Continuous Probability Distributions

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Sep 23, 2019

On the Agenda

1. Continuous Probability Distributions
2. Uniform Probability Distribution
3. Normal Probability Distribution

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2. Uniform Probability Distribution
3. Normal Probability Distribution

Continuous Probability Distributions

- A <u>continuous random variable</u> can assume any value in an interval on the real line or in a collection of intervals

- It is not possible to talk about the probability of the random variable assuming a particular value

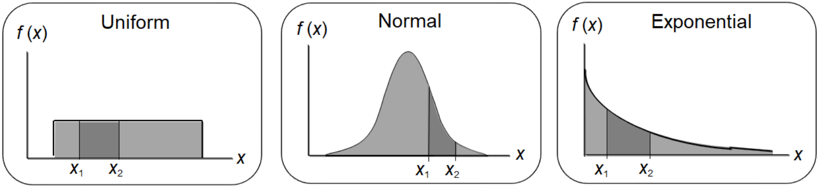
- Instead, we talk about the probability of the random variable assuming a value within a given interval

Continuous Probability Distributions

- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is:

  By definition:

  - the area under the probability density function between x_1 and x_2



On the Agenda

Uniform Probability Distribution

- A random variable is uniformly distributed whenever the probability is proportional to the interval's length
- The probability density function is:

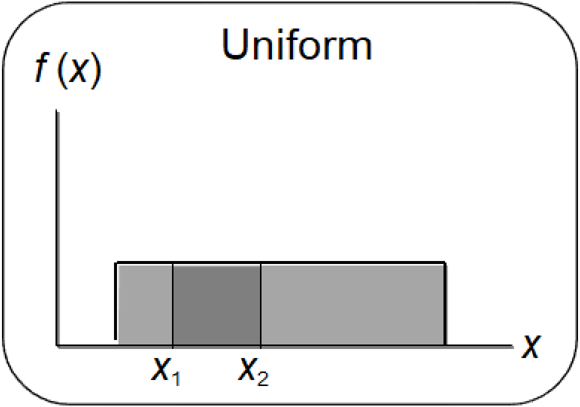
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

where

$a=$ smallest value the variable can assume

$b=$ largest value the variable can assume

Uniform Probability Distribution



Uniform Probability Distribution

- Expected Value of X

$$E(x) = \frac{a+b}{2}$$

- Variance of x

$$Var(x) = \frac{(b-a)^2}{12}$$

Example: Uniform Probability Distribution

- Flight time of an airplane traveling from Chicago to New York

- Suppose the flight time can be any value in the interval from 120 minutes to 140 minutes

Example: Uniform Probability Distribution

- Uniform Probability Density Function

$$f(x) = \begin{cases} \frac{1}{20} & \text{if } 120 \leq x \leq 140 \\ 0 & \text{elsewhere} \end{cases}$$

where:

x = Flight time of an airplane traveling from Chicago to New York

Example: Uniform Probability Distribution

- Expected Value of X

$$
\begin{aligned}
E(x) &= \frac{a+b}{2} \\
&= \frac{120+140}{2} \\
&= 130
\end{aligned}
$$

- Variance of x

$$
\begin{aligned}
Var(x) &= \frac{(b-a)^2}{12} \\
&= \frac{20^2}{12} \\
&= 33.3
\end{aligned}
$$

Example: Uniform Probability Distribution

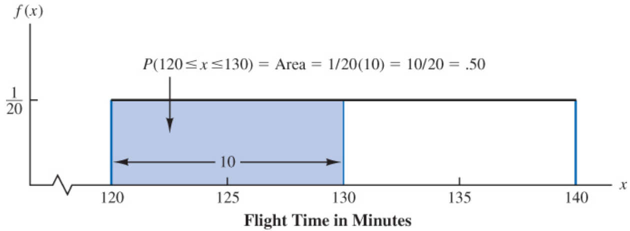
- What is the probability of a flight time between 120 and 130 minutes?

- Mathematically:

$$P(120 \leq x \leq 130) = ?$$

Example: Uniform Probability Distribution

- What is the probability of a flight time between 120 and 130 minutes?
- Graphically:



On the Agenda

Normal Probability Distribution

- The normal probability distribution is the most important distribution for describing a continuous random variable
- It is widely used in statistical inference
- Abraham de Moivre, a French mathematician, published The Doctrine of Chances in 1733. He derived the normal distribution

Normal Probability Distribution

- Normal Probability Density Function

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

$\mu =$ Mean

$\sigma =$ Standard Deviation

Normal Probability Distribution

- Characteristics:
  - The distribution is symmetric; its skewness measure is zero
  - The entire family of normal probability distributions is defined by:

    its mean μ, and

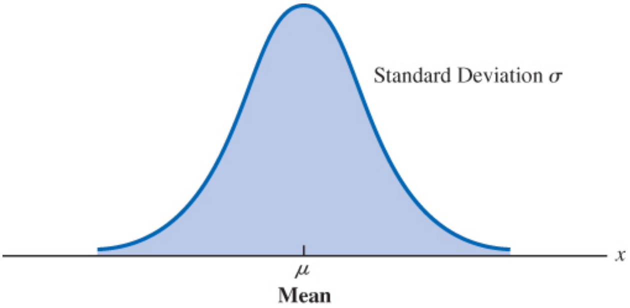
    its standard deviation σ
  - The highest point on the normal curve is:

    at the mean,

    which is also the median,

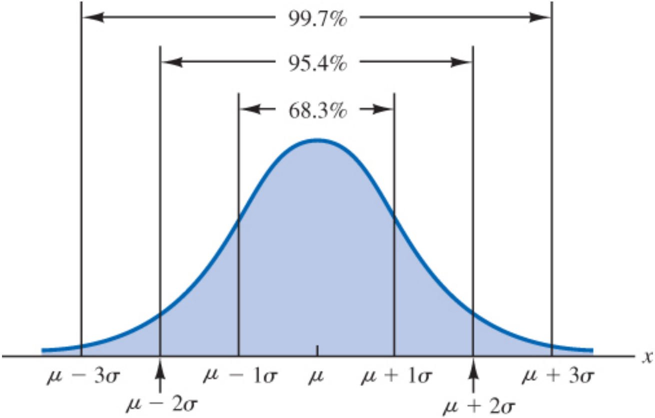
    and mode

Normal Probability Distribution

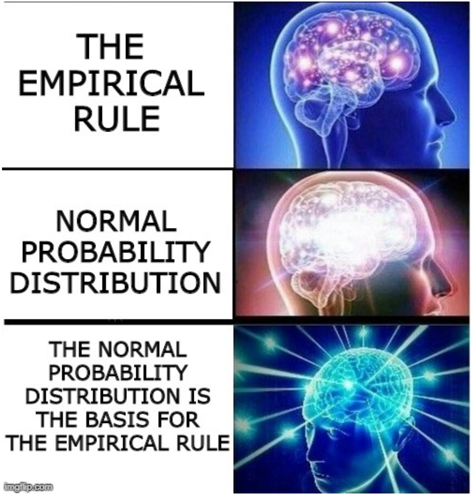


Normal Probability Distribution

- Characteristics:
  - basis for the empirical rule



Normal Probability Distribution



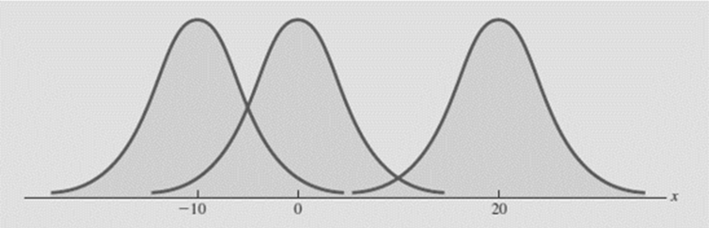
Normal Probability Distribution

- Characteristics:
  - The mean can be any numerical value:

    negative,

    zero,

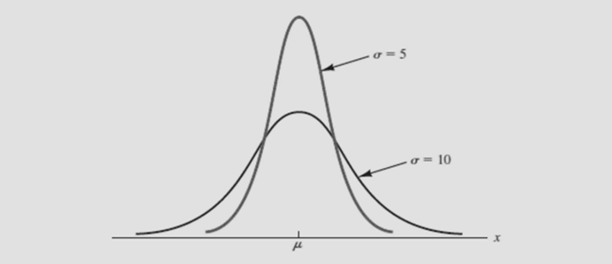
    or positive



Normal Probability Distribution

- Characteristics:
  - The standard deviation determines the width of the curve:

    larger values result in wider, flatter curves

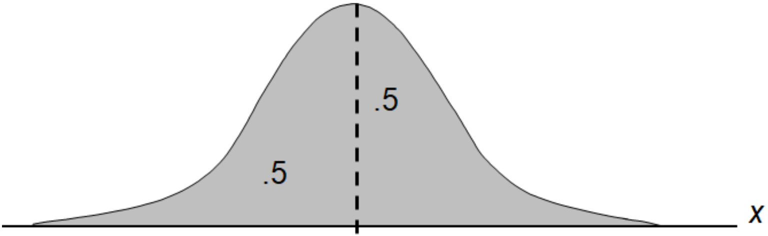


Normal Probability Distribution

- Characteristics:
  - Probabilities for the normal random variable are given by areas under the curve
  - The total area under the curve is 1

    .5 to the left of the mean

    .5 to the right of the mean



Normal Probability Distribution

- Characteristics:
  - Standard Normal Probability Distribution: A random variable having a normal distribution with a mean of 0 and a standard deviation of 1
  - The standard normal probability distribution is:

$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}$$

- To convert a normal random variable, x, into a standard normal random variable, Z, we transform:

$$Z = \frac{x - \mu}{\sigma}$$

Normal Probability Distribution

- Wait, what? Isn't it the z-score?
- Yes, it is the z-score but for a normal random variable
- We can think of Z as:
  - a measure of the number of:

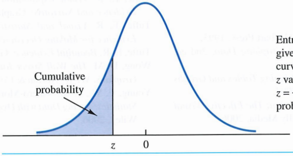
    the number of standard deviations that x is from the mean μ

Normal Probability Distribution

- How do we get the probabilities of a Z standard normal probability distribution?

- We use the z-table

TABLE 1 CUMULATIVE PROBABILITIES FOR THE STANDARD NORMAL DISTRIBUTION



Entries in the table give the area under the curve to the left of the z value. For example, for $z = -.85$, the cumulative probability is .1977.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Normal Probability Distribution

- The z-table gives us the cumulative probability: $P(z < a) = P(z \leq a)$
- Useful Values of the z-table
  * $P(z < -2.01) = P(z \leq -2.01) = 2.22\%$
  * $P(z < -1.64) = P(z \leq -1.64) = 5.05\%$
  * $P(z > 2.01) = P(z \geq 2.01) = 2.22\%$ (Why?)
  * $P(z > 1.64) = P(z \geq 1.64) = 5.05\%$ (Why?)
- Implications:
  - If $Z = -2.01 \implies$ P-value $= 2.22\%$
  - If $Z = 2.01 \implies$ P-value $= 2.22\%$
  - If $Z = -1.64 \implies$ P-value $= 5.05\%$
  - If $Z = 1.64 \implies$ P-value $= 5.05\%$
- Useful formula for symmetric distributions:

$$P(x_1 \leq z \leq x_2) = P(x_1 < z < x_2) = P(z < x_2) - P(z < x_1)$$

Example: Normal Probability Distribution

- Grear Tire Company Problem
- The company has developed a new tire to be sold through a chain of discount stores
- Before finalizing the tire mileage guarantee policy, the manager wants probability information about the number of miles of tires will last
- It was estimated that the mean tire mileage is 36,500 miles with a standard deviation of 5,000
- The manager now wants to know the probability that the tire mileage x will exceed 40,000

Example: Normal Probability Distribution

- Grear Tire Company Problem
- Solving for the probability
  1. What is the question? $P(x > 40,000)$
  2. Convert x to standard normal distribution

$$P\left(\frac{x-\mu}{\sigma} > \frac{40,000-\mu}{\sigma}\right)$$
$$P\left(Z > \frac{40,000-36,500}{5,000}\right)$$
$$P(Z > 0.7)$$

  3. Find the area under the standard normal curve to the right of Z = 0.7 (How?)

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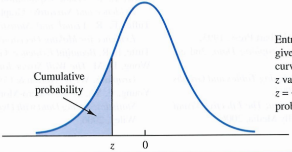

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−.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Example: Normal Probability Distribution

- Grear Tire Company Problem
- What should be the guaranteed mileage if Grear wants no more than 10% of tires to be eligible for the discount guarantee?

1. Find the z value that cuts off an area of 10% in the left tail of the standard normal distribution
2. From the table we see that z = -1.28 cuts off an area of 0.1 in the lower tail
3. Solve for x to get x=36,500 - 1.28 (5,000) = 30,100