



Games With Incomplete Information: Bayesian Nash Equilibrium

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On the Agenda

- 1 Private vs. Public Information
- 2 Bayesian game
- 3 How do we model Bayesian games?
- 4 Bayesian Nash equilibrium
- 5 Exercises

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Introduction

- ▶ In many game theoretic situations, one agent is unsure about the payoffs or preferences of others
- ▶ Examples:
 - Auctions: How much should you bid for an object that you want, knowing that others will also compete against you?
 - Market competition: Firms generally do not know the exact cost of their competitors
 - Signaling games: How should you infer the information of others from the signals they send
 - Social learning: How can you leverage the decisions of others in order to make better decisions

Introduction

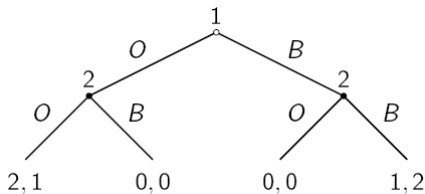
- ▶ We would like to understand what is a game of incomplete information, a.k.a. Bayesian games.
- ▶ First, we would like to differentiate private vs. public information.
- ▶ Example: Battle of Sex (BoS): "Coordination Game" (public information)

In Sequential BoS, all information is public, meaning everyone can see all the same information:

Introduction

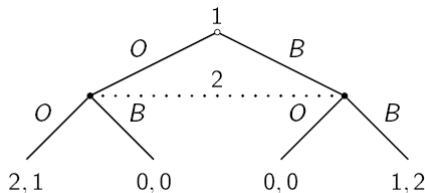
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In Sequential BoS, all information is public, meaning everyone can see all the same information:



Private vs. Public Information

- ▶ In this extensive-form representation of regular BoS, Player 2 cannot observe the action chosen by Player 1.



- ▶ The previous is a game of (complete but) imperfect information because players are unaware of the actions chosen by other player.
- ▶ However, they know who the other players are and their possible strategies/actions are. (The information is complete or public)
- ▶ Imagine that player 1 does not know whether player 2 wishes to meet or wishes to avoid player 1. Therefore, this is a situation of *incomplete information*, also sometimes called *asymmetric* or *private information*.

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Bayesian game

- ▶ In games of incomplete information players may or may not know some information about the other players, e.g. their "type", their strategies, payoffs or preferences.
- ▶ Example: Tinder BoS

Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1's preferences. So Player 1 thinks that with probability $1/2$ she is playing the game on the left and with probability $1/2$ she is playing the game on the right.

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	<i>O</i>	<i>B</i>
<i>O</i>	2, 1	0, 0
<i>B</i>	0, 0	1, 2

2 wishes to meet

	<i>O</i>	<i>B</i>
<i>O</i>	2, 0	0, 2
<i>B</i>	0, 1	1, 0

2 wishes to avoid

- ▶ This is an example of a game in which one player does not know the payoffs of the other.

Bayesian game

- ▶ More examples:
 - Bargaining over a surplus and you aren't sure of the size
 - Buying a car of unsure quality
 - Job market: candidate is of unsure quality
 - Juries: unsure whether defendant is guilty
 - Auctions: sellers, buyers unsure of other buyers' valuations
- ▶ When some players do not know the payoffs of the others, a game is said to have incomplete information. It's also known as a Bayesian game.

Bayesian game

- ▶ Example: First-price auction (game with incomplete information)
 1. I have a copy of the Mona Lisa that I want to sell for cash
 2. Each of you has a private valuation for the painting, only known to you
 3. I will auction it off to the highest bidder
 4. Everyone submits a bid (sealed → simultaneous)
 5. Highest bidder wins the painting, *pays their bid*
 6. If tie, I will flip a coin

Bayesian game

- ▶ Example: Second-price auction (game with incomplete information)
 1. I have a copy of the Mona Lisa that I want to sell for cash
 2. Each of you has a private valuation for the painting, only known to you
 3. I will auction it off to the highest bidder
 4. Everyone submits a bid (sealed → simultaneous)
 5. Highest bidder wins the painting, *pays the second-highest bid*
 6. If tie, I will flip a coin

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How do we model Bayesian games?

- ▶ Formally, we can define Bayesian games, or "incomplete information games" as follows:
 - A set of players: $I = \{1, 2, \dots, n\}$
 - A set of States: Ω e.g. good or bad car.
 - A signaling function that goes into type space and is one-to-one: $\tau_i : \Omega \rightarrow T_i$.
 - Pure strategies that are profile of actions conditional on player's type: $\sigma_i : T_i \rightarrow A_i$
 - Individual Utility of outcome y , given actions $(a_1, a_2, \dots, a_n)$:

$$U_i(\sigma_1, \dots, \sigma_n | t_i) = \sum_{a_1} \dots \sum_{a_n} u_i(y, \sigma_1(t_1(\omega)), \dots, \sigma_n(t_1(\omega))) \cdot p_i(y | t_i(\omega))$$

- ▶ What would be the BR of player i ? $\max_{\sigma_i} U_i(\sigma_1, \dots, \sigma_n | t_i)$ (not solvable)
- ▶ Player i needs to know what $-i$ knows about him. Also, Player i needs to know what i knows about $-i$. Moreover, i needs to know what $-i$ know about him conditional on what i know about $-i$, and so on

How do we model Bayesian games?

- ▶ Harsanyi (1968) doctrine: There is a prior about the states of the nature that is common knowledge
- ▶ This is also known as the common prior assumption
- ▶ With this assumption we can turn Bayesian games into games with imperfect information.
- ▶ This is a very strong assumption, but very convenient because any private information is included in the description of the types.
- ▶ With the common prior players can form beliefs about others' type and each player understands others' beliefs about his or her own type, and so on
- ▶ When players are not sure about the game they are playing you may consider:
 - Random events are considered an act of nature (that determine game structure)
 - Treat nature as another (non-strategic) player
 - Draw nature's decision nodes in extensive form
- ▶ Treat game as extensive form game with imperfect info: players may/may not observe nature's action

How do we model Bayesian games?

- Recall: BoS variant

Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1's preferences. So Player 1 thinks that with probability $1/2$ she is playing the game on the left and with probability $1/2$ she is playing the game on the right.

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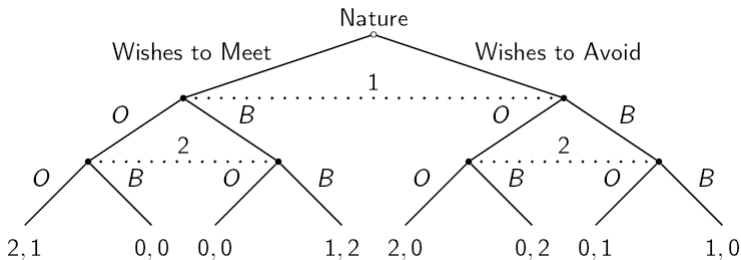
	<i>O</i>	<i>B</i>
<i>O</i>	2, 0	0, 2
<i>B</i>	0, 1	1, 0

2 wishes to avoid

- Let's put this into extensive form.

How do we model Bayesian games?

- BoS variant in extensive form:



How do we model Bayesian games?

- ▶ When players are not sure about other players' preferences:
 - Consider a game where each players has private information about his preferences.
 - That can be model as $u_i(\sigma_i, \sigma_{-i}, \theta_i)$ where $\theta_i \in T_i$.
 - Here we are assuming that θ_i is the type of player i .
 - Note that we are assuming that each player knows its own type, but that information is not public

How do we model Bayesian games?

- ▶ When players are not sure about other players' preferences:
 - An example of a game where players don't know the preferences of the others can be the one represented by the following normal form:

1\2	L	R
T	$2\theta_1, 3\theta_2$	1,1
B	1,0	0,0

- Each player i knows his own type, but types are not public information

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Bayesian Nash equilibrium

- ▶ Bayesian Nash equilibrium is a straightforward extension of NE:
- ▶ Each type of player chooses a strategy that maximizes expected utility given the actions of all types of other players and that player's beliefs about others' types.
- Example: Let us consider the previous game:

1\2	L	R
T	$2\theta_1, 3\theta_2$	1,1
B	1,0	0,0

- It is common knowledge among the two players that each player i 's type θ_i is independently drawn from the uniform distribution on $[0, 1]$.
- Let us derive a pure strategy Bayesian Nash Equilibrium in this game.

Bayesian Nash equilibrium

1\2	L	R
T	$2\theta_1, 3\theta_2$	1,1
B	1,0	0,0

- We first note that player 1 has a dominant strategy to choose T when his type is $\theta_1 > \frac{1}{2}$
- Player 2 has a dominant strategy to choose R when his type is $\theta_2 < \frac{1}{3}$.
- We therefore conjecture the following form of equilibrium strategies

$$P1 : \begin{cases} T & \text{if } \theta_1 \geq \theta_1^* \\ B & \text{if } \theta_1 < \theta_1^* \end{cases}$$

$$P2 : \begin{cases} L & \text{if } \theta_2 \geq \theta_2^* \\ B & \text{if } \theta_2 < \theta_2^* \end{cases}$$

- Solving for the equilibrium requires solving for the constants θ_1^* and θ_2^*

Bayesian Nash equilibrium

1\2	L	R
T	$2\theta_1, 3\theta_2$	1,1
B	1,0	0,0

- In a Nash Equilibrium each player must be indifferent between each of his pure strategies (Why?)
- player 1 plays T with probability $1 - \theta_1^*$ (Why?)
- player 2 plays L with probability $1 - \theta_2^*$ (Why?)
- Hence,

$$2\theta_1^* \cdot (1 - \theta_2^*) + 1 \cdot \theta_2^* = 1 \cdot (1 - \theta_2^*) + 0 \cdot \theta_2^*$$

$$3\theta_2^* \cdot (1 - \theta_1^*) + 0 \cdot \theta_1^* = 1 \cdot (1 - \theta_1^*) + 1 \cdot \theta_1^*$$

- From where we can determine that

$$\theta_1^* = \frac{1}{6}$$

$$\theta_2^* = \frac{2}{5}$$

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Exercises

- ▶ You and a friend are playing a 2×2 matrix game, but you're not sure if it's BoS or PD. Both are equally likely.

	A	B
A	2, 2	0, 3
B	3, 0	1, 1

PD

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

BoS

Put this game into Bayesian normal form.

- ▶ Consider the following two person game of incomplete information:

1\2	L	R
T	θ_1, θ_2	$1, \frac{1}{2}$
B	$\frac{1}{2}, 0$	$-\frac{1}{4}, -\frac{1}{4}$

It is common knowledge among the two players that player 1's type θ_1 and player 2's type θ_2 are independently drawn from the uniform distribution on $[0, 1]$.

Derive a pure strategy Bayesian-Nash equilibrium in this game.