ and $\mathbf{v}$. [For example, $F_x(x, y, z, v_x, v_y, v_z, t)$.]

In analogy to our derivation in (2.3) to (2.9) of energy conservation in one-dimensional motion, we take the dot product with $\mathbf{v}$ of both sides of (2.89) to obtain

$$\mathbf{v}\cdot\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)\cdot\mathbf{v} \tag{2.90}$$

or equivalently,

$$d\left(\tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)\cdot d\mathbf{r} \tag{2.91}$$

From (2.44), the dot product $\mathbf{v}\cdot\mathbf{v}$ is

$$\mathbf{v}\cdot\mathbf{v} = v^2 = v_x^2 + v_y^2 + v_z^2 \tag{2.92}$$

Thus the differential on the left-hand side of (2.91) is the kinetic energy for three-dimensional motion. Integrating, we obtain the *work-energy theorem* in three dimensions:

$$\Delta K = K_2 - K_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}(\mathbf{r}, \mathbf{v}, t)\cdot d\mathbf{r} = \text{Work} \tag{2.93}$$

An integral of the above form is called a *line integral.* Using the definitions of the dot product it can be expressed as

$$\begin{aligned}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = &\int_{x_1}^{x_2} F_x(\mathbf{r}(x), \mathbf{v}(x), t(x))dx + \int_{y_1}^{y_2} F_y(\mathbf{r}(y), \mathbf{v}(y), t(y))dy \\ &+ \int_{z_1}^{z_2} F_z(\mathbf{r}(z), \mathbf{v}(z), t(z))dz\end{aligned} \tag{2.94}$$

where in the integral over dx the line along which the integral is carried out is described by the functions $y = y(x)$ and $z = z(x)$, and similarly for the integrals over dy and dz.

As in the one-dimensional case, we define a potential energy $V(\mathbf{r})$ by the line integral

$$V(\mathbf{r}) = -\int_{\mathbf{r}_s}^{\mathbf{r}} \mathbf{F}(\mathbf{r}')\cdot d\mathbf{r}' \tag{2.95}$$

This line integral is illustrated in Fig. 2-11. By the same reasoning as in the one-dimensional case, a necessary condition that the potential energy is a unique function of coordinate is that the force be a function of coordinate only. But it is also necessary that the value of the integral in (2.95) be independent of the integration path. Assuming this, we obtain the energy conservation condition as before:

$$E = \tfrac{1}{2}mv^2 + V(\mathbf{r}) = \text{constant} \tag{2.96}$$

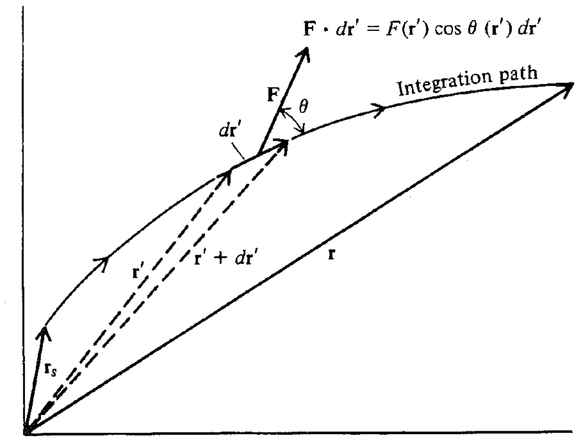


FIGURE 2-11. Geometrical interpretation of the line integral $\int_{\mathbf{r}_s}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$, where $\mathbf{r}'$ is the integration variable and $\mathbf{r}_s$, $\mathbf{r}$ are the limits of integration. The projection angle θ generally varies along the path.

Before proceeding further, we investigate the condition on the force for the above line integral to be path-independent. To find this condition on **F**, we first consider integration paths which include two adjacent sides of an infinitesimal rectangle in the y, z plane, as shown in Fig. 2-12. We locate a corner of the rectangle at the point (y_s, z_s) and calculate $V(\mathbf{r})$ at the diagonal corner $(y_s + dy,\ z_s + dz)$ by two different paths:

$$\text{Path I}: \quad (y_s,\ z_s) \to (y_s,\ z_s + dz) \to (y_s + dy,\ z_s + dz)$$
$$\text{Path II}: \quad (y_s,\ z_s) \to (y_s + dy,\ z_s) \to (y_s + dy,\ z_s + dz)$$

The value of $V(\mathbf{r})$ calculated from Path I is

$$V(\mathbf{r}) = -F_z(x_s,\ y_s,\ z_s)dz - F_y(x_s,\ y_s,\ z_s + dz)dy \tag{2.97}$$

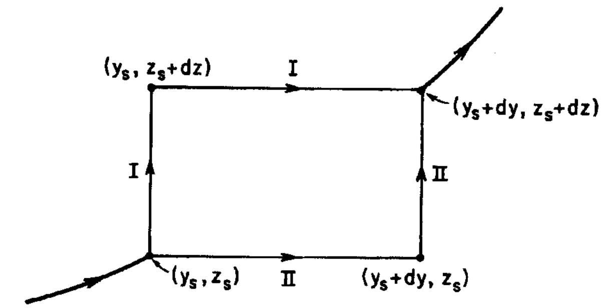


FIGURE 2-12. Integration path in the y, z plane including two alternative routes around an infinitesimal rectangle.

The corresponding result from Path II is

$$V(\mathbf{r}) = -F_y(x_s,\ y_s,\ z_s)dy - F_z(x_s,\ y_s + dy,\ z_s)dz \tag{2.98}$$

Demanding that the same $V(\mathbf{r})$ results from both integration paths yields by subtraction

$$\begin{aligned} &\left[F_y(x_s, y_s, z_s{+}dz) - F_y(x_s, y_s, z_s)\right]dy \\ -&\left[F_z(x_s, y_s{+}dy, z_s) - F_z(x_s, y_s, z_s)\right]dz = 0 \end{aligned} \tag{2.99}$$

The quantities in brackets are immediately recognizable in terms of partial derivatives as

$$\left[\frac{\partial F_y}{\partial z}(x_s,\ y_s,\ z_s)\right] dz$$

and

$$\left[\frac{\partial F_z}{\partial y}(x_s,\ y_s,\ z_s)\right] dy$$

Canceling the factor $dy\,dz$ in (2.99) gives the condition,

$$\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = 0 \tag{2.100}$$

for the force **F** to be conservative. This condition must hold for any choice of (x_s, y_s, z_s) on the curve. To derive the above condition on an energy-conserving force, we have used an integration path in the y, z plane. If instead we integrate along a differential rectangle in the x, y plane, we get

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0 \tag{2.101}$$

and similarly for a rectangle in the x, z plane.

At this point it is convenient to introduce the vector differentiation operator $\boldsymbol{\nabla}$, defined as

$$\boldsymbol{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{2.102}$$

This is called the *del operator*, also known as the *gradient* or *grad.* In vector notation the requirement in (2.100) can then be written

$$(\boldsymbol{\nabla} \times \mathbf{F})_z = 0 \tag{2.103}$$

where $\boldsymbol{\nabla}\times\mathbf{F}$ is called the *curl* of $\mathbf{F}$, sometimes written curl $\mathbf{F}$. To verify this assertion we recall the cross-product definition from (2.54):

$$(\boldsymbol{\nabla} \times \mathbf{F})_x = \nabla_y F_z - \nabla_z F_y = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \tag{2.104}$$

In general, the requirement for a force to be conservative (*i.e.*, derivable from a path-independent potential) is

$$\boldsymbol{\nabla}\times\mathbf{F} = 0 \tag{2.105}$$

where from (2.55) with $\mathbf{a} = \boldsymbol{\nabla}$ and $\mathbf{b} = \mathbf{F}$ we obtain the complete expansion for $\boldsymbol{\nabla} \times \mathbf{F}$ in cartesian coordinates:

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}} \tag{2.106}$$

To summarize the preceding discussion, we have shown that if $\boldsymbol{\nabla} \times \mathbf{F} = 0$ then $\int_{\mathbf{r}_s}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$ is path-independent and there is therefore a unique potential energy.

Conversely, if the potential-energy function $V(\mathbf{r})$ exists, then from (2.95) we can express $\mathbf{F}(\mathbf{r})$ in terms of it. The differential of (2.95) reads

$$dV(\mathbf{r}) = -\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \tag{2.107}$$

Comparing the right-hand side with the total differential dV

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = (\boldsymbol{\nabla}V)\cdot d\mathbf{r} \tag{2.108}$$

we see

$$\mathbf{F}(\mathbf{r}) = -\boldsymbol{\nabla}V(\mathbf{r}) \tag{2.109}$$

In component form we can write

$$F_k(\mathbf{r}) = -\nabla_k V(\mathbf{r}) = -\frac{\partial V(\mathbf{r})}{\partial x_k} \tag{2.110}$$

Forming the curl of this $\mathbf{F}$, we find

$$\boldsymbol{\nabla} \times \mathbf{F} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla}\, V(\mathbf{r}) = 0 \tag{2.111}$$

since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} = 0$. Hence, there is a potential energy if and only if $\boldsymbol{\nabla} \times \mathbf{F} = 0$.

Among the most important physical examples of conservative forces are central forces. The magnitude of a central force at each point depends only on the distance from a certain point, the force center, and the direction of the force is radial to the force center, *i.e.*, towards or away from it. The gravitational and Coulomb forces are both of this type. If the force center is at the origin of the coordinate system, the force field has the form

$$\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}} \tag{2.112}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$. If $F(r) < 0$ the force is towards the center and is called *attractive* (at that value of r), while if $F(r) > 0$ the force is *repulsive.* More generally, the center can be at any point $\mathbf{r}_0$ and the expression for the force looks like (2.112) with $\mathbf{r}$ replaced by $\mathbf{r} - \mathbf{r}_0$ (and r by $|\mathbf{r} - \mathbf{r}_0|$). The superposition of several such central forces with arbitrary centers is also conservative.

To prove explicitly that central forces are conservative, it suffices to take the center at the origin, (2.112). Using cartesian components

$$F_x = \frac{x}{r}F(r), \qquad F_y = \frac{y}{r}F(r), \qquad F_z = \frac{z}{r}F(r) \tag{2.113}$$

we construct dV according to (2.107)

$$\begin{aligned} dV &= -(F_x dx + F_y dy + F_z dz) \\ &= -\frac{F(r)}{r}(x\,dx + y\,dy + z\,dz) = -F(r)dr \end{aligned} \tag{2.114}$$

In the last step we have used the differential of

$$r^2 = x^2 + y^2 + z^2 \tag{2.115}$$

Since the right-hand side of (2.114) depends only on the radial coordinate r (not on θ or ϕ), its integral is path-independent. This establishes the conservative nature of a central force; equivalently we could have directly shown that $\nabla \times \mathbf{F} = 0$.

From (2.114) we obtain the central potential energy from the force law as

$$V(r) = -\int_{r_s}^{r} F(r')dr' \tag{2.116}$$

The above formula could have been found directly from the line integral (2.95). For instance, from the gravitational force law

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{2.117}$$

the gravitational potential energy due to a mass M at $\mathbf{r} = 0$ is

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \left(-\frac{GMm}{r'^2}\hat{\mathbf{r}}'\right) \cdot d\mathbf{r}' \tag{2.118}$$

and with $d\mathbf{r}' = \hat{\mathbf{r}}'dr' + \hat{\boldsymbol{\theta}}'r'd\theta'$ we have $\hat{\mathbf{r}}' \cdot d\mathbf{r}' = dr'$ and therefore

$$V(\mathbf{r}) = \int_{\infty}^{\mathbf{r}} \frac{GMm}{r'^2}dr' = -\frac{GMm}{r} \tag{2.119}$$

Only the magnitude of the velocity enters into the three dimensional energy conservation law and hence the launch direction of a rocket is arbitrary as long as the rocket does not hit the earth. If $v \geq v_{\text{esc}}$, the spacecraft will not return.

2.6 Motion in a Plane

For the analysis of the mechanical motion of some systems cartesian coordinates are not the most convenient choice. For example, some kinds of motion in a plane can frequently be described more simply in terms of polar coordinates (r, θ) than (x, y). Since Newton's equations of motion do not have the same form in polar coordinates, we cannot just substitute $\ddot{r}$ and $\ddot{\theta}$ for $\ddot{x}$ and $\ddot{y}$ in Newton's equations; Newton's equations have the same form only in different *cartesian* coordinate systems. Therefore we must do some algebra to express Newton's equations in terms of polar coordinates.

In cartesian coordinates we have

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \tag{2.120}$$

which is written in vector form as

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{2.121}$$

with

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y \qquad \text{and} \qquad \mathbf{F} = \hat{\mathbf{x}}F_x + \hat{\mathbf{y}}F_y \tag{2.122}$$

In polar coordinates the vector $\mathbf{r}$ is given by

$$\mathbf{r} = \hat{\mathbf{x}}r\cos\theta + \hat{\mathbf{y}}r\sin\theta \equiv \hat{\mathbf{r}}r \tag{2.123}$$

The unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in the cartesian system do not change with time. The differential $d\mathbf{r}$ is thus

$$\begin{aligned} d\mathbf{r} &= \hat{\mathbf{x}}(\cos\theta dr - r\sin\theta d\theta) + \hat{\mathbf{y}}(\sin\theta dr + r\cos\theta d\theta) \\ &= (\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta)dr + (-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)rd\theta \\ &= \hat{\mathbf{r}}dr + \hat{\boldsymbol{\theta}}rd\theta \end{aligned} \tag{2.124}$$

where in the last form we have defined the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ to be along the direction of $d\mathbf{r}$ when only r or θ, respectively, are increased:

$$\begin{aligned} \hat{\mathbf{r}} &= \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta \\ \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta \end{aligned} \tag{2.125}$$

[see Fig. 2-13 for a geometrical representation.] By direct calculation it is seen that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = 1$.

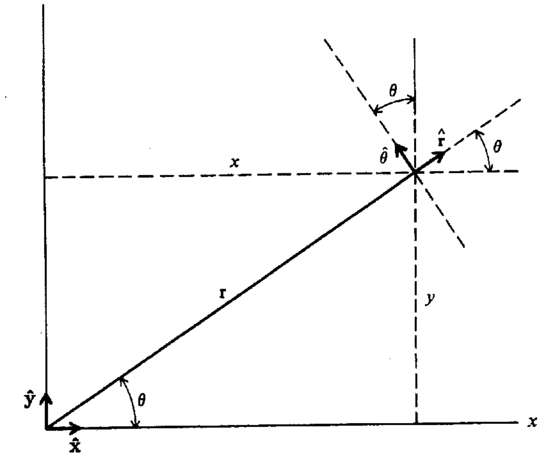


FIGURE 2-13. Polar variables and unit vectors for motion in a plane.

Dividing $d\mathbf{r}$ by dt we obtain

$$\mathbf{v} = \dot{\mathbf{r}} = \hat{\mathbf{r}}\dot{r} + \hat{\boldsymbol{\theta}} r\dot{\theta} \tag{2.126}$$

Differentiating $\mathbf{v}$ with respect to time, we find

$$\dot{\mathbf{v}} = \hat{\mathbf{r}}\ddot{r} + \dot{\hat{\mathbf{r}}}\dot{r} + \hat{\boldsymbol{\theta}} r\ddot{\theta} + (\dot{\hat{\boldsymbol{\theta}}} r + \hat{\boldsymbol{\theta}}\dot{r})\dot{\theta} \tag{2.127}$$

The derivatives of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are found from (2.125) to be

$$\begin{aligned} \dot{\hat{\mathbf{r}}} &= \dot{\theta}\hat{\boldsymbol{\theta}} \\ \dot{\hat{\boldsymbol{\theta}}} &= -\dot{\theta}\hat{\mathbf{r}} \end{aligned} \tag{2.128}$$

Substituting these results for $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ into the expressions for $\mathbf{v}$ and $\dot{\mathbf{v}}$ above, we arrive at

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + \dot{\theta} r\hat{\boldsymbol{\theta}} \\ \mathbf{a} &= \dot{\mathbf{v}} = \hat{\mathbf{r}}(\ddot{r} - r\dot{\theta}^2) + \hat{\boldsymbol{\theta}}(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{aligned} \tag{2.129}$$

In polar coordinates we write $\mathbf{F} = \hat{\mathbf{r}}F_r + \hat{\boldsymbol{\theta}}F_\theta$, so Newton's law $m\ddot{\mathbf{r}} = \mathbf{F}$ is

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= F_r \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= F_\theta \end{aligned} \tag{2.130}$$

Notice the difference between the left-hand side of these equations and the cartesian equations (2.30). In other coordinate systems the structure of the equations in motion can be even more complicated, and the derivation of results similar to (2.130) correspondingly more difficult. In the next chapter we will see that the derivation of the equations of motion is greatly simplified using Lagrange's method.

2.7 Simple Pendulum

The plane pendulum is a familiar system of historical importance whose motion cannot be described in terms of elementary functions. Nevertheless, we can easily find the equations of motion and an approximate solution for small oscillations.

The simple plane pendulum consists of a point mass m at the end of a weightless rod or string of constant length ℓ which swings back and forth in a vertical plane. We take the origin of the coordinate system at the pivot point, with x positive down and y positive to the right. The two forces acting upon the mass m are gravity and the tension in the rod T_r, as shown in Fig. 2-14. If T_r is positive, the force on m is radially inward. In terms of polar coordinates the force components are

$$\begin{aligned} F_r &= mg\cos\theta - T_r \\ F_\theta &= -mg\sin\theta \end{aligned} \tag{2.131}$$

Newton's law in polar coordinates, (2.130), with $r = \ell$, together with the above forces gives

$$-m\ell\dot{\theta}^2 = mg\cos\theta - T_r \tag{2.132}$$

$$m\ell\ddot{\theta} = -mg\sin\theta \tag{2.133}$$

The first equation can be ignored if we are not interested in the value of $T_r(t)$, but only in the motion $\theta(t)$. After solving the second equation for $\theta(t)$, the first equation then gives $T_r(\theta)$. The second equation, (2.133), can be written as

$$\ddot{\theta} + \omega_0^2\sin\theta = 0 \tag{2.134}$$

where $\omega_0 = \sqrt{g/\ell}$.

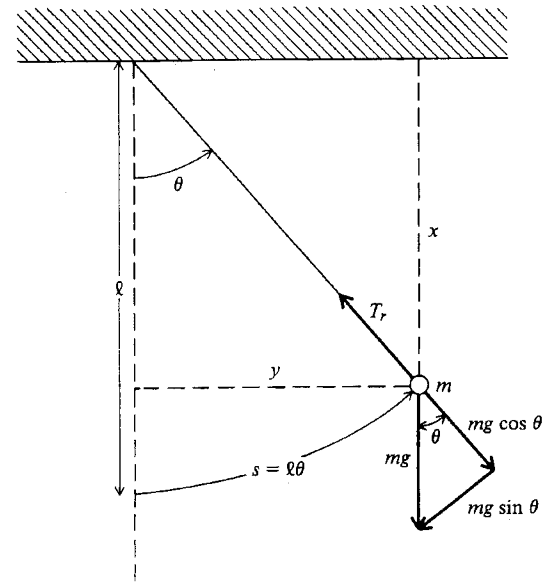


FIGURE 2-14. Simple plane pendulum.

For small oscillations, $|\theta| \ll 1$, we can approximate $\sin\theta \approx \theta$, with θ in radians, to obtain

$$\ddot{\theta} + \omega_0^2 \theta = 0 \tag{2.135}$$

This is readily recognizable as simple harmonic motion in θ with angular frequency ω_0 [see (1.61) through (1.64)]. The general solution of (2.135) is

$$\theta = a\cos(\omega_0 t + \alpha) \tag{2.136}$$

where the arbitrary constants a and α are to be fixed by the initial conditions. The period of small oscillations,

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\ell}{g}} \tag{2.137}$$

is independent of a and α. This approximately amplitude-independent feature of the period of motion, called *isochronism*, is incorporated in the pendulum clock.

To solve for the motion exactly, without a small displacement approximation, we can use the energy method. Using the chain rule,

$$\ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{\dot{\theta}d\dot{\theta}}{d\theta} = \frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right) \tag{2.138}$$

we integrate (2.134)

$$\int_0^{\dot{\theta}} d\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{g}{\ell}\int_{\theta_0}^{\theta} \sin\theta d\theta \tag{2.139}$$

where θ_0 is the angle at which $\dot{\theta} = 0$ (θ_0 is the maximum angle of the motion). The evaluation of the integrals gives

$$\dot{\theta}^2 = \frac{2g}{\ell}(\cos\theta - \cos\theta_0) \tag{2.140}$$

If we multiply by $\frac{1}{2}m\ell^2$, we can recognize this as the statement of energy conservation.

Before finding the motion, we make the observation that we can use (2.140) to eliminate $\dot{\theta}$ from the formula (2.132) for the tension

$$T_r = mg\cos\theta + m\ell\dot{\theta}^2 \tag{2.141}$$

to get

$$T_r = 3mg\cos\theta - 2mg\cos\theta_0 \tag{2.142}$$

Equation (2.140) gives the angular velocity to be

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta - \cos\theta_0} \tag{2.143}$$

This differential equation is separable; it can be written as

$$\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \pm\sqrt{\frac{2g}{\ell}}\,dt \tag{2.144}$$

and integrated

$$\sqrt{\frac{2g}{\ell}}\int_0^t dt' = -\int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\cos\theta' - \cos\theta_0}} \tag{2.145}$$

The minus sign is required if θ_0 is positive (noting that $\theta < \theta_0$ and the θ

integral is negative). This integral can be cast into a standard form by substitution of the identity

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \tag{2.146}$$

in (2.145)

$$2\sqrt{\frac{g}{\ell}}\,t = -\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta'/2)}} \tag{2.147}$$

If we now introduce a new variable

$$\sin\beta' = \frac{\sin(\theta'/2)}{\sin(\theta_0/2)} \tag{2.148}$$

the solution in (2.147) becomes

$$\sqrt{\frac{g}{\ell}}\,t = \int_{\beta}^{\pi/2}\frac{d\beta'}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\beta'}} \tag{2.149}$$

Setting $\theta = 0$, hence $\beta = 0$, gives a quarter period, so the period is given by

$$\tau = 4\sqrt{\frac{\ell}{g}}\int_{0}^{\pi/2}\frac{d\beta}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\beta}} \tag{2.150}$$

In terms of the simple harmonic period $\tau_0 = 2\pi\sqrt{\frac{\ell}{g}}$,

$$\frac{\tau}{\tau_0} = \frac{2}{\pi}\int_{0}^{\pi/2}\frac{d\beta}{\sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2\beta}} \tag{2.151}$$

This integral is known as the complete elliptic integral of the first kind. It cannot be evaluated in closed form, but numerical evaluations are available in tabular form or from computer software. To find an approximation to the period for small angular displacements, the integrand can be

expanded by power series and then integrated term by term

$$\begin{aligned}\frac{\tau}{\tau_0} &= \frac{2}{\pi}\int_0^{\pi/2} d\beta\left[1 + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\sin^2\beta + \cdots\right]\\ &= \frac{2}{\pi}\left[\beta + \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right)\left(\beta - \frac{\sin 2\beta}{2}\right) + \cdots\right]_0^{\pi/2}\\ &= \left[1 + \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right)\right] + \cdots\end{aligned} \tag{2.152}$$

Again using the approximation of θ_0 small and $\sin^2(\theta_0/2) \approx \theta_0^2/4$, we find

$$\tau = \tau_0\left(1 + \frac{\theta_0^2}{16} + \cdots\right) \tag{2.153}$$

The period is increased over the simple harmonic period. The fractional lengthening of the period is

$$\frac{\tau - \tau_0}{\tau_0} = \frac{\theta_0^2}{16} \tag{2.154}$$

For a 30° maximal displacement, the fractional lengthening of the period

$$\frac{\tau - \tau_0}{\tau_0} = \frac{1}{16}\left(\frac{30^\circ}{57.3^\circ}\right)^2 = 0.017 \tag{2.155}$$

is less than 2 percent. For a pendulum clock of period 1 s and $\theta_0 = 5°$, the amplitude θ_0 must be regulated to $\pm 3°$ if the clock is to be accurate to 1 min/day. If the clock is desired to have an accuracy of 1 s/day, θ_0 must be regulated to 0.06 degrees.

2.8 Coupled Harmonic Oscillators

In physical problems that can be approximated by several small oscillations there is usually a coupling between the oscillators. As a specific example, we investigate the motion of two simple pendulums whose bobs are connected by a spring, as indicated in Fig. 2-15.

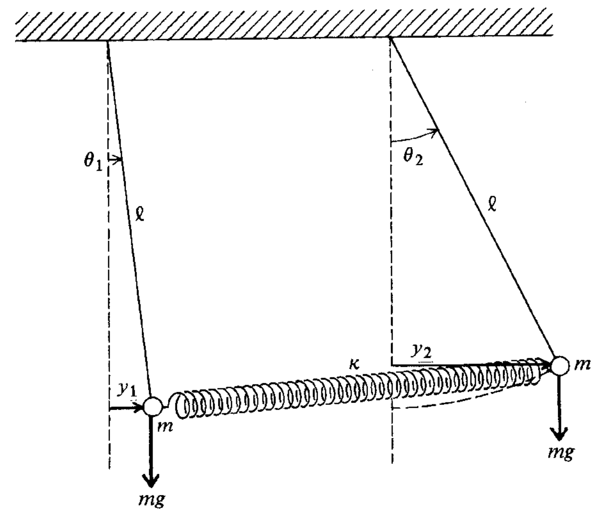


FIGURE 2-15. Two simple pendulums coupled by a spring.

For small angular displacements the equation of motion of a single isolated pendulum is given by (2.135).

$$\ddot{\theta} + \omega_0^2 \theta = 0 \tag{2.156}$$

This equation can be alternatively expressed in terms of the x and y coordinates of the pendulum bob. For $\theta \ll 1$, we have

$$\begin{aligned} x &= \ell \cos\theta \approx \ell \\ y &= \ell \sin\theta \approx \ell\theta \end{aligned} \tag{2.157}$$

and (2.135) becomes

$$\ddot{y} + \omega_0^2 y = 0 \tag{2.158}$$

In this approximation the pendulum executes simple harmonic motion in the horizontal direction. The pendulum spring system of Fig. 2-15 is therefore equivalent for small displacements to the three-spring system of Fig. 2-16, with spring constants $k = m\omega_0^2 = mg/\ell$ for the outer springs and κ for the inner spring.

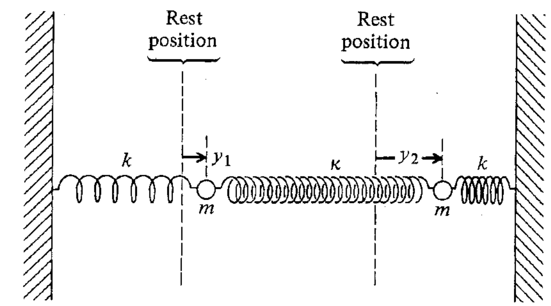


FIGURE 2-16. Equivalent three-spring system for the coupled-pendulum system of Fig. 2-15.

The equations of motion can be obtained by considering the forces on each mass separately. First fix y_2 at $y_2 = 0$ and imagine a positive y_1 displacement. The restoring forces on m_1 are $-ky_1$ due to the extension of left spring and $-\kappa y_1$ due to the compression of the right spring. Now include a positive y_2 displacement and consider its effect on m_1. The stretching of the middle spring gives a positive force κy_2 on m_1. Combining these forces the equation of motion for m_1 is

$$m\ddot{y}_1 = -ky_1 - \kappa y_1 + \kappa y_2 \tag{2.159}$$

Note that the restoring force due to the middle spring depends only on the difference $y_1 - y_2$. A similar exercise applied to the right-hand mass yields

$$m\ddot{y}_2 = -ky_2 - \kappa y_2 + \kappa y_1 \tag{2.160}$$

The differential equations of motion for the system are thus

$$\begin{aligned} m\ddot{y}_1 &= -ky_1 - \kappa(y_1 - y_2) \\ m\ddot{y}_2 &= -ky_2 + \kappa(y_1 - y_2) \end{aligned} \tag{2.161}$$

To solve these differential equations, we look for linear combinations of y_1 and y_2 that yield differential equations of simple harmonic form. Later in this Section we discuss the solution to coupled equations in some generality. In the present case it suffices to take the sum and the difference of the equations to uncouple them. If we add the equations, we find

$$m(\ddot{y}_1 + \ddot{y}_2) = -k(y_1 + y_2) \tag{2.162}$$

and if we subtract

$$m(\ddot{y}_1 - \ddot{y}_2) = -(k + 2\kappa)(y_1 - y_2) \tag{2.163}$$

The solutions of these two uncoupled equations are found directly from (1.64) to be

$$\begin{aligned} y_1 + y_2 &= a_+ \cos(\omega_+ t + \alpha_+) \\ y_1 - y_2 &= a_- \cos(\omega_- t + \alpha_-) \end{aligned} \tag{2.164}$$

where the angular frequencies are

$$\begin{aligned} \omega_+ &= \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\ell}} \\ \omega_- &= \sqrt{\frac{k + 2\kappa}{m}} = \sqrt{\frac{g}{\ell} + \frac{2\kappa}{m}} \end{aligned} \tag{2.165}$$

The combinations $(y_1 + y_2)$ and $(y_1 - y_2)$ oscillate independently, simple harmonically, and are called the *normal modes.* The motion of the pendulum bobs is in general a superposition of the two normal modes of vibration and from (2.164) we have

$$\begin{aligned} y_1 &= \tfrac{1}{2}a_+ \cos(\omega_+ t + \alpha_+) + \tfrac{1}{2}a_- \cos(\omega_- t + \alpha_-) \\ y_2 &= \tfrac{1}{2}a_+ \cos(\omega_+ t + \alpha_+) - \tfrac{1}{2}a_- \cos(\omega_- t + \alpha_-) \end{aligned} \tag{2.166}$$

The four constants a_+, a_-, α_+, and α_- in (2.166) are to be determined by the initial conditions. If only the amplitude of one normal mode is excited, that is, only a_+ or a_- is non-zero, the bobs swing in phase with frequency ω_+ or out of phase with frequency ω_-, as illustrated in Fig. 2-17. We note that ω_+ does not depend on κ since the middle spring is never stretched.

In the weak coupling limit $\kappa \ll k$, the coupling between the two pendulums causes a gradual interchange of energy between the two oscillators. To demonstrate this we suppose that both bobs are initially at rest and the motion of the system is started by displacing the first bob

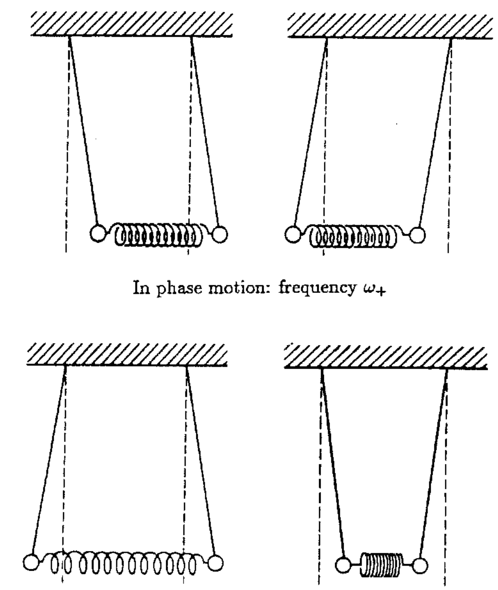


FIGURE 2-17. Normal modes for the coupled-pendulum system.

by a distance a. When these initial conditions are imposed on (2.166), we obtain

$$\begin{aligned} y_1 &= \frac{a}{2}(\cos\omega_+ t + \cos\omega_- t) \\ y_2 &= \frac{a}{2}(\cos\omega_+ t - \cos\omega_- t) \end{aligned} \tag{2.167}$$

From trigonometric identities for the sum and difference of cosine functions, (2.167) can be written as

$$\begin{aligned} y_1 &= a\cos\left(\frac{\omega_- + \omega_+}{2}t\right)\cos\left(\frac{\omega_- - \omega_+}{2}t\right) \\ y_2 &= a\sin\left(\frac{\omega_- + \omega_+}{2}t\right)\sin\left(\frac{\omega_- - \omega_+}{2}t\right) \end{aligned} \tag{2.168}$$

At time $t = \pi/(\omega_- - \omega_+)$, the first pendulum has come to rest and all the energy has been transferred through the coupling to the second oscillator. For the weak coupling limit, $\omega_- - \omega_+ \ll \omega_+$, the last factors in (2.168) are slowly varying functions of time. These slowly varying factors constitute an envelope for the rapidly oscillating sinusoidal factors of argument $[(\omega_- + \omega_+)/2]t$, as illustrated in Fig. 2-18. This phenomenon is known as *beats*. The beat frequency is $(\omega_- - \omega_+)/2$, and the period of the envelope of the amplitude is $2\pi/(\omega_- - \omega_+)$.

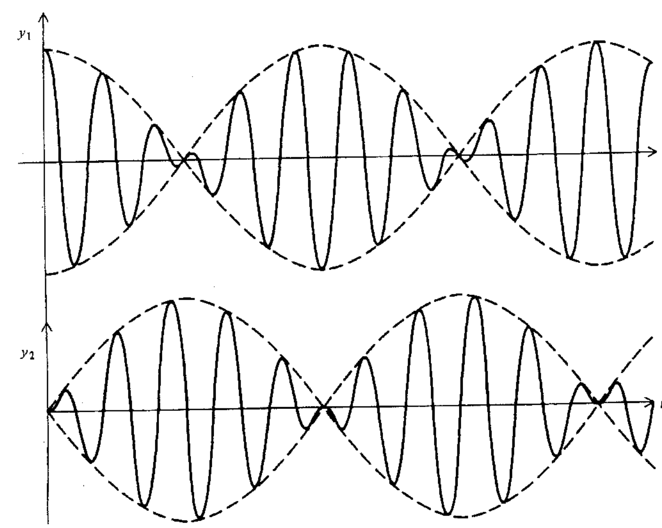


FIGURE 2-18. Beats and envelope exhibited by the coordinates of two weakly coupled oscillators.

We solved the coupled differential equations (2.161) by adding and subtracting to obtain uncoupled equations. We now discuss an alternative method of solution which can be more straightforwardly generalized to more complex coupled systems with different masses and spring constants. Denoting

$$\omega_0^2 \equiv \frac{k}{m} \qquad \text{and} \qquad \Delta^2 \equiv \frac{\kappa}{m} \tag{2.169}$$

the equations (2.161) become

$$\begin{aligned} \ddot{y}_1 + (\omega_0^2 + \Delta^2)y_1 - \Delta^2 y_2 = 0 \\ \ddot{y}_2 + (\omega_0^2 + \Delta^2)y_2 - \Delta^2 y_1 = 0 \end{aligned} \tag{2.170}$$

This coupled set of differential equations is homogeneous and linear with constant coefficients. It has complex solutions of the form

$$\begin{aligned} y_1 &= C_1 e^{i\omega t} \\ y_2 &= C_2 e^{i\omega t} \end{aligned} \tag{2.171}$$

where C_1 and C_2 are constants that are, in general, complex. The physical solutions are the real parts of these complex y_1, y_2 solutions. Substitution of (2.171) into (2.170) gives a pair of coupled linear equations for C_1 and C_2 that can be written in matrix form as

$$\begin{pmatrix} -\omega^2 + \omega_0^2 + \Delta^2 & -\Delta^2 \\ -\Delta^2 & -\omega^2 + \omega_0^2 + \Delta^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0 \tag{2.172}$$

For a non-trivial solution the determinant of the matrix must vanish, giving

$$\left(-\omega^2 + \omega_0^2 + \Delta^2\right)^2 = \Delta^4 \tag{2.173}$$

Solving the quadratic equation for ω^2, in this simple case by taking the square root, we find two solutions, $\omega^2 = \omega_+^2$ and $\omega^2 = \omega_-^2$, where

$$\begin{aligned} \omega_+^2 &= \omega_0^2 \\ \omega_-^2 &= \omega_0^2 + 2\Delta^2 \end{aligned} \tag{2.174}$$

Then solving for C_2/C_1 from (2.172) with these ω^2 values gives

$$\begin{aligned} C_2/C_1 &= +1 \qquad \text{for} \quad \omega^2 = \omega_+^2 \\ C_2/C_1 &= -1 \qquad \text{for} \quad \omega^2 = \omega_-^2 \end{aligned} \tag{2.175}$$

We shall parameterize the complex constant C_1 for solutions $\omega_\pm$ by

$$(C_1)_\pm = \tfrac{1}{2} a_\pm e^{i\alpha_\pm} \tag{2.176}$$

where $a_\pm$ and $\alpha_\pm$ are real. Then the most general motion is given by the linear superpositions

$$\begin{aligned} y_1 &= \tfrac{1}{2} a_+ e^{i(\omega_+ t + \alpha_+)} + \tfrac{1}{2} a_- e^{i(\omega_- t + \alpha_-)} \\ y_2 &= \tfrac{1}{2} a_+ e^{i(\omega_+ t + \alpha_+)} - \tfrac{1}{2} a_- e^{i(\omega_- t + \alpha_-)} \end{aligned} \tag{2.177}$$

The general physical solution obtained by taking the real part of the superposition is the same as (2.166). If only one mode is excited, by a particular choice of initial conditions, the system will oscillate with a single frequency—the normal mode frequency ω_+ or ω_-.

Equation (2.172) is an example of an eigenvalue problem with matrix equation

$$\begin{pmatrix} \omega_0^2+\Delta^2 & -\Delta^2 \\ -\Delta^2 & \omega_0^2+\Delta^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \omega^2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad (2.178)$$

The eigenvalues are $\omega^2 = \omega_+^2$ and $\omega^2 = \omega_-^2$. The vectors $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}_+$ and $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}_-$ are known as eigenvectors. Two eigenvectors corresponding to different eigenvalues are orthogonal, or normal. The modes of motion corresponding to ω_+^2 and ω_-^2 are accordingly called normal modes.

PROBLEMS

2.1 Potential Energy

2-1. The potential energy of a mass element dm at a height z above the earth's surface is $dV = (dm)gz$. Compute the potential energy in a pyramid of height h, square base $b \times b$, and mass density ρ. The Great Pyramid of Khufu is 147 m high and has a base of 234×234 m. Estimate its potential energy using $\rho = 2.5$ g/cm^3 for the density of its material. If an average worker lifted 50 kg through a distance of 1 m each minute of a 10 hr work day, estimate the person-years of labor expended in the construction of the Great Pyramid. This ignores friction and the considerable effort required to quarry and transport the stone.

2-2. The Turkish bow of the 15th and 16th centuries greatly outperformed western bows. The draw force $F(x)$ of the Turkish bow versus the bowstring displacement x (for x negative) is approximately represented by a quadrant of the ellipse

$$\left(\frac{F(x)}{F_{\text{max}}}\right)^2 + \left(\frac{x+d}{d}\right)^2 = 1$$

Calculate the work done by the bow in accelerating the arrow, taking $F_{\text{max}} = 360$ N, $d = 0.7$ m, and arrow mass $m = 34$ g. Assuming that all of the work ends up as arrow kinetic energy, determine the maximum range R of the arrow. (The actual range is about 430 m.) Compare with the range for a bow that acts like a simple spring force with the same F_{max} and d.

2.2 Gravitational Escape

2-3. From the radius and mass ratios

$$R\,(\text{moon})/R\,(\text{earth}) \simeq 1/3.66$$
$$M\,(\text{moon})/M\,(\text{earth}) \simeq 1/81.6$$

show that the gravitational acceleration on the moon and earth are related by

$$g\,(\text{moon})/g\,(\text{earth}) \simeq 1/6$$

Find the escape velocity from the surface of the moon.

2-4. A projectile is fired from the surface of the earth to the moon. Neglecting the orbital motion of the moon, what is the minimum velocity of impact on the surface of the moon? Take into account the gravitational pull of both the moon and the earth.

2-5. An iron meteor enters the earth's atmosphere at the escape velocity. Compute the kinetic energy per molecule and compare with the rough vaporization energy of 1 eV/molecule.

2.3 Small Oscillations

2-6. A particle of mass m moves under the action of a force

$$F = -F_0 \sinh(ax) = -\frac{F_0}{2}(e^{ax} - e^{-ax})$$

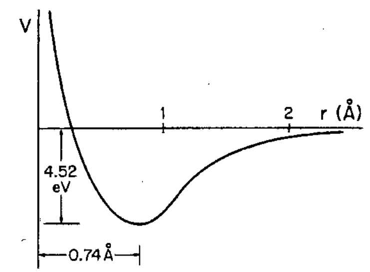
where $a > 0$. Sketch the potential energy, discuss the motion, and solve for the frequency of small oscillation if there exists a point of stability.

2-7. A particle moves subject to the potential energy

$$V(x) = V_0\left(\frac{a}{x} + \frac{x}{a}\right)$$

where V_0 and a are positive. Locate any equilibrium points, determine which are stable and obtain the frequency of small oscillations about those points.

2-8. Estimate the spring constant in units of eV/Å^2 for the hydrogen (H_2) molecule from the potential energy curve shown below, where r is the distance between protons. From the spring constant and the "reduced mass" $m = \frac{1}{2}m_{\text{proton}}$, compute the vibrational frequency ν. This frequency corresponds to infrared light.



2.4 Three Dimensional Motion: Vector Notation

2-9. Given the vectors

$$\mathbf{A} = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}} \qquad \mathbf{B} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - 2\hat{\mathbf{z}}$$

find

a) $A = |\mathbf{A}|$ and $B = |\mathbf{B}|$,

b) $\mathbf{A} \cdot \mathbf{B}$ and the angle θ between $\mathbf{A}$ and $\mathbf{B}$,

c) $\mathbf{A} \times \mathbf{B}$ and the angle θ between $\mathbf{A}$ and $\mathbf{B}$.

From b) and c) deduce the consistent choice of the angle θ.

2-10. A force field is given by

$$F_x = kyz \sin kxy$$
$$F_y = kxz \sin kxy$$
$$F_z = -\cos kxy$$

a) Evaluate $\nabla \times \mathbf{F}$ to show that $\mathbf{F}$ is conservative.

b) If the reference potential energy at $(x = 0,\ y = 0,\ z = 0)$ is zero, compute the potential energy at the point $(x = 1.0,\ y = 1.0,\ z = 1.0)$. Use any convenient path, such as along the axes.

c) Using a different path, compute the potential energy at the same point to check path independence.

2-11. Consider the following force:

$$\mathbf{F} = -K(x - z)^2(\hat{\mathbf{x}} - \hat{\mathbf{z}})$$

a) Show that it is conservative.

b) Find the potential energy $V(\mathbf{r})$ assuming $V(\mathbf{0}) = 0$.

c) Calculate $\nabla V(\mathbf{r})$ to verify that it gives $\mathbf{F}$ correctly.

2-12. For a central force $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$ show directly that $\nabla \times \mathbf{F} = 0$ for $r \neq 0$.

2-13. Determine whether or not the force $\mathbf{F} = \mathbf{r} \times \mathbf{a}$ (where $\mathbf{a}$ is a constant vector) leads to a conservative potential energy. Compute $\int \mathbf{F} \cdot d\mathbf{r}$ around a circle of radius R in the x, y plane centered at $\mathbf{r} = 0$.

2-14. Show that a force consisting of a superposition of N central forces with centers at $\mathbf{r} = \mathbf{r}_k$, $k = 1$ to N, is also a conservative force. *Hint: the* $\mathbf{r}_k$ *are constant vectors so* $\nabla_{\mathbf{r}} = \nabla_{\mathbf{r}-\mathbf{r}_k}$. *Use problem 2-12.*

2-15. Show that the ∇ operator can be expressed in spherical coordinates as

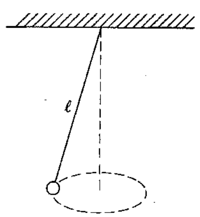
$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta}\frac{\partial}{\partial \phi}$$

where $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ are perpendicular unit vectors in the direction of increasing $(r, \theta, \phi,)$. (*Hint: Use* $df = d\mathbf{r} \cdot \nabla f$ *where* $d\mathbf{r}$ *is given by* $d\mathbf{r} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\theta}}r d\theta + \hat{\boldsymbol{\phi}}r \sin\theta d\phi$ *and* f *is an arbitrary scalar function. Express* df *in terms of partial derivatives.*) Show that the ∇ operator in cylindrical coordinates (ρ, ϕ, z) is

$$\nabla = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

2.6 Motion in a Plane

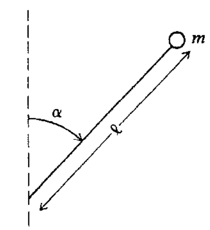
2-16. The bob of a pendulum moves in a horizontal circle as illustrated. Find the angular frequency of the circular motion in terms of the angle θ and the length ℓ of the string. This is known as a conical pendulum



2.7 Simple Pendulum

2-17. A hemispherical thin glass goblet of radius $R = 5\,\mathrm{cm}$ will withstand a perpendicular force of up to $2\,\mathrm{N}$. If a 100-g steel ball is released from rest at the lip of the goblet and allowed to slide down the inside, at what point on the goblet will the ball break through? Neglect the radius of the ball.

2-18. A mass m is attached at one end of a massless rigid rod of length ℓ, and the rod is suspended at its other end by a frictionless pivot, as illustrated. The rod is released from rest at an angle $\alpha_0 < \pi/2$ with the vertical. At what angle α does the force in the rod change from compression to tension?



2-19. A ball of mass m is suspended by a string of length ℓ. For what ranges of the total energy will the string remain taut when the ball swings in an arc in a vertical plane? Choose the lowest point on the arc as the reference point for the potential energy.

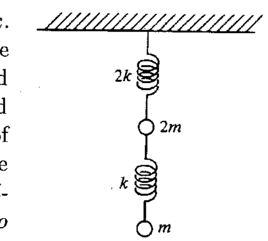
2-20. A physics professor holds a bowling ball suspended as a pendulum. The ball is initially 1.9 m above the floor, the pendulum wire is 7 m in length and the ceiling height is 7.5 m. The bowling ball has diameter 0.15 m, mass 15 kg; the drag parameter is $C_D = 0.4$ and the air density is 1 kg/m^3. The professor gently releases the ball just in front of her nose and confidently expects that it will return short of its original position.

a) Estimate the work done by friction over one period, using (1.17) for the drag coefficient. Approximate the motion by that of a simple pendulum and use (2.140) in your calculation of this work.

b) Using the work-energy theorem, estimate the change in height when the ball swings back to the professor and by the given geometry find how close the pendulum returns to its release point.

2.8 Coupled Harmonic Oscillators

2-21. A mass $2m$ is suspended from a fixed support by a spring with spring constant $2k$. A second mass m is suspended from the first mass by a spring of constant k. Find the equation of motion for this coupled system and determine the frequencies of oscillation of normal modes. Neglect the masses of the springs. *Hint: It is easiest to choose the coordinates of the two masses at their equilibrium positions.*



2-22. A mass m is suspended from a support by a spring with spring constant $m\omega_1^2$. A second mass m is suspended from the first by a spring with spring constant $m\omega_2^2$. A vertical harmonic force $F_0 \cos \omega t$ is applied to the upper mass. Find the steady-state motion for each mass. Examine what happens when $\omega = \omega_2$.