This assignment covers conditional probability and expected value. You must cite any conversations that have contributed to your solutions and turn in only work that you understand and have written up yourself. Solutions to optional questions do not need to be submitted, but you are responsible for any material covered. You must submit this homework in \LaTeX. Only submit your final pdfs, not your .tex source files.

1. (10 points) Say Frosty is a good boy with probability 0.7, or a bad boy with probability 0.3. If Frosty is a good boy, he gets a treat with probability 0.5, but if Frosty is a bad boy, he gets a treat with probability 0.2.

You see Dr. Szajda giving Frosty a treat. Based on your observation, what is the probability that Frosty was a good boy?

Solution:

2. (20 points) You flip a coin that is Heads with probability $p$ 100 times.
   
   (a) (Example) What is the probability that exactly $k$ of these flips are heads?

   Solution: Call the number of heads $H$. From the Binomial distribution,
   
   $$ \Pr[H = k] = \binom{100}{k} p^k (1-p)^{n-k} $$

   (b) What is the probability that there is at least 1 heads among the first 25 flips?

   Solution:

   (c) Conditioned on the fact that there is at least 1 heads among the first 100 flips, what is the probability that there is at least 1 heads among the first 25 flips? Use the law of conditional probability.

   Solution:

   (d) Conditioned on the fact that there is exactly 1 heads among the first 100 flips, what is the probability that there is exactly 1 heads among the first 25 flips? Use the law of conditional probability.

   Solution:

   (e) How could you have derived the above answer more easily?

   Solution:

3. (20 points) You flip a coin that is Heads with probability $p$ repeatedly.
(a) (Example) What is the probability that the first Heads occurs on the kth flip.

**Solution:** Call the number of flips needed until you see the first heads \( X \). From the Geometric distribution,
\[
P[X = k] = (1 - p)^{k-1} p
\]

(b) What is the probability that the first Heads occurs after the first 300 flips? For full credit, give the simplest expression possible (no summation notation).

**Solution:**

(c) Conditioned on the fact that the first Heads happens in the first 300 flips, what is the probability that it occurs in the first 100 flips? Use the law of conditional probability. For full credit, give the simplest expression possible (no summation notation).

**Solution:**

(d) Conditioned on the fact that the first Heads doesn’t appear in the first 100 flips, what is the probability that it occurs in the first 300 flips? Use the law of conditional probability.

**Solution:**

(e) How could you have derived the above answer more easily?

**Solution:**

4. (50 points) For this experiment, I throw \( n \) balls into \( b \) buckets, with each ball equally likely to land in each bucket. Let \( B_i \) be the number of balls that land in bucket \( i \).

(a) (10 points) Find the distribution of \( B_1 \). That is, find an expression for \( P(B_1 = k) \) in terms of \( n, b, k \). In particular, what is the probability that \( B_1 \) is empty?

**Solution:**

(b) (10 points) Find \( P[B_1 = k \land B_2 = l] \) for general \( k, l \) such that \( k + l \leq n \). Use this answer to find an unsimplified expression for \( P[B_1 = k | B_2 = l] \). Are \( B_1 \) and \( B_2 \) independent?

**Solution:**

(c) (10 points) Find an easier way to derive \( P[B_2 = l | B_1 = k] \) using symmetry. Think about it this way: just assume \( k \) balls went into bucket 1. What’s left?

**Solution:**

(d) (10 points) Find \( E[B_1] \).

**Solution:**
(e) (10 points) Find the expected number of empty buckets.

Solution:

5. (25 points) (Bonus) This problem illustrates a famous “paradox” in probability theory.

In the mystical land of Twotwen-Teetoo, you can assume that every pet is either a cat or dog with equal probability, and that pet is equally likely to be born on any day of the week, and that these two attributes are independent, and the species and birthdays of different pets are also independent of each other.

Two residents of Twotwen-Teetoo, Prateek and Doug, have the following conversation. Prateek says “I have two pets.”

(a) Without knowing anything more, what is the probability that Prateek has two cats?

Solution:

(b) Doug asks “Do you have a cat?” Prateek replies “Yes.”

Now, with this new information, what is the probability that Prateek has two cats?

Solution:

(c) (10 points) Doug asks further “Do you have a cat that was born on a thursday?” Prateek replies “Yes.”

Now what is the probability that Prateek has two cats?

Solution:

Hint: The answer to all three of these questions are different, and none are 1/2. Keep carefully in mind what information you are assuming in the conditional probabilities for each of the above.