A Online Appendix for

Population Aging, Economic Growth, and the Importance of Capital

(Not For Publication)

A.1 Proof of Proposition 1

Assume capital at time $t+1$ is not a function of time $t+1$ variables. Output per capita at time $t+1$ is

$$y_{t+1} = \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}}.$$

The partial derivative of y_{t+1} with respect to $N_{w,t+1}$ is

$$\frac{\partial y_{t+1}}{\partial N_{w,t+1}} = \frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} - \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2}.$$

Note, this implicitly holds the number of non-workers constant. The partial derivative is negative if

$$\frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} < \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2}.$$

Algebra shows this is negative if

$$\underbrace{\frac{F_N(K_{t+1}, N_{w,t+1})N_{w,t+1}}{F(K_{t+1}, N_{w,t+1})}}_{ls_{t+1}} < \underbrace{\frac{N_{w,t+1}}{N_{t+1}}}_{ws_{t+1}}.$$

A.2 Example with Log Utility

Suppose households make consumption/saving decisions beginning in their working years. Agents within the working cohort at time t maximize utility over current consumption $c_{w,t}$, and during retirement $c_{o,t+1}$

$$\max_{c_{w,t},c_{o,t+1}} \mathbb{U} = log(c_{w,t}) + log(c_{o,t+1}).$$

We abstract from time discounting as it does not impact the analysis. The households face two budget constraints

$$c_{w,t} + s_t = w_t \tag{1}$$

$$c_{o,t+1} = R_{t+1}s_t \tag{2}$$

where w is the wage, s is saving, and R is the gross real interest rate.

If households have log utility, then they save a constant fraction of wages

$$s_t = \frac{1}{2}w_t. \tag{3}$$

Capital completely depreciates at the end of each period of use (depreciation rate $\delta = 1$).[1] Therefore, the next generation's physical capital stock depends on the number of people in the current working cohort and their per-person gross saving

$$K_{t+1} = N_{w,t}s_t.$$

Setting wages equal to the marginal product of labor, $s_t = \frac{1}{2}F_N(K_t, N_t)$. Therefore, capital at time $t+1$ is

$$K_{t+1} = \frac{1}{2}N_{w,t}F_N(K_t, N_t).$$

Thus, K_{t+1} is not a function of time $t+1$ variables and Proposition 1 follows as outlined above.

[1]Each period of life represents 20 or more years, so the assumption of full depreciation is not without merit.

A.3 Proposition 1: A More General Case

With log utility, the income and substitution effects cancel out leaving K_{t+1} a function of time t variables only. If the elasticity of substitution is not 1, then this is not necessarily the case. Here, we consider the general functional form of (additively separable) preferences. Agents within the working cohort at time t maximize lifetime utility over current consumption and during retirement, based on utility function $\mathbb{U} = u(c_{w,t}) + u(c_{o,t+1})$ and subject to intertemporal budget constraints (1) and (2).[2]

The Euler Equation with the budget constraints results in

$$R_{t+1}u'(R_{t+1}s_t) = u'(w_t - s_t), \tag{4}$$

which implicitly gives the saving function $s_t = s(w_t, R_{t+1})$. We first show the sign of the response to saving from a change in interest rates, which we will use in the derivation of Proposition 1 under general preferences. Let s_R be the derivative of saving with respect to interest rates. Using the Euler Equation (4) we find through implicit differentiation

$$s_R = -\frac{u'(c_{o,t+1})(1 - \sigma(c_{o,t+1}))}{R_{t+1}^2 u''(c_{o,t+1}) + u''(c_{w,t})}$$

where $\sigma(c) = -u''(c)c/u'(c)$ and $1/\sigma$ is the elasticity of substitution. The equation is a well known result for this type of model. If preferences are, for example, represented with the commonly used Constant Relative Risk Aversion (CRRA) form, most empirical estimates place $\sigma > 1$. Under this restriction, $s_R < 0$. If $\sigma < 1$ then $s_R > 0$.

Next, recall the capital stock at time $t + 1$ equals the saving by the previous working generation, $K_{t+1} = N_{w,t}s(w_t, R_{t+1})$. Thus, per capita GDP at time $t + 1$ is

$$y_{t+1} = \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}} = \frac{F(N_{w,t}s(w_t, R_{t+1}), N_{w,t+1})}{N_{t+1}} = \frac{F(N_{w,t}s(w_t, F_K(K_{t+1}, N_{w,t+1})), N_{w,t+1})}{N_{t+1}}.$$

[2] For simplicity letting the subjective discount factor $\beta = 1$.

The partial derivative of y_{t+1} with respect to $N_{w,t+1}$ is

$$\frac{\partial y_{t+1}}{\partial N_{w,t+1}} = \frac{F_N(K_{t+1}, N_{w,t+1})}{N_{t+1}} - \frac{F(K_{t+1}, N_{w,t+1})}{N_{t+1}^2} + \frac{1}{N_{t+1}} F_K(K_{t+1}, N_{w,t+1}) \frac{\partial K_{t+1}}{\partial N_{w,t+1}}.$$

This is negative if

$$ws_{t+1} > ls_{t+1} + \frac{N_{w,t+1} R_{t+1}}{F(K_{t+1}, N_{w,t+1})} \frac{\partial K_{t+1}}{\partial N_{w,t+1}}. \tag{5}$$

If the elasticity of substitution $\sigma < 1$, the sign of the right-most term is positive $\left(\frac{\partial K_{t+1}}{\partial N_{w,t+1}} > 0\right)$. If the elasticity of substitution $\sigma > 1$, the sign of the last (right-most) term is ambiguous. To see this, let F_{KK} and F_{KN} be the second partial derivatives and consider

$$\frac{\partial K_{t+1}}{\partial N_{w,t+1}} = \frac{s_R N_{w,t} F_{KN}(K_{t+1}, N_{w,t+1})}{1 - s_R N_{w,t} F_{KK}(K_{t+1}, N_{w,t+1})}.$$

The sign is ambiguous since, with $\sigma > 1$, $s_R < 0$ and $F_{KK} < 0$. Thus, the sign of the last term in Equation (5) depends on the relative curvature of the utility and the production functions. If the last term is negative, the range in which a decline in the working age cohort leads to higher output increases. As the sign and magnitude depend on the functional forms of the utility and production functions, as well as an empirical determination of the parameter values in these functions, finding an exact solution lies beyond the scope of this paper.