Rutherford Scattering

1. For a two-particle collision the center of mass is defined by

$$\vec{p}\,' = m_1\vec{v}_{1i}\,' + m_2\vec{v}_{2i}\,' = m_1\vec{v}_{1f}\,' + m_2\vec{v}_{2f}\,' = 0$$

   where the primes refer to quantities in the center-of-mass (CM) frame. In the laboratory frame

$$\vec{v}_1 = \vec{v}_1\,' + \vec{V}_0$$

   where $\vec{V}_0$ is the CM velocity in the lab frame. Show that in the lab frame

$$K_{lab} = \frac{1}{2}\left(m_1 {v_1'}^2 + m_2 {v_2'}^2\right) + \frac{1}{2}MV_0^2$$

   where K is the kinetic energy and $M = m_1 + m_2$.

2. The relative velocity between the two particles in the collision is

$$\vec{v}_{rel} = \vec{v}_1\,' - \vec{v}_2\,'$$

   in the CM. Show the following relationships.

$$\vec{v}_1\,' = \frac{m_2}{m_1+m_2}\vec{v}_{rel} \qquad \text{and} \qquad \vec{v}_2\,' = -\frac{m_1}{m_1+m_2}\vec{v}_{rel}$$

   Use these last results and show

$$K_{lab} = \frac{1}{2}\mu v_{rel}^2 + \frac{1}{2}MV_0^2$$

   where

$$\mu = \frac{m_1 m_2}{m_1+m_2} \qquad .$$

3. The kinetic energy in the CM is the following.

$$K_{cm} = \frac{1}{2}\mu v_{rel}^2$$

   Show that in polar coordinates K_{rel} can be written as

$$K_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

   where r and θ are the polar components of the relative vector between the two particles in the CM.

4. Starting from the energy integral for central force motion

$$\int \frac{dr}{\sqrt{r^2(\frac{2\mu E}{l^2}r^2 + \frac{2\mu\alpha}{l^2}r - 1)}} = \theta + C$$

make the change of variable $u = 1/r$ to obtain a modified integral.

$$-\int \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu\alpha}{l^2}u - u^2}} = \theta + C$$

Integrate this result and show the final expression (known as the orbit equation) can be written in the following way

$$\frac{1}{r} = \frac{\mu\alpha}{l^2} + \frac{\mu\alpha}{l^2}\epsilon\cos(\theta - \theta_0)$$

where μ is the reduced mass (referred to as m in your text), E is the energy, l is the angular momentum, $\alpha = Gm_1m_2$ or $\alpha = -Z_1Z_2e^2/4\pi\epsilon_0$, and ϵ is the eccentricity

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha^2}} \qquad .$$

5. When the projectile starts its journey it is essentially infinitely far away and its angle is equal to $\theta = 180°$. Use this information and the orbit equation (see Problem 1) to show that $\theta_0 = -\arccos(1/\epsilon)$. The negative sign is required for the projectile to start at $\theta = 180°$.

6. When the projectile finally reaches the detector the target-detector distance r is essentially infinite compared to the sizes of the particles and distances between them when they are strongly interacting. Let θ_s be the final, asymptotic scattering angle of the projectile and show the following two statements are true using the orbit equation.

$$\theta_s = \theta_0 + \arccos\left(-\frac{1}{\epsilon}\right) \qquad\qquad \sin\frac{\theta_s}{2} = \frac{1}{\epsilon}$$

Clearly point out any sign choices you make. You may find the trigonometric identity $\arccos(-x) = \pi - \arccos(x)$ helpful.

7. Show that

$$\epsilon = \sqrt{1 + \left(\frac{\mu v_{1i}^2 b}{\alpha}\right)^2}$$

where v_{i1} is the initial velocity of the projectile (referred to as v_0 in your text) and b is the impact parameter. Start with the equation in Problem 1 for ϵ.

8. Combine the results of problems 6 and 7 to prove the following expression.

$$\sin\frac{\theta_s}{2} = \frac{1}{\sqrt{1 + \left(\frac{\mu v_{1i}^2 b}{\alpha}\right)^2}}$$

Invert this relationship to get the impact parameter b in terms of the scattering angle θ_s and show the following expression is true.

$$b = \frac{|\alpha|}{\mu v_{1i}^2}\cot\frac{\theta_s}{2}$$

9. The data in the figure below show the result of a measurement of the differential cross section $d\sigma/d\omega$ of ^{9}Li scattering off a lead target (blue points in upper panel, blue and white points in the lower panel). Starting from the equations here

$$\frac{1}{r} = \frac{\mu\alpha}{l^2}\left[1 + \epsilon\cos\left(\theta_s - \theta_0\right)\right] \qquad \frac{1}{\epsilon} = \sin\frac{\theta_s}{2}$$

where

$$\epsilon = \sqrt{1 + \frac{2E_{cm}l^2}{\mu\alpha^2}} \quad \alpha = Z_1 Z_2 e^2$$

generate an expression for the distance-of-closest-approach (DOCA) in terms of the angle θ_{DOCA} associated that point on the trajectory of the ^{6}Li. Explain how you arrive at your result. Pick one of the data sets and extract the DOCA.