# Introduction to Probability

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Sep 11, 2019

# On the Agenda

1. Experiments
2. Counting Rules
3. Assigning Probabilities
4. Some Basic Relationships of Probability
5. Conditional Probability

# On the Agenda

# Uncertainties

- Managers often base their decisions on an analysis of uncertainties:
  - What are the chances that sales will decrease if we increase prices?
  - What is the likelihood a new assembly method will increase productivity?
  - What are the odds that a new investment will be profitable?

# Probability

- Probability: is a numerical measure of the likelihood that an event will occur.
- Probability values are always assigned on a scale from 0 to 1
  - A probability near zero indicates an event is quite unlikely to occur
  - A probability near one indicates an event is almost certain to occur

# Statistical Experiments

- Statistical Experiments: a.k.a. random experiments, are process that generate well-defined outcomes
- Sample Space: Is the set of all experimental outcomes
- Sample Point: Is an experimental outcome

# Examples: Statistical Experiments and Sample Space

| Experiment | Experiment Outcomes |
| --- | --- |
| Toss a coin | Head, tail |
| Inspection a part | Defective, non-defective |
| Conduct a sales call | Purchase, no purchase |
| Roll a die | 1, 2, 3, 4, 5, 6 |
| Play a football game | Win, lose, tie |

# On the Agenda

# Multiple-Step Experiments

- If an experiment consists of a sequence of $k$ steps in which:
  - there are $n_1$ possible results for the first step
  - there are $n_2$ possible results for the second step
  - and so on
- The total number of experimental outcomes is given by:

$$n_1 \times n_2 \cdots n_k$$

- Example: The order of students sitting on the first row of the classroom

# Counting Rule for Combinations

- A combination is a selection of items from a collection, such that the order of selection does not matter.
- For example: given three letters A, B, C, how many combinations of two can be drawn from this set?
  - A, B
  - A, C
  - B, C
- More formally, a $k$-combination of a set $S$ is a subset of $k$ distinct elements of $S$
- the number of $k$-combinations is equal to the binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = {}_nC_k$$

## Size of Sample Space

- The total number of experimental outcomes of a multiple-step experiment and the combinations can be written using factorials
- In mathematics, the factorial of a positive integer $n$, denoted by $n!$, is the product of all positive integers less than or equal to $n$:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1$$

- For example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

- The value of $0!$ is 1, by convention

# Size of Sample Space

- The total number of experimental outcomes of a multiple-step experiment and the combinations can be written using factorials

- Multiple-Step Experiment:

$$P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}_{k \text{ factors}} = \frac{n!}{(n-k)!}$$

- Combinations:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$

# On the Agenda

# Assigning Probabilities

- Basic Requirements for Assigning Probabilities:

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively

$$0 \leq P(E_i) \leq 1 \quad \forall i$$

where: $E_i$ is the $i^{th}$ experimental outcome and $P(E_i)$ is its probability

# Assigning Probabilities

- Basic Requirements for Assigning Probabilities:

2. The sum of the probabilities for all experimental outcomes must equal 1

$$P(E_1) + P(E_1) + \cdots + P(E_n) = \sum_{i=1}^{n} P(E_i) = 1$$

where: $n$ is the number of experimental outcomes, $E_i$ is the $i^{th}$ experimental outcome, and $P(E_i)$ is its probability

# Assigning Probabilities

- Classical Method:
  - Assigning probabilities based on the assumption of equally likely outcomes
- Relative Frequency Method:
  - Assigning probabilities based on experimentation or historical data
- Subjective Method:
  - Assigning probabilities based on judgment

# Example: Classical Method

- Rolling a Die:
  - If an experiment has n possible outcomes, the classical method would assign a probability of $1/n$ to each outcome
  - Experiment: Rolling a die
  - Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
  - Probabilities: Each sample point has a $1/6$ chance of occurring

# Example: Relative Frequency Method

- Waiting time in the X-ray department for a local hospital
  - Consider a study of waiting times in the X-ray department for a local hospital
  - A clerk recorded the number of patients waiting for service at 9:00 A.M on 20 successive days and obtained the following results

# Example: Relative Frequency Method

- Waiting time in the X-ray department for a local hospital
  - Consider a study of waiting times in the X-ray department for a local hospital
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| Number waiting | Number of days outcome occurred |
|---|---|
| 0 | 2 |
| 1 | 5 |
| 2 | 6 |
| 3 | 4 |
| 4 | 3 |
| **Total** | **20** |

# Example: Relative Frequency Method

- Waiting time in the X-ray department for a local hospital
  - Each probability assignment is given by dividing the frequency (number of days) by the total frequency (total number of days)

| Number waiting | Number of days outcome occurred | Probability |
|---|---|---|
| 0 | 2 | 2/20 = 0.1 |
| 1 | 5 | 0.25 |
| 2 | 6 | 0.30 |
| 3 | 4 | 0.20 |
| 4 | 3 | 0.15 |
| **Total** | **20** | **1.00** |

# Example: Subjective Method

- When economic conditions or a company's circumstances change rapidly it might be inappropriate to assign probabilities based solely on historical data
- We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur
- The best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with the subjective estimate

# Example: Subjective Method

- Tom and Judy make an offer to purchase a house
- Possible outcomes: $E_1$ =offer accepted, $E_2$ =offer rejected

| Judy's Probability Estimates | Tom's Probability Estimates |
| --- | --- |
| $P(E_1) = 0.8$ | $P(E_1) = 0.6$ |
| $P(E_2) = 0.2$ | $P(E_2) = 0.4$ |

# Events and Their Probabilities

- Event: is a collection of sample points
- Probability of any event: Is equal to the sum of the probabilities of the sample points in the event
- If we can identify all the sample points of an experiment and assign a probability to each, we can compute the probability of an event
- Example: Rolling a die and getting an even number

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# Some Basic Relationships of Probability

# Some Basic Relationships of Probability

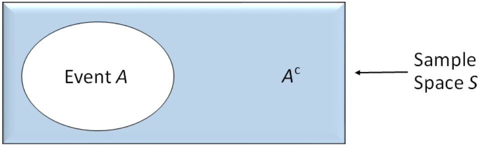
- There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities
  - Complement of an Event
  - Intersection of Two Events
  - Union of Two Events
  - Mutually Exclusive Events

# Complement of an Event

- The complement of event $A$ is defined to be the event consisting of all sample points that are not in $A$

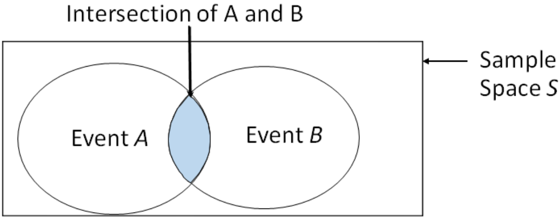
- The complement of $A$ is denoted by $A^c$ and

$$P(A) + P(A^c) = 1 \qquad \text{(Why?)}$$



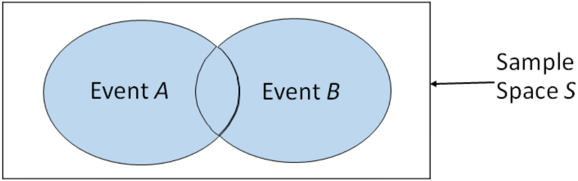
# Intersection of Two Events

- The intersection of events $A$ and $B$ is the set of all sample points that are in both $A$ and $B$
- The intersection of events $A$ and $B$ is denoted by $A \cap B$



# Union of Two Events

- The union of events $A$ and $B$ is the event containing all sample points that are in $A$ or $B$ or both
- The intersection of events $A$ and $B$ is denoted by $A \cup B$

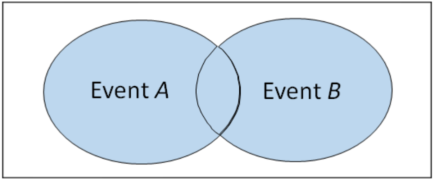


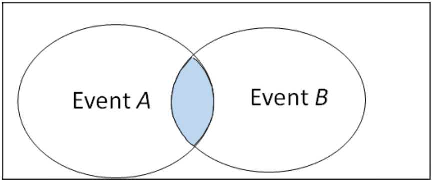
# Addition Law

- The <u>addition law</u> provides a way to compute the probability of event $A$, or $B$, or both $A$ and $B$ occurring

- The law is written as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



# Example: Addition Law

- A small assembly plant with 50 employees is carrying out performance evaluation
- Each worker is expected to complete work assignments on time and in such a way that the assembled product will pass a final inspection
- The results were as follows:

| Result | Number of Employees | Relative Frequency |
|---|---|---|
| Late completion of work | 5 | 5/50 = 0.1 |
| Assembled a defective work | 6 | 0.12 |
| Completed work late and assembled defective work | 2 | 0.04 |

# Example: Addition Law

- Event $L$ = the work is completed late
- Event $D$ = the assembled product is defective
- The production manager decided to assign poor performance rating to any employee whose work is either late or defective
- What is the probability that the production manager assigns a poor performance rating for a given employee?

## Example: Addition Law

- Event $L \cup D =$ the production manager assigned an employee a poor performance rating

- Using the addition law:

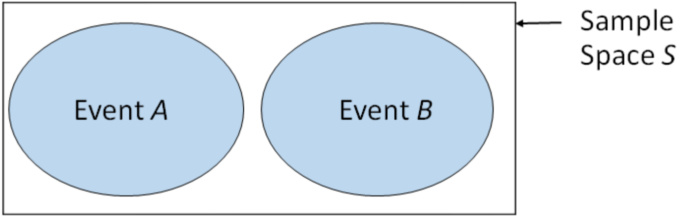
$$P(L \cup D) = P(L) + P(D) - P(L \cap D)$$

- Using the probabilities from the relative frequency

$$P(L \cup D) = 0.1 + 0.12 - 0.04 = 0.18$$

# Mutually Exclusive Events

- Two events are said to be mutually exclusive if the events have no sample points in common
- Two events are mutually exclusive if, when one event occurs, the other cannot occur
- Examples?



# Mutually Exclusive Events

- If events $A$ and $B$ are mutually exclusive, $P(A \cap B) = 0$
- The addition law for mutually exclusive events is:

$$P(A \cup B) = P(A) + P(B)$$

There is no need to include "$-P(A \cap B)$"

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# Conditional Probability

- A conditional probability of an event is the probability of an event given that another event has occurred
- The conditional probability of $A$ given $B$ is denoted by $P(A|B)$
- Example: Probability of promotion status given gender
- A conditional probability is computed as follows

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

# Conditional Probability

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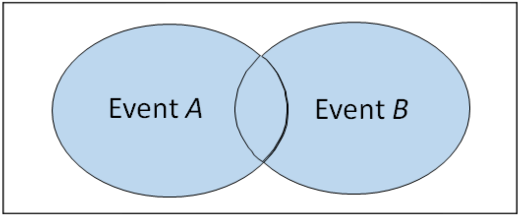

# Exmple: Conditional Probability

- Promotion status of police officers over the past two years

| | Men (M) | Women (W) | Total |
|---|---|---|---|
| Promoted (A) | 288/1200 = 0.24 | 0.03 | 0.27 |
| Not Promoted ($A^c$) | 0.56 | 0.17 | 0.73 |
| **Total** | **0.80** | **0.20** | **1.00** |

# Exmple: Conditional Probability

- Promotion status of police officers over the past two years
  - Event $A =$ An officer is promoted
  - Event $M =$ The promoted officer is a man
  - $P(A|M) =$ An officer is promoted given that the officer is a man

$$P(A|M) = \frac{P(A \cap M)}{P(M)}$$

- From the table:
  - $P(A \cap M) = 0.24$
  - $P(M) = 0.8$
  - $P(A|M) = 0.28/0.8 = 0.3$

# Multiplication Law

- The multiplication law provides a way to compute the probability of the intersection of two events
- It comes from solving for $P(A \cap B)$ in the conditional probability equation
- The law is written as:

$$P(A \cap B) = P(A|B) \times P(B)$$

# Example: Multiplication Law

- Newspaper circulation department
- Event $D =$ A household subscribes to the daily edition
- Event $S =$ The household already holds subscription to the Sunday edition
- Given: $P(D) = 0.84$ and $P(S|D) = 0.75$

# Example: Multiplication Law

- Newspaper circulation department

- What is the probability that the household subscribes to both the Sunday and daily editions of the newspaper?

$$
\begin{aligned}
P(S \cap D) &= P(S|D) \times P(D) \\
&= 0.75 \times 0.84 \\
&= 0.63
\end{aligned}
$$

# Joint Probability

- Joint probabilities appear in the center of the table
- Marginal probabilities appear in the margins of the table

| | Men (M) | Women (W) | Total |
|---|---|---|---|
| Promoted (A) | 288/1200 = 0.24 | 0.03 | 0.27 |
| Not Promoted ($A^c$) | 0.56 | 0.17 | 0.73 |
| **Total** | **0.80** | **0.20** | **1.00** |

# Independent Events

- If the probability of event $A$ is not changed by the existence of event $B$, we would say that events $A$ and $B$ are independent

- Two events A and B are independent if:

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

# Multiplication Law for Independent Events

- The multiplication law also can be used as a test to see if two events are not independent

- The law is written as:

$$P(A \cap B) = P(A) \times P(B)$$

# Example: Multiplication Law for Independent Events

- Use of credit card for purchase of gasoline
- From past experience it is known 80% of customers use credit card for the purchase of gasoline
- The service station manger wants to determine the probability that the next two customers purchasing gasoline will each use a credit card

# Example: Multiplication Law for Independent Events

- Use of credit card for purchase of gasoline
- Event $A =$ First customer uses a credit card
- Event $B =$ Second customer uses a credit card
- Hence,

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ &= 0.8 \times 0.8 \\ &= 0.64 \end{aligned}$$

# Mutual Exclusiveness and Independence

- Do not confuse the notion of mutually exclusive events with that of independent events
- Two events with nonzero probabilities cannot be both mutually exclusive and independent
- If one mutually exclusive event is known to occur, the other cannot occur
- Hence, the probability of the other event occurring is reduced to zero (and they are therefore dependent)
- Two events that are not mutually exclusive, might or might not be independent