Rutherford Scattering

1. For a two-particle collision the center of mass is defined by

$$\vec{p}\,' = m_1\vec{v}_{1i}' + m_2\vec{v}_{2i}' = m_1\vec{v}_{1f}' + m_2\vec{v}_{2f}' = 0$$

where the primes refer to quantities in the center-of-mass (CM) frame. In the laboratory frame

$$\vec{v}_1 = \vec{v}_1\,' + \vec{V}_0$$

where $\vec{V}_0$ is the CM velocity in the lab frame. Show that in the lab frame

$$K_{lab} = \frac{1}{2}\left(m_1 {v_1'}^2 + m_2 {v_2'}^2\right) + \frac{1}{2}MV_0^2$$

where K is the kinetic energy and $M = m_1 + m_2$.

2. The relative velocity between the two particles in the collision is

$$\vec{v}_{rel} = \vec{v}_1\,' - \vec{v}_2\,'$$

in the CM. Show the following relationships.

$$\vec{v}_1\,' = \frac{m_2}{m_1 + m_2}\vec{v}_{rel} \qquad \text{and} \qquad \vec{v}_2\,' = -\frac{m_1}{m_1 + m_2}\vec{v}_{rel}$$

Use these last results and show

$$K_{lab} = \frac{1}{2}\mu v_{rel}^2 + \frac{1}{2}MV_0^2$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad .$$

3. The kinetic energy in the CM is the following.

$$K_{cm} = \frac{1}{2}\mu v_{rel}^2$$

Show that in polar coordinates K_{rel} can be written as

$$K_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

where r and θ are the polar components of the relative vector between the two particles in the CM.

4. Starting from the energy integral for central force motion

$$\int \frac{dr}{\sqrt{r^2(\frac{2\mu E}{l^2}r^2 + \frac{2\mu\alpha}{l^2}r - 1)}} = \theta + C$$

make the change of variable $u = 1/r$ to obtain a modified integral.

$$-\int \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu\alpha}{l^2}u - u^2}} = \theta + C$$

Integrate this result and show the final expression (known as the orbit equation) can be written in the following way

$$\frac{1}{r} = \frac{\mu\alpha}{l^2} + \frac{\mu\alpha}{l^2}\epsilon\cos(\theta - \theta_0)$$

where μ is the reduced mass (referred to as m in your text), E is the energy, l is the angular momentum, $\alpha = Gm_1m_2$ or $\alpha = -Z_1Z_2e^2/4\pi\epsilon_0$, and ϵ is the eccentricity

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha^2}} \quad .$$

5. When the projectile starts its journey it is essentially infinitely far away and its angle is equal to $\theta = 180°$. Use this information and the orbit equation (see Problem 4) to show that $\theta_0 = -\arccos(1/\epsilon)$. The negative sign is required for the projectile to start at $\theta = 180°$.

6. When the projectile finally reaches the detector the target-detector distance r is essentially infinite compared to the sizes of the particles and distances between them when they are strongly interacting. Let θ_s be the final, asymptotic scattering angle of the projectile and show the following two statements are true using the orbit equation.

$$\theta_s = \theta_0 + \arccos\left(-\frac{1}{\epsilon}\right) \qquad\qquad \sin\frac{\theta_s}{2} = \frac{1}{\epsilon}$$

Clearly point out any sign choices you make. You may find the trigonometric identity $\arccos(-x) = \pi - \arccos(x)$ helpful.

7. Show that

$$\epsilon = \sqrt{1 + \left(\frac{\mu v_{1i}^2 b}{\alpha}\right)^2}$$

where v_{i1} is the initial velocity of the projectile (referred to as v_0 in your text) and b is the impact parameter. Start with the equation in Problem 4 for ϵ.

8. Combine the results of problems 6 and 7 to prove the following expression.

$$\sin\frac{\theta_s}{2} = \frac{1}{\sqrt{1 + \left(\frac{\mu v_{1i}^2 b}{\alpha}\right)^2}}$$

Invert this relationship to get the impact parameter b in terms of the scattering angle θ_s and show the following expression is true.

$$b = \frac{|\alpha|}{\mu v_{1i}^2}\cot\frac{\theta_s}{2}$$

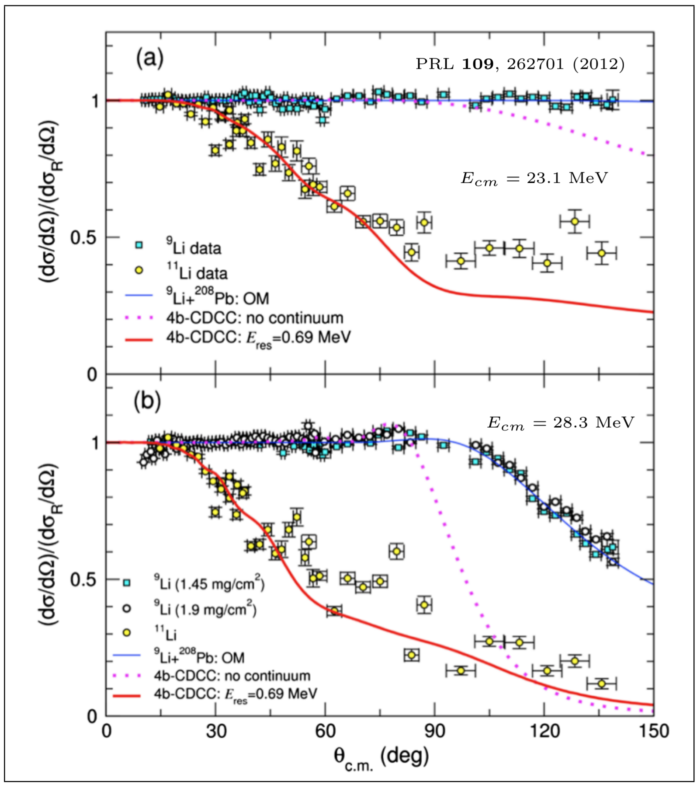
9. The data in the figure below show the result of a measurement of the differential cross section $d\sigma/d\omega$ of ^{9}Li scattering off a lead target (blue points in upper panel, blue and white points in the lower panel). Starting from the equations here

$$\frac{1}{r} = \frac{\mu\alpha}{l^2}\left[1 + \epsilon\cos\left(\theta - \theta_0\right)\right] \qquad \frac{1}{\epsilon} = \sin\frac{\theta_s}{2}$$

where

$$\epsilon = \sqrt{1 + \frac{2E_{cm}l^2}{\mu\alpha^2}} \qquad \alpha = Z_1 Z_2 e^2$$

generate an expression for the distance-of-closest-approach (DOCA) in terms of the angle θ_{DOCA} associated that point on the trajectory of the ^{9}Li. Explain how you arrive at your result. Pick one of the ^{9}Li data sets and extract the DOCA.



10. The website below contains links to *Mathematica* notebooks with data for elastic scattering of ^{4}He on gold (^{197}Au) at energies of $E_1 = 23.65\ MeV$ and $E_2 = 27.96\ MeV$. Pick one of the links, download the *Mathematica* notebook and generate a plot of the data including the uncertainties. Another link on the website is to a notebook entitled `plotting1.nb` which has samples of the commands you will need. Compare your data with the Rutherford cross section on the data plot. Clearly show your calculations. What is the size of the gold nucleus? You should start from the orbit equation to answer this question.

If your last name starts with the letters A-J pick the 23.65-MeV notebook, otherwise pick the 27.95-MeV one.

`https://facultystaff.richmond.edu/~ggilfoyl/cm/notebooks.html`