**Physics 303**

**Normal Modes**

1. Consider the following matrices.

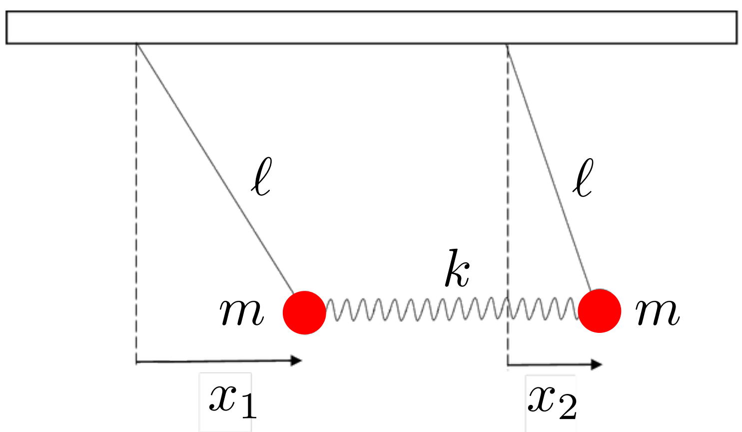
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \tag{1}$$

   What is the secular equation for each? What are the eigenvalues and eigenvectors for each? You should be able to do these problems with just paper and pencil except perhaps factoring the polynomial you get when you calculate the determinant. An introduction to some of the tools in *Mathematica* that are useful here is on the course website here: `https://facultystaff.richmond.edu/~ggilfoyl/cm/notebooks/eigen1.nb`.

2. The secular equation for the identical, coupled pendula shown in the figure below is the following.

$$\begin{bmatrix} \frac{\kappa+k}{m} - \omega^2 & -k \\ -k & \frac{\kappa+k}{m} - \omega^2 \end{bmatrix} \tag{2}$$

   What are the eigenvalues and eigenvectors in terms of $k$, $\kappa$, and $m$?




3. When the masses of the coupled pendulum shown in the figure above are no longer equal (call the left mass $m_1$ and the right mass $m_2$) the equations of motion become the following.

$$m_1\ddot{x}_1 = -m_1\left(\frac{g}{l}\right)x_1 - k(x_1 - x_2) \tag{3}$$

$$m_2\ddot{x}_2 = -m_2\left(\frac{g}{l}\right)x_2 + k(x_1 - x_2) \tag{4}$$

   If one of the normal coordinates is

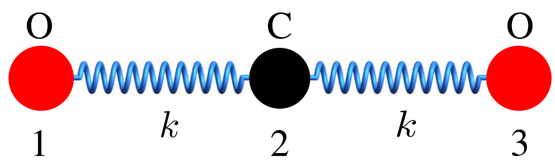
$$x_+ = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{5}$$

then what is the eigenvalue $\omega_1$? If the other normal coordinate is

$$x_- = x_1 - x_2 \tag{6}$$

then what is the eigenvalue $\omega_2$?

4. Our classical model of the $CO_2$ molecule is shown below. What is the Lagrangian for this system? What are the equations of motion?




5. The equations of motion for our classical model of the $CO_2$ molecule are

$$m_O\ddot{x}_1 - k(x_2 - x_1) = 0 \tag{7}$$

$$m_C\ddot{x}_2 - k(x_1 + x_3 - 2x_2) = 0 \tag{8}$$

$$m_O\ddot{x}_3 + k(x_3 - x_2) = 0 \tag{9}$$

where $x_i$ are the displacements from equilibrium (see figure below), $m_O$ is the oxygen mass, $m_C$ is the carbon mass, and $k$ is the spring constant parameterizing the C – O bond. Assume that $x_i = c_i e^{i\omega t}$ where $i = 1, 2, 3$ and apply Equations 7-9.

   (a) What is the secular equation for this system of equations?

   (b) Show the eigenvalues of the system are the following.

$$\omega_1 = 0 \qquad \omega_2 = \pm\sqrt{\frac{k}{m_O} + \frac{2k}{m_C}} \qquad \omega_3 = \pm\sqrt{\frac{k}{m_O}} \tag{10}$$

6. Use the results of the previous problem to find the eigenvectors that correspond to each eigenvalue. Make a sketch of the different motions that each eigenvector describes. What is the general solution for this system?

7. Consider the following set of initial conditions.

$$\text{for } t = 0 \qquad x_1 = -0.2 \text{ Å and } x_2 = 0.0 \text{ Å and } x_3 = 0.2 \text{ Å} \tag{11}$$

Get the particular solution for this set of initial conditions in terms of the eigenvectors you found in the previous problem.

8. Vary the set of initial conditions in the previous problem so you excite the asymmetric mode in $CO_2$. Get the particular solution for this set of initial conditions in terms of the eigenvectors you found in Problem 5.

9. Determine the eigenvalues, eigenvectors, and general solution of a system with two coordinates with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 + x_2^2) + V_{12}x_1x_2 \tag{12}$$

and describe in words the motion for each eigenvector. Normal coordinates are combinations of the positions of particles in a system that oscillate at one of the eigenvalues of the system. Consider the following combinations of the coordinates of the Lagrangian here.

$$x_- = \frac{x_1 - x_2}{\sqrt{2}} \qquad \text{and} \qquad x_+ = \frac{x_1 + x_2}{\sqrt{2}}$$

Rewrite the Lagrangian in these new coordinates. Comment on any special features of the result.