Product Bundling in the Presence of Vertical Differentiation

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Abstract. Product bundling is frequently employed to exploit the heterogeneity in consumers' willingness to pay for different products and extract more consumer surplus. Meanwhile, vertical differentiation is employed to exploit the heterogeneity in consumers' willingness to pay for different qualities of a product. When consumers exhibit both types of heterogeneities, which is frequently observed in practice, the combined use of product bundling and vertical differentiation makes perfect sense. Although empirical evidence indicates the combined use of the two strategies leads to many successes, the academic community is falling behind in analyzing and optimizing them. We employ a simple analytical model to study a firm who sells two product types, with the ability to vertically differentiate each product type and bundle products across types to sell to the consumers. We investigate different levels of vertical differentiation that is coupled with or without product bundling, from single quality to two qualities and to more than two. For each level of vertical differentiation, we derive the optimal product quality and price deicsions for both component and bundling strategies, and identify the conditions under which bundling outperforms component or vice versa. Our results suggest that as the level of vertical differentiation is increased, product bundling dominates the component strategy over a broader range of market conditions. As a result, product bundling becomes more favorable to the firm to be coupled with vertical differentiation, as the firm seeks to increase its product line depth (or variety) through vertical differentiation.

Key words: product bundling, vertical differentiation, product quality, pricing, component strategy.

1. Introduction

Product bundling, the practice of selling two or more products together at a discount, is a common marketing strategy that enables firms to exploit the heterogeneity of consumers' willingness to pay for different products and extract more surplus. Different forms of the strategy have been observed in practice and investigated in the marketing literature, which includes (1) the pure bundling strategy, where only bundles of products are offered for consumers to purchase, and (2) the mixed bundling strategy, where product bundles, as well as the individual components, are offered. Previous research on product bundling has focused on studying its ability to extract more surplus from consumers through price optimization, without changing the underlying anchor products. The main advantage of bundling is that it allows firms to reduce the aggregate heterogeneity in consumers' willingness-to-pay due to the non-perfect correlation between products. The seminal work of Stigler (1963) and Adams and Yellen (1976) study and compare the bundling and unbundling strategies, where they find both can be optimal depending on the specific settings. Since then, a rich literature on product bundling has been developed and the strategy has been shown to be effective in various settings. For example, Schmalensee (1984) shows that product bundling is optimal under certain market conditions when consumers' reservation prices follow a bivariate Gaussian distribution. McAfee et al. (1989) extend the analysis of Schmalensee (1984) to consider a general distribution of consumers' reservation prices, and show that mixed bundling is almost always optimal. McCardle et al. (2007) study and compare the pure bundling and pure component strategies under a uniform distribution of reservation prices, while Bhargava (2013) studies mixed bundling under the same setting. Chen and Riordan (2013) use cupolas to show that mixed bundling is more profitable than component selling when the reservation prices of products are negatively correlated, independent, or somewhat positively correlated. Cao et al. (2015b) explore the benefit of bundling when product inventory is limited. Prasad et al. (2015) compare bundling to reserved pricing where a firm sells to a mix of myopic and strategic consumers. Bhargava (2012), Chakravarty et al. (2013), Girju et al. (2013), Cao et al. (2015a), Ma and Mallik (2017), and Cao et al. (2022) study bundling in the context of a distribution channel and investigate the competitive interplay between retailers and their suppliers.

Another stream of research related to this work is vertical differentiation, where firms leverage consumers' heterogeneity in their willingness to pay for product quality to determine their product qualities and prices. Vertical differentiation allows firms to extract more consumer surplus by offering varied qualities at different prices, and consumers self-select product and segment themselves into different groups. Mussa and Rosen (1978) and Moorthy (1984) demonstrate how a monopolist can maximize the profit of a product line by setting the qualities and prices of two products within the product line, where consumers choose among the products and the no purchase option according to some utility model. Moorthy (1988) extends the study to investigate the quality and price competition between two firms, where each firm offers one product and consumers choose among available options under the same utility framework. Desai (2001) examines how product cannibalization impacts a firm's quality and price decisions when consumers vary in their quality valuation and taste preference. Bhargava and Choudhary (2001) show that when the cost of product increases concavely in quality, the highest quality product should be offered. Bhargava and Choudhary (2008) identify conditions under which a monopolist should add a lower quality product along with an existing product. Pan and Honhon (2012) solve the optimal assortment and pricing problem of a firm by selecting products from a vertically differentiated product set. Zeithammer and Thomadsen (2013) investigate the quality and price competition in a duopoly where consumers also have a preference for variety. Our work incorporates the vertical differentiation strategy into the implementation of the product bundling strategy, and we seek to uncover the intricate interplay between the two strategies.

The two strategies, product bundling and vertical differentiation, though studied extensively in the marketing and operations literature, have not been investigated jointly for their intricate interplay that may improve firm profit or social surplus. Specifically, the product bundling literature has mostly studied how to bundle products to exploit the heterogeneity in consumers' willingness to pay for different products, by assuming the underlying products are exogenously given. Meanwhile, the vertical differentiation literature has studied how to design the qualities of products to exploit consumers' heterogeneity in their willingness to pay for quality, without considering the viability of bundling. When consumers exhibit heterogeneity in their willingness to pay for both different types of products and for different qualities within a product type, the joint deployment of the two strategies may create more value but it is not fully understood from an analytical perspective. Some recent works, such as Banciu et al. (2010) and Honhon and Pan (2017), have studied product bundling and vertical differentiation jointly to some extent, but they study them in a single product type setting and assume consumers will buy more than one product with different qualities from the same product type, which departs from the standard assumption in the vertical differentiation literature, where it is assumed that a consumer buys at most one unit of product from a product type. Our work follows the standard assumption in the vertical differentiation literature and studies product bundling across two product types, that is, a consumer buys at most one product from a product type and the firm manages two product types (with vertical differentiation). To the best of our knowledge, our work is among the first to study the joint deployment of product bundling and vertical differentiation in a two-product type setting.

In practice, the joint deployment of the two strategies in a multi-product type setting is prevailing. Automobile manufacturers frequently bundle cars with warranties (i.e., cars and warranties are considered as two types of "products"), where they strategically differentiate the qualities of cars and warranties before bundling them. For example, Tesla offers multiple quality options for the same model (i.e., Model Y consists of Rear-Wheel Drive, All-Wheel Drive, Long Range Rear-Wheel Drive, and Long Range All-Wheel Drive) and bundles each option with a specific warranty (i.e., the Rear-Wheel Drive option is bundled with a 100,000 miles battery and drive unit warranty, while the others are bundled with a 120,000 miles warranty. See Tesla.com (2024)). Other automobile manufacturers adopt a similar practice and offer cars of different trims (i.e., Toyota offers LE and XLE trims on several of their models, Cadillac offers Luxury and Premium trims on several of theirs) and bundle each trim with a specific warranty (i.e., basic or enhanced). In the travel industry, airlines offer cabin classes of different qualities (i.e., premium economy, economy, basic economy) and bundle each with curated benefits/restrictions, vacation packages bundle flights of varied qualities (i.e., non-stop vs connections, major vs cheap airlines, premium vs standard classes) with hotels of varied qualities (i.e., prime vs less-desired locations, luxury vs plain brands), and cruise lines bundle cruise cabins of different qualities (i.e., balcony, ocean-view, interior) with experiences or excursions of different qualities. In the entertainment industry, high profile entertainers sell VIP and standard packages that are a bundle of services from multiple categories, including food, beverage, seat, filming, etc. Each category can be differentiated by quality offerings and one quality is selected into a package. Some restaurants offer bundled menus that consist of a starter, a main course and a dessert (of varied qualities) and use them to extract more consumer surplus. In all these examples, the firms bundle products from different types, where each type is vertically differentiated by qualities of different levels. The fundamental task of the firms is

Table 1 Closely Related Literature on Product Bundling and Vertical Differentiation

		Quality Levels in a Product Type	
		Single quality level	Two quality levels (or more)
Number of Product Types (or Product Categories)	1	No bundling options	Vertical differentiation (w/o bundling): • Mussa and Rosen (1978) • Moorthy (1984) Vertical differentiation (with bundling): • Banciu et al. (2010) • Honhon and Pan (2017)
	2	Core bundling research: • Adams and Yellen (1976) • Schmalensee (1984) • ...	Vertical differentiation & bundling of two product types **(This paper)**

to choose the quality offerings for each product type and bundle products of different types to sell to their customers. Though the joint deployment of product bundling and vertical differentiation proves successful in these examples, it is not understood how the bundles should be designed and under the optimal design, whether (and when) the bundling strategy outperforms the component strategy.

Motivated by the above discussion, we study the joint product bundling and vertical differentiation problem for a firm that sells two product types. The firm decides how to choose product qualities and whether to sell products as individual components or as bundles (across product types). Our work adds an importance piece to the extant literature by filling a void on the interplay between vertical differentiation and product bundling (See Table 1). To the best of our knowledge, our work is among the first to study product bundling with vertical differentiation, with both product bundling and vertical differentiation drawn ample attention in the literature. We investigate the endogenous quality and price decisions of a firm, who sells products in two product types as either individual components or as bundles. We aim to identify the optimal product and bundling design as well as the conditions under which the bundling strategy outperforms the component strategy. Given the widespread use of product bundling and vertical differentiation in practice, our study contributes to the marketing and operations management literature by proposing an analytical framework to examine their joint deployment, offering insights of practical relevance.

The rest of the paper is organized as follows. In Section 2, we study the product bundling problem across two product types without considering vertical differentiation. Unlike the bundling literature, which also does not consider vertical differentiation, we endogenize the product quality decisions. We analyze and compare the bundling and the component strategy under their corresponding optimal decisions. In Section 3, we examine the case with vertical differentiation, where we assume the firm offers a maximum of two qualities in each product type, as is most common in the vertical differentiation literature. We analyze and compare the bundling and the component strategy and identify the conditions under which each is optimal. In Section 4, we extend the vertical differentiation case to precise targeting, where a maximum of more than two qualities can be offered. Finally, we conclude with some closing remarks in Section 5.

2. Single Quality for Each Product Type

Consider a firm who sells two types of products in a market, where each consumer is interested in buying at most one product of each type. In this section, we study the case that the firm offers one quality level in a product type. Unlike the bundling literature, which assumes products are fixed and given, we endogenize the product quality decisions. To tackle this decision, we follow the standard models in the vertical differentiation literature and employ a utility framework to model consumers' purchase decisions. Specifically, for a single product type, a consumer with quality preference index θ has a willingness to pay θq for a product with quality q. When the product is priced at p, the consumer enjoys a utility $\theta q - p$ if buying the product. The parameter θ captures the heterogeneity in consumers' willingness to pay for quality and can be modeled by $\theta \in [a, b]$ for continuous consumer types and $\theta \in \{\theta_L, \theta_H\}$ for discrete consumer types. We assume $\theta \in \{\theta_H, \theta_L\}$, with $\text{prob}(\theta = \theta_H) = \alpha$ and $\text{prob}(\theta = \theta_L) = 1 - \alpha$ for some $\alpha \in [0, 1]$. When the market size is normalized to 1 and each consumer is infinitesimal, the above assumption implies that α proportion of the consumers have the high quality index (θ_H) and $(1 - \alpha)$ proportion have the low quality index (θ_L). Such a discrete model is frequently employed in the marketing and operations management literature (see Iyer 1998, Acquisti and Varian 2005, Girju et al. 2013, etc). Our work treats product quality as an endogenous decision of the firm. Higher product quality increases consumers' willingness to pay for the product, but it also increases production cost. We assume the unit production cost is a convex function of quality, i.e., $c(q) = q^2$, which is commonly assumed in the vertical differentiation literature[1].

For simplicity of analysis, we consider two symmetric product types where θ is independent and identically distributed (i.i.d) over the two product types. We refer to the consumers with quality index θ_H and θ_L in one product type as the H and L consumers, respectively. Combining two product types, we refer to the consumers with joint quality index (θ_H, θ_H), (θ_H, θ_L), (θ_L, θ_H), (θ_L, θ_L) as the HH, HL, LH, and LL consumers. Due to the i.i.d. assumption, the sizes of the four segments of consumers are α^2, $\alpha(1 - \alpha)$, $\alpha(1 - \alpha)$, $(1 - \alpha)^2$, respectively, as illustrated in Figure 1.

In what follows, we analyze and compare two selling strategies of the firm: (1) component strategy, where the two products are sold independently, and (2) bundling strategy, where the two products are sold together as a bundle. We first derive the optimal product quality and price decisions of the firm under each strategy and then compare them to identify the conditions under which each is optimal. As shall be seen later, each of the two strategies can be optimal under some market conditions.

2.1. Component Strategy

When the firm employs the component strategy, the products from the two product types are sold independently. Since the two product types are symmetric, our analysis below is carried out for an arbitrary type. Suppose the firm offers a product at quality q and at price p. Consumers with quality index $\theta \geq \frac{p}{q}$ will buy

[1] Our results for $c(q) = q^2$ can be easily extended to $c(q) = aq^2$ by replacing q, θ with q', θ', where $q' \triangleq \sqrt{a}q$ and $\theta' \triangleq \frac{1}{\sqrt{a}}\theta$.

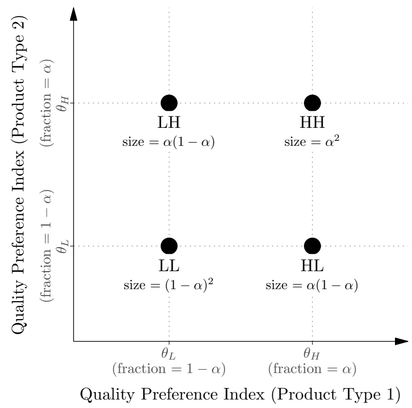


Figure 1 Consumer Segments by the Joint Quality Preference Index

the product and with $\theta < \frac{p}{q}$ will not buy (i.e., buying the product requires $\theta q - p \geq 0$). Given any θ_H and θ_L (with $\theta_H > \theta_L$), the firm may choose q and p such that one of the three cases occurs: (1) $\frac{p}{q} > \theta_H > \theta_L$ (nobody buys the product); (2) $\theta_H \geq \frac{p}{q} > \theta_L$ (the H consumers buy the product), and (3) $\theta_H > \theta_L \geq \frac{p}{q}$ (both H and L consumers buy the product). The first case is clearly sub-optimal. We analyze and compare the latter two cases below.

(1) If the firm attempts to target the H consumers (Case 2), for any quality q, the firm should set the price at $p = \theta_H q$ in order to maximize profit. Since the unit production cost is $c = q^2$ and the size of the H consumers is α (for one product type), the total profit of the firm for two product types is

$$2\alpha(\theta_H q - q^2)$$

(2) If the firm attempts to target both H and L consumers (Case 3), for any quality q, the firm should set the price at $p = \theta_L q$ in order to maximize profit. Since the toal size of the H and L consumers is 1 (for one product type), the total profit of the firm for two product types is

$$2(\theta_L q - q^2)$$

The above two cases are analyzed to determine the optimal quality that maximizes their corresponding total profit, as summarized in Table 2. When the firm targets the H consumers only, the optimal product quality is $\frac{1}{2}\theta_H$ and the optimal product price is $\frac{1}{2}\theta_H^2$, yielding a profit of $\frac{1}{2}\alpha\theta_H^2$. When the firm targets both the H and the L consumers, the optimal product quality is $\frac{1}{2}\theta_L$ and the optimal product price is $\frac{1}{2}\theta_L^2$, yielding a profit of $\frac{1}{2}\theta_L^2$. Comparing the optimal profit of the two cases reveals that when the proportion of the H consumers is large (i.e., $\alpha \geq z^2$, or equivalently $z \leq \sqrt{\alpha}$), where $z \triangleq \frac{\theta_L}{\theta_H} \in (0, 1)$, the firm should target the H consumers only. When the proportion of the H consumers is not large (i.e., $z > \sqrt{\alpha}$), the firm should target both H and L consumers. We refer to the above-defined z as the similarity score between the H and the L

Table 2 Optimal quality, price and profit for the component strategy (single quality)

Targeting Consumers	Product Demand	Optimal Quality (q^*)	Optimal Price (p^*)	Optimal Profit	Optimality Condition
H	α	$\frac{1}{2}\theta_H$	$\frac{1}{2}\theta_H^2$	$\frac{1}{2}\alpha\theta_H^2$	$z \le \sqrt{\alpha}$
H&L	1	$\frac{1}{2}\theta_L$	$\frac{1}{2}\theta_L^2$	$\frac{1}{2}\theta_L^2$	$z > \sqrt{\alpha}$

consumers, where a high z indicates a high similarity in the willingness to pay for quality between the H and L consumers. As shall be seen later, z plays a critical role in characterizing the optimal strategy of the firm when we compare within and across the component and the bundling strategies.

2.2. Bundling Strategy

Alternatively, the firm may employ the bundling strategy to sell the two products as a bundle. In this case, the firm should determine which market segments to target, by deciding the bundle's quality and price decisions. We assume that the willingness to pay of consumers for a bundle is additive across the two product types. Therefore for a consumer with joint quality index (θ_1, θ_2), her willingness to pay for a bundle (q_1, q_2) is $\theta_1 q_1 + \theta_2 q_2$, where q_1(or q_2) is the quality of the product from type 1(or 2). Without loss of generality, we assume $q_1 \ge q_2$[2]. Given any bundle (q_1, q_2) with $q_1 \ge q_2$, we can order the willingness to pay for the bundle of the four consumer segments by

$$\theta_H q_1 + \theta_H q_2 \ge \theta_H q_1 + \theta_L q_2 \ge \theta_L q_1 + \theta_H q_2 \ge \theta_L q_1 + \theta_L q_2$$

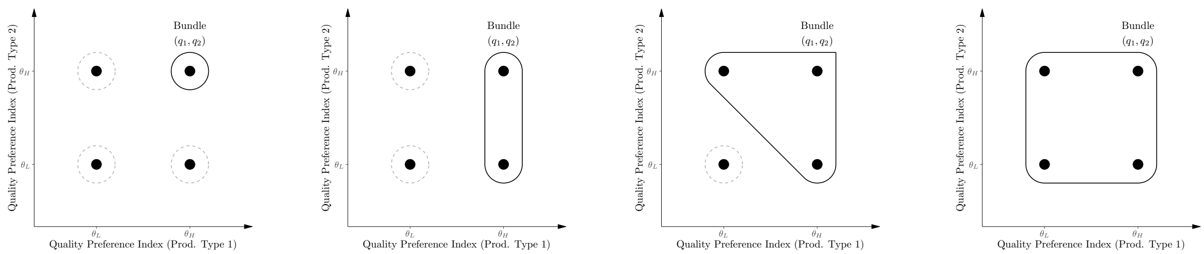
The firm can then set the bundle price (denoted by p_B) at one of the four segments' willingness to pay levels and any segment whose willingness to pay level is higher than or equal to the price will purchase. This gives rise to the following four targeting options for the firm: (1) targeting the {HH} segment only by setting $p_B = \theta_H q_1 + \theta_H q_2$, (2) targeting the {HH, HL} segments by setting $p_B = \theta_H q_1 + \theta_L q_2$, (3) targeting the {HH, HL, LH} segments by setting $p_B = \theta_L q_1 + \theta_H q_2$, and (4) targeting all segments by setting $p_B = \theta_L q_1 + \theta_L q_2$. An illustration of these targeting options is provided in Figure 2. For each one of the targeting options, we provide the bundle demand, bundle price, and the firm's total profit in Table 3.

Table 3 Bundle demand, price and profit for the bundling strategy (single quality)

Option	Bundle Demand	Bundle Price	Total Profit
1	α^2	$\theta_H q_1 + \theta_H q_2$	$\alpha^2(\theta_H q_1 + \theta_H q_2 - q_1^2 - q_2^2)$
2	α	$\theta_H q_1 + \theta_L q_2$	$\alpha(\theta_H q_1 + \theta_L q_2 - q_1^2 - q_2^2)$
3	$\alpha(2-\alpha)$	$\theta_L q_1 + \theta_H q_2$	$\alpha(2-\alpha)(\theta_L q_1 + \theta_H q_2 - q_1^2 - q_2^2)$
4	1	$\theta_L q_1 + \theta_L q_2$	$\theta_L q_1 + \theta_L q_2 - q_1^2 - q_2^2$

[2] This is without loss of generality because we can swap the indices of the two product types if $q_1 \le q_2$.

Figure 2 **Targeting options for the bundling strategy (single quality)**



(a) Option 1 (b) Option 2 (c) Option 3 (d) Option 4

We next derive the optimal quality of the bundle for the four options and provide the associated profit. For Option (1), the optimal bundle quality is $(\frac{1}{2}\theta_H, \frac{1}{2}\theta_H)$ and the associated profit is $\frac{1}{2}\alpha^2\theta_H^2$; for Option (2), the optimal bundle quality is $(\frac{1}{2}\theta_H, \frac{1}{2}\theta_L)$ and the associated profit is $\frac{1}{4}\alpha(\theta_H^2 + \theta_L^2)$; for Option (3), the optimal bundle quality is $(\frac{1}{4}(\theta_H + \theta_L), \frac{1}{4}(\theta_H + \theta_L))$ and the associated profit is $\frac{1}{8}\alpha(2-\alpha)(\theta_L + \theta_H)^2$; and for Option (4), the optimal bundle quality is $(\frac{1}{2}\theta_L, \frac{1}{2}\theta_L)$ and the associated profit is $\frac{1}{2}\theta_L^2$. It is worth noting that only Option (2) yields asymmetric qualities across product types while the others yield symmetric ones. Moreover, under the optimal solution, Option (4) becomes equivalent to the optimal component strategy that targets both H and L consumers, because the two strategies offer the same product qualities and charge the same total price, and all consumers buy both products (as two components or as one bundle).

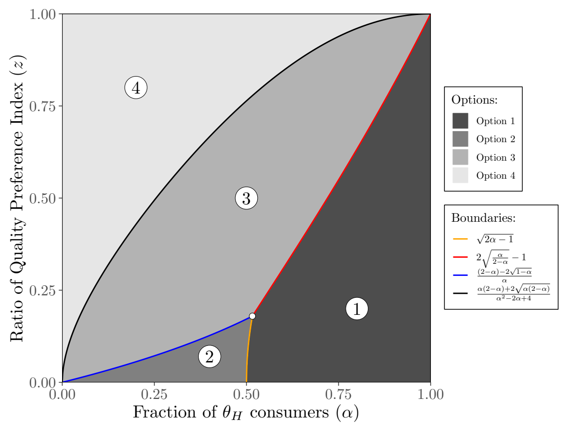
We then compare Option $(1-4)$ to identify the conditions under which each is optimal. The results show that all four options can be optimal under some market conditions, which are provided in Table 4. An illustration of the optimal conditions for the four options is provided in Figure 3. We summarize the results with some key observations. First, Option (2) is not completely dominated by the other options. Given that the two product types are symmetric, one might expect that the optimal qualities for the two product types should be symmetric as well under the optimal bundling design. This is indeed true under most market conditions, except for when Option (2) is optimal. In the parameter region $\sqrt{2\alpha-1} \le z \le \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}$, the firm should target the $\{$HH, HL$\}$ consumer segments with bundle $(\frac{1}{2}\theta_H, \frac{1}{2}\theta_L)$ and abandon the other segments. As one can see from Figure 3, this region corresponds to the lower center-to-left part of the (α, z) space which presents the firm with a particular challenge: a low z indicates the willingness to pay of the L consumers is much lower than that of the H consumers. Therefore targeting a segment with an L (i.e., LH) requires the firm to significantly reduce the bundle price, compared to not targeting any segment with an L. As a result, the firm may be advised to abandon any segment with an L. However, when α is less than approximately 0.5, the size of a segment with an L is larger than the segment without (i.e., the HH segment), so it may not be wise to abandon all segments with an L. The compromised solution for the firm is to serve only the HL segment[3] (Option (2)), but not both the HL, LH segments (Option (3)). Compared

[3] Here, the firm covers HL because we assume $q_1 \ge q_2$. Alternatively, the firm can cover LH which requires $q_1 \le q_2$.

Table 4 Optimal quality, price and profit for the bundling strategy (single quality)

Option	Bundle	Optimal Quality		Bundle Price	Bundle Profit	Optimality Condition
(#)	Demand	q_1^*	q_2^*	p_B^*	Π_B^*	
1	α^2	$\frac{1}{2}\theta_H$	$\frac{1}{2}\theta_H$	θ_H^2	$\frac{1}{2}\alpha^2\theta_H^2$	$z \leq \min\{\sqrt{2\alpha-1}, 2\sqrt{\frac{\alpha}{2-\alpha}}-1\}$
2	α	$\frac{1}{2}\theta_H$	$\frac{1}{2}\theta_L$	$\frac{1}{2}(\theta_H^2+\theta_L^2)$	$\frac{1}{4}\alpha(\theta_H^2+\theta_L^2)$	$\sqrt{2\alpha-1} \leq z \leq \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}$
3	$\alpha(2-\alpha)$	$\frac{1}{4}(\theta_H+\theta_L)$	$\frac{1}{4}(\theta_H+\theta_L)$	$\frac{1}{4}(\theta_H+\theta_L)^2$	$\frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2$	$\max\{2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}\}$ $\leq z \leq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$
4	1	$\frac{1}{2}\theta_L$	$\frac{1}{2}\theta_L$	θ_L^2	$\frac{1}{2}\theta_L^2$	$z \geq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$

Figure 3 The region of the optimal options for the bundling strategy (single quality)



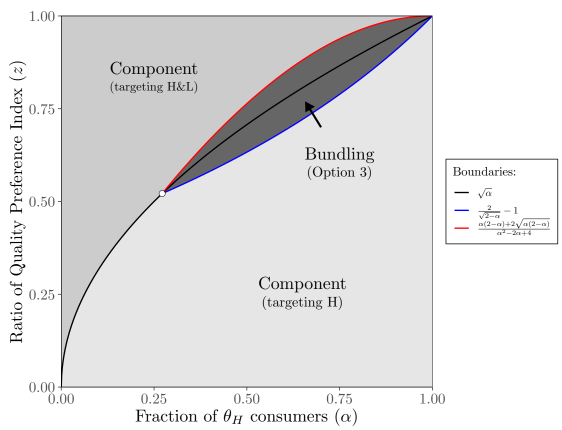
with Option (3), Option (2) yields a smaller market share (demand) but it allows the firm to sustain a higher price (i.e., $\frac{1}{2}(\theta_H^2+\theta_L^2) \geq \frac{1}{4}(\theta_H+\theta_L)^2$), which leads to a higher profit margin. Balancing market share with profit margin, Option (2) dominates Option (3) within the specified region.

The results also confirm that our common intuition holds regarding how the firm's strategy should change as the market condition evolves: (1) as the similarity score (z) between the H and L consumers increases, the firm should serve more types of consumers and cover more segments; (2) as the proportion of the H consumers (α) increases, the firm focus more on the H consumers and cover fewer segments. The former occurs because, as z increases, the difference in willingness to pay between the H and L consumers decreases. Consequently, the two types of consumers become less distinguishable, and the firm benefits from serving more of them simultaneously. For example, when $\alpha = 0.25$, increasing z from 0 to 1 causes the firm's optimal targeting option to change from 2 to 3 to 4, indicating more segments with an L are covered. There is a very narrow range of α for which, as z is increased from 0 to 1, the firm's optimal option traverses all four options (i.e., $\alpha \in [0.500, 0.516]$. See Figure 3). For the latter case, when the proportion of H consumers (α) increases, serving a market segment with an L erodes the bundle's profit margin. As the size of a segment with an L shrinks, serving it becomes less attractive to the firm. Therefore the firm should progressively

Table 5 The optimal strategy between component and bundling (Single Quality)

Selling Strategy	Targeting (option)	Optimal Quality q_1^*	Optimal Quality q_2^*	Optimal Price p_1^*	Optimal Price p_2^*	Total Profit Π^*	Optimality Condition
Component	H	$\frac{1}{2}\theta_H$	$\frac{1}{2}\theta_H$	$\frac{1}{2}\theta_H^2$	$\frac{1}{2}\theta_H^2$	$\frac{1}{2}\alpha\theta_H^2$	$z \le \min(\sqrt{\alpha}, \frac{2}{\sqrt{2-\alpha}} - 1)$
Component	H&L	$\frac{1}{2}\theta_L$	$\frac{1}{2}\theta_L$	$\frac{1}{2}\theta_L^2$	$\frac{1}{2}\theta_L^2$	$\frac{1}{2}\theta_L^2$	$z > \max(\sqrt{\alpha}, \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4})$
Bundling	Option 3	$\frac{1}{4}(\theta_H+\theta_L)$	$\frac{1}{4}(\theta_H+\theta_L)$	$\frac{1}{4}(\theta_H+\theta_L)^2$		$\frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2$	$\frac{2}{\sqrt{2-\alpha}} - 1 < z \le \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$

Figure 4 The region of the optimal strategy between component and bundling (Single Quality)



abandon more segments with an L. For example, when $z = 0.25$, increasing α from 0 to 1 leads the firm's optimal targeting option to change from 4 to 3 to 1. There is a much wider range of z (i.e., $z \in [0, 0.179]$) over which the firm's optimal option traverses all four possibilities as α is increased from 0 to 1.

2.3. Component vs. Bundling

We next compare the component and the bundling strategy to identify the conditions under which each is optimal, along with the corresponding quality and price decisions, as stated in Proposition 1.

PROPOSITION 1. *The optimal strategy (component vs. bundling) and the corresponding quality and price decisions of the firm when offering a single quality in each product type is summarized in Table 5.*

Table 5 summarizes the firm's optimal strategy, which can be characterized by three cases: (1) when $z \le \min(\sqrt{\alpha}, \frac{2}{\sqrt{2-\alpha}} - 1)$, the firm should adopt the component strategy targeting the H consumers; (2) when $z > \max(\sqrt{\alpha}, \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4})$, the firm should adopt the component strategy targeting both H and L consumers; (3) when $\frac{2}{\sqrt{2-\alpha}} - 1 < z \le \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$, the firm should adopt the bundling strategy targeting the {HH, HL, LH} segments (Option 3). Figure 4 provides a graphical illustration of these optimality conditions.

According to the results, both optimal options derived for the component strategy (Section 2.1) can remain optimal when compared against the bundling strategy. However, of the four optimal options derived for the bundling strategy (Section 2.2), only Option (3) can remain optimal when compared against the component strategy. As shown in Figure 4, Option (3) dominates the component strategy when z is moderately larger than α and α is not too small. We note that the black line ($z = \sqrt{\alpha}$) is the boundary that separates the two targeting options of the component strategy. Therefore Option (3) can dominate both options of the component strategy. Since Option (3) is the only option from the bundling strategy that is not dominated by the component strategy, this implies that the bundling strategy provides incremental value to the component strategy by allowing the firm to abandon only the LL segment, which is not achievable by the component strategy. Under such a bundling design, the firm offers quality $(\frac{1}{4}(\theta_H + \theta_L), \frac{1}{4}(\theta_H + \theta_L))$ that caters to neither the H nor the L consumers. The quality $\frac{1}{4}(\theta_H + \theta_L)$ can be regarded as a compromise between the ideal quality for the H consumers ($\frac{1}{2}\theta_H$) and the ideal quality for the L consumers ($\frac{1}{2}\theta_L$). The firm leverages the heterogeneity in consumers' willingness to pay across the two product types from the HL&LH segments, which is aligned with the principle found in Adams and Yellen (1976) and other bundling works.

3. Vertical Differentiation

Vertical differentiation refers to the practice of offering products at different quality levels such that all consumers agree on the ranking of products (higher quality is better), but differ in how much they are willing to pay for a fixed quality level. Under this strategy, firms jointly choose quality and price for two (or more) products, trading off higher production costs against the ability to charge higher prices, and consumers self-select some product that maximizes their individual utility. The strategy enables the firm to segment the market and extract more surplus (see Mussa and Rosen (1978), Moorthy (1984), etc). We incorporate vertical differentiation into the selling strategy (component or bundling) of a firm, where the firm has the option to offer more than one quality level for a product type and decides whether and how to bundle products to segment the market. In the main stream bundling literature, only one bundle is considered and can be offered by a firm, since there is no product differentiation being considered. As we incorporate differentiation, the firm can offer several bundles simultaneously, making the problem more intricate and intriguing.

We study the quality and price decision of the firm with vertical differentiation under two selling strategies: (1) component strategy; (2) bundling strategy. We seek to understand how will vertical differentiation impact these two strategies and when will bundling add value in the presence of vertical differentiation. In this section, we focus on two quality levels with regards to the differentiation, which is the most common setup in the vertical differentiation literature. In Section 4, we extend the analysis to consider more than two quality levels. We provide the analysis for the two selling strategies below.

3.1. Component Strategy

Under the component strategy, the firm sells products as individual components and consumers select some product (or none) within a product type that maximizes her individual utility. Given that the two product types are symmetric, we conduct the analysis for an arbitrary type. With vertical differentiation, the firm offers two products within each product type, one at a higher quality q_H (product H) and the other at a lower quality q_L (product L), with prices p_H and p_L, respectively. Each consumer chooses among product H, product L, and the no purchase option to maximize her individual utility. For a consumer with quality index θ (toward the product type of interest), her utilities for the three purchase options (H, L and no purchase) are $\theta q_H - p_H$, $\theta q_L - p_L$, and 0, respectively. Consequently, consumers with $\theta \geq \max(\frac{p_H - p_L}{q_H - q_L}, \frac{p_H}{q_H})$ will purchase product H, with $\frac{p_L}{q_L} \leq \theta < \frac{p_H - p_L}{q_H - q_L}$ will purchase product L, and with $\theta < \min(\frac{p_H}{q_H}, \frac{p_L}{q_L})$ will not purchase[4]. When θ follows a two-point distribution and in order for both products to have a nonnegative demand (such that the model does not degenerate to the single quality case by design), the firm must choose quality and price decisions such that $\frac{p_L}{q_L} \leq \theta_L < \frac{p_H - p_L}{q_H - q_L} \leq \theta_H$. With this, the H consumers will purchase product H and the L consumers will purchase product L. The demands for product H and L become α and $1 - \alpha$, respectively. Hence the firm's total profit (for two product types) is

$$2\left[\alpha(p_H - q_H^2) + (1-\alpha)(p_L - q_L^2)\right]$$

subject to the constraint $\frac{p_L}{q_L} \leq \theta_L < \frac{p_H - p_L}{q_H - q_L} \leq \theta_H$.

For any fixed q_H and q_L (with $q_H > q_L$), the optimal price decision of the firm is $p_H^* = \theta_H q_H - (\theta_H - \theta_L)q_L$ and $p_L^* = \theta_L q_L$. Intuitively, product L should be priced at the willingness to pay of the L consumers, who are intended by the firm to buy product L, and product H should be priced such that the H consumers are indifferent between buying product H and product L. Under the optimal price, the L consumers will enjoy a zero surplus (i.e., $\theta_H q_L - p_L^* = 0$) and the H consumers will enjoy a positive surplus (i.e., $\theta_H q_H - p_H^* = (\theta_H - \theta_L)q_L > 0$) when vertical differentiation is adopted.

Plugging in the optimal price, we can derive the optimal quality decision of the firm, which is summarized in Table 6. The results indicate that there are two cases for the quality decisions of the products. First, when the size of the H consumers is sufficiently large (i.e., $\alpha \geq z$), the firm should offer $q_L^* = 0$. This implied the firm should not offer product L and only the H consumers are covered. Second, when the size of the H consumers is not sufficiently large (i.e., $\alpha < z$), the firm should offer $q_H^* = \frac{\theta_H}{2}$ and $q_L^* = \frac{\theta_L - \alpha\theta_H}{2(1-\alpha)} \leq \frac{\theta_L}{2}$, where $\frac{\theta_L}{2}$ is the ideal quality for the L consumers if the firm can perfectly discriminate the two types of consumers. Comparing across the two cases, the optimal quality of product H is always set at $q_H^* = \frac{\theta_H}{2}$, which is the ideal quality for the H consumers. The optimal quality of product L, when offered, is $q_L^* = \frac{\theta_L - \alpha\theta_H}{2(1-\alpha)}$. q_L^* is decreasing in α and $q_L^* = \frac{\theta_L}{2}$ when $\alpha = 0$. This implies that as the market consists of more H consumers, the optimal quality of product L should be pulled away from the ideal quality for the L consumers. The reduced quality of the product helps the firm mitigate the cannibalization effect of product L on product H. This is manifested by the setting of the optimal price of product H, which is $p_H^* = \theta_H q_H - (\theta_H - \theta_L)q_L^*$. Reducing q_L^* allows the firm to increase the price for product H. As the market consists of more H consumers, the firm

[4] If the utilities of the purchase options are tied, we assume consumers have the following preference list: H > L > no purchase.

Table 6 Optimal quality, price and profit for the component strategy (vertical differentiation)

Targeting Consumers	Optimal Quality		Optimal Price		Optimal Profit	Optimality Condition
	q_H^*	q_L^*	p_H^*	p_L^*	Π^*	
H	$\frac{1}{2}\theta_H$	0	$\frac{1}{2}\theta_H^2$	0	$\frac{1}{2}\alpha\theta_H^2$	$z \le \alpha < 1$
H&L	$\frac{1}{2}\theta_H$	$\frac{z-\alpha}{2(1-\alpha)}\theta_H$	$\frac{1-(1+\alpha)z+z^2}{2(1-\alpha)}\theta_H^2$	$\frac{z^2-\alpha z}{2(1-\alpha)}\theta_H^2$	$\frac{\alpha-2\alpha z+z^2}{2(1-\alpha)}\theta_H^2$	$0 < \alpha < z$

benefits from being able to charge a higher price for product H, enabling the firm to extract more surplus from the H consumers. The profit loss resulting from the disservice to the L consumers is exceeded by the gains from serving the H consumers, who yield higher profit margins and represent a larger segment.

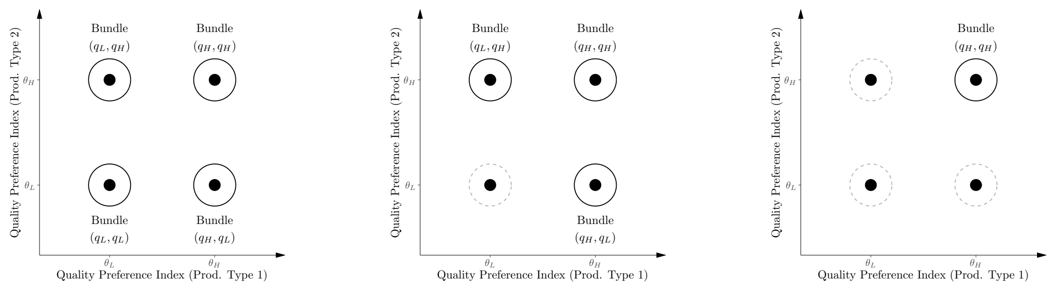
3.2. Bundling Strategy

Under the bundling strategy, the firm offers product bundles to target the market segments. With two qualities $\{q_H, q_L\}$ for each product type, the firm may offer a maximum of four bundles to target the four segments shown in Figure 1. Specifically, the firm can offer bundles (q_L, q_L), (q_H, q_L), (q_L, q_H), and (q_H, q_H) that are targeted at the segment LL, HL, LH, and HH, respectively (illustrated in Figure 5(a)). According to Section 3.1, when selling vertically differentiated products within a product type, the low quality product has a cannibalization effect on the high quality product, and consequently the firm may be better off not selling the low quality product under certain market conditions. Similarly, when selling bundles of products of differentiated qualities, the lower quality bundles may have a cannibalization effect on the higher quality ones, and hence the firm may be better off not selling some of the lower quality bundles. The firm may drop bundle (q_L, q_L) first and offer the rest to target the HL, LH and HH segments (illustrated in Figure 5(b)), and proceed to further drop bundles (q_H, q_L), (q_L, q_H) and target only the HH segment[5] (illustrated in Figure 5(c)). We number these options $(1-3)$. Under each option, the price of a bundle should be set such that the bundle provides the highest utility for the targeted segment and hence the segment purchases it. We provide the details of the price constraints and the resulting profit function of the firm in Table 7. The price of a bundle (q_i, q_j) is denoted by p_{ij}, for any $i, j \in \{L, H\}$.

For any fixed q_H and q_L (with $q_H > q_L$), the optimal prices of the bundles can be obtained by solving a linear program. We provide the optimal price results for the three options in Table 7. The principles for setting the optimal prices across the three options are consistent and can be summarized as follows: (1) the bundle with the lowest quality is priced at the willingness to pay of consumers targeted by the bundle, (2) the bundle with the next highest quality is priced such that the targeted consumer segment for the bundle is indifferent between buying it and the immediate lower quality one. For example, under Option (1), the optimal price of the lowest quality bundle (q_L, q_L) is set at the willingness to pay for the bundle by the LL segment, which is $2\theta_L q_L$. The optimal price of the next highest quality bundle (q_H, q_L) is set such that the

[5] The bundles (q_H, q_L), (q_L, q_H) are dropped simultaneously due to symmetry.

Figure 5 Targeting options ($1-3$) for the bundling strategy (vertical differentiation)



(a) Option (1) (b) Option (2) (c) Option (3)

Table 7 Profit functions and optimal prices for bundling Option ($1-3$) (vertical differentiation)

Option	Profit function	Price Constraints	p^*_{HH}	p^*_{HL}	p^*_{LL}
(1)	$\alpha^2(p_{HH}-2q_H^2)$ $+2\alpha(1-\alpha)(p_{HL}-q_H^2-q_L^2)$ $+(1-\alpha)^2(p_{LL}-2q_L^2)$	$2\theta_H q_H - p_{HH} \geq \theta_H q_H + \theta_H q_L - p_{HL}$ $2\theta_H q_H - p_{HH} \geq 2\theta_H q_L - p_{LL}$ $2\theta_H q_H - p_{HH} \geq 0$ $\theta_H q_H + \theta_L q_L - p_{HL} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_H + \theta_L q_L - p_{HL} \geq \theta_H q_L + \theta_L q_L - p_{LL}$ $\theta_H q_H + \theta_L q_L - p_{HL} \geq 0$ $2\theta_L q_L - p_{LL} > 2\theta_L q_H - p_{HH}$ $2\theta_L q_L - p_{LL} > \theta_L q_H + \theta_L q_L - p_{HL}$ $2\theta_L q_L - p_{LL} \geq 0$	$2(\theta_H q_H + \theta_L q_L - \theta_H q_L)$	$\theta_H q_H + 2\theta_L q_L - \theta_H q_L$	$2\theta_L q_L$
(2)	$\alpha^2(p_{HH}-2q_H^2)$ $+2\alpha(1-\alpha)(p_{HL}-q_H^2-q_L^2)$	$2\theta_H q_H - p_{HH} \geq \theta_H q_H + \theta_H q_L - p_{HL}$ $2\theta_H q_H - p_{HH} \geq 0$ $\theta_H q_H + \theta_L q_L - p_{HL} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_H + \theta_L q_L - p_{HL} \geq 0$	$2\theta_H q_H + \theta_L q_L - \theta_H q_L$	$\theta_H q_H + \theta_L q_L$	-
(3)	$\alpha^2(p_{HH}-2q_H^2)$	$2\theta_H q_H - p_{HH} \geq 0$	$2\theta_H q_H$	-	-

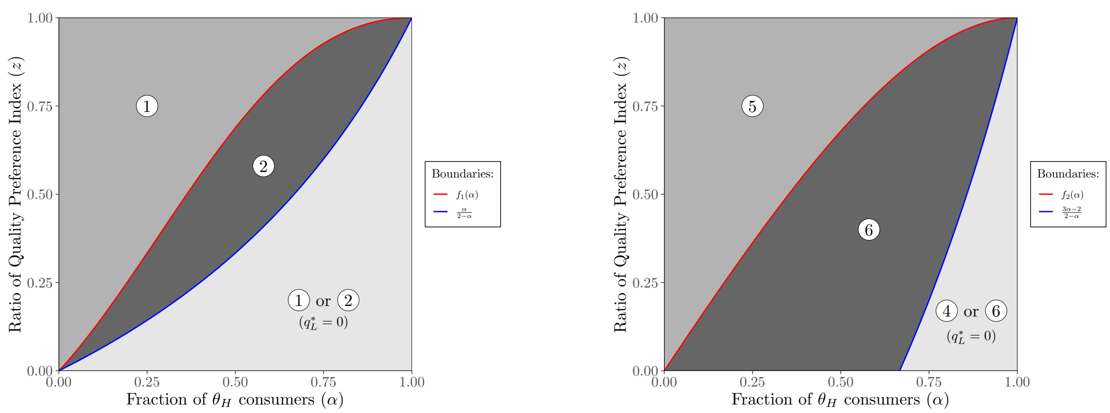
HL segment (targeted by the bundle) is indifferent between buying (q_H, q_L) and (q_L, q_L). Due to symmetry, the optimal price of bundle (q_L, q_H) is the same as that of (q_H, q_L) . Finally, the optimal price of bundle (q_H, q_H) is set such that the HH segment is indifferent between buying (q_H, q_H) and (q_H, q_L) .

Plugging in the optimal prices, we can derive the optimal quality decisions of the firm for the three options. Across all options, $q_H^* = \frac{1}{2}\theta_H$. That is, the optimal quality of the component of the bundle addressing the H component of a market segment should be set at the ideal quality for the H consumers. The lower quality q_L^* varies across options. Under Option (1), $q_L^* = \frac{(z-\alpha)}{2(1-\alpha)}\theta_H$ if $z > \alpha$, and $q_L^* = 0$ otherwise. Under Option (2), $q_L^* = \frac{(2-\alpha)z-\alpha}{4(1-\alpha)}\theta_H$ if $z > \frac{\alpha}{2-\alpha}$, and $q_L^* = 0$ otherwise. Under Option (3), q_L does not exist since only (q_H, q_H) is offered. Similar to the case in the component strategy, q_L^* is a decreasing function of α for both Option (1) and (2), and $q_L^* = \frac{1}{2}\theta_L$ when $\alpha = 0$. This means that to mitigate the cannibalization effect, the firm downgrades the quality of the L product from the ideal level ($\frac{1}{2}\theta_L$). Furthermore, we observe that $\frac{(z-\alpha)}{2(1-\alpha)}\theta_H \leq \frac{(2-\alpha)z-\alpha}{4(1-\alpha)}\theta_H$ with equality holds when $\alpha = 0$ or $z = 1$. This implies that dropping the LL segment in its market coverage allows the firm to increase the quality of product L. Hence the HL and LH segments

Table 8 Optimal quality and profit for bundling Option $(1-3)$ (vertical differentiation)

Option (#)	Cases (within option)	Optimal Quality q_H^*	Optimal Quality q_L^*	Total Profit Π^*	Optimality Condition (across options 1 - 3)
(1)	$z > \alpha$	$\frac{1}{2}\theta_H$	$\frac{(z-\alpha)}{2(1-\alpha)}\theta_H$	$\frac{\alpha(1-2z)+z^2}{2(1-\alpha)}\theta_H^2$	$z \geq f_1(\alpha)$
	$z \leq \alpha$	$\frac{1}{2}\theta_H$	0	$\frac{\alpha}{2}\theta_H^2$	$z \leq \frac{\alpha}{2-\alpha}$ $(*)$
(2)	$z > \frac{\alpha}{2-\alpha}$	$\frac{1}{2}\theta_H$	$\frac{(2-\alpha)z-\alpha}{4(1-\alpha)}\theta_H$	$\frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)-2\alpha z\right]}{8(1-\alpha)}\theta_H^2$	$\frac{\alpha}{2-\alpha} < z \leq f_1(\alpha)$
	$z \leq \frac{\alpha}{2-\alpha}$	$\frac{1}{2}\theta_H$	0	$\frac{\alpha}{2}\theta_H^2$	$z \leq \frac{\alpha}{2-\alpha}$ $(*)$
(3)	all z, α	$\frac{1}{2}\theta_H$	-	$\frac{\alpha^2}{2}\theta_H^2$	-

Figure 6 The region of the locally optimal options for the bundling strategy (vertical differentiation)



(a) Among Option $(1-3)$ (b) Among Option $(4-6)$

enjoy a higher quality for product L under Option (2) and under Option (1). We provide these optimal results and the corresponding firm profit in Table 8.

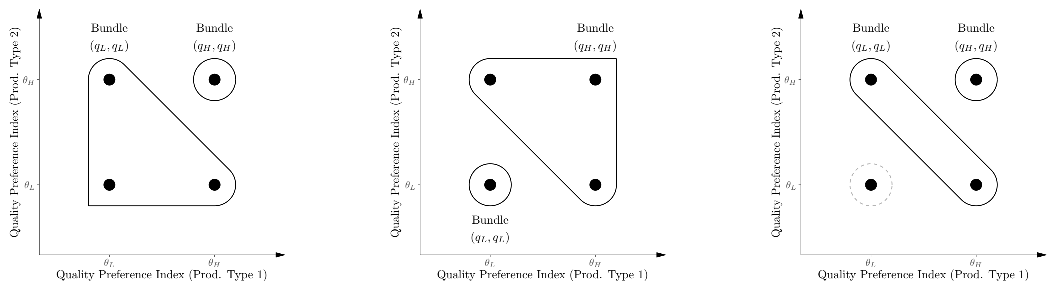
We then compare the firm profit across the three options to identify the conditions under which each option is optimal. To facilitate presentation of the results, we define the following function:

$$f_1(\alpha) \triangleq \frac{4\alpha - 2\alpha^2 + \alpha^3 + 2\alpha(1-\alpha)\sqrt{\alpha}}{4 - 4\alpha + 4\alpha^2 - \alpha^3}.$$

$f_1(\alpha)$ is strictly increasing for $\alpha \in [0,1]$, with $f_1(0) = 0$ and $f_1(1) = 1$. The optimality result of the three options (1-3) can be summarized as follows: (i) when $z \geq f_1(\alpha)$, Option (1) with $q_L^* > 0$ is optimal; (ii) when $\frac{\alpha}{2-\alpha} < z \leq f_1(\alpha)$, Option (2) with $q_L^* > 0$ is optimal; and (iii) when $z \leq \frac{\alpha}{2-\alpha}$, Option (1) or (2) with $q_L = 0$ is optimal. Option (3) is strictly dominated. We provide an illustration of the optimal conditions in Figure 6.

Equivalence between bundling Option (1) and the component strategy. Under close scrutiny, we find that the optional solution of bundling Option (1) where $q_L^* > 0$ is equivalent to the optimal solution of the component strategy where $q_L^* > 0$, and the optional solution of bundling Option (1) or (2) where $q_L^* = 0$ is equivalent to the optimal solution of the component strategy where $q_L^* = 0$. For each of these two cases, q_H^*

Figure 7 **Targeting options ($4-6$) for the bundling strategy (vertical differentiation)**



(a) Option (4) (b) Option (5) (c) Option (6)

and q_L^* are equal across the two strategies, and the bundling strategy can be regarded as "simply stapling two products together" and adding up their prices from the component strategy. Option (2) where $q_L^* > 0$ cannot be replicated by and hence is not equivalent to any component strategy.

While it is natural that (q_H, q_L) and (q_L, q_H) are used to target the HL and LH segments, it is intriguing to ask whether (q_H, q_H) or (q_L, q_L) can be viable options for these segments as well. We investigate three additional cases and seek to address if they are dominated by the previously studies three options. Under Option (4), the HL and LH segments are targeted by (q_L, q_L), and under Option (5), these segments are targeted by (q_H, q_H). A special case of Option (4), which is not mathematically equivalent to the analysis of (4), is when the firm drops the LL segment from the market coverage for (q_L, q_L). This is defined as Option (6). We provide an illustration of these additional cases in Figure 7.

Similar to the analysis of Option (1-3), we provide the firm's profit functions and price constraints for Option (4-6) in Table 9. We solve the price decision first for any fixed q_H and q_L. The optimal price results are provided in Table 9 as well. Plugging in the optimal prices, we then solve the optimal quality decision for the three options and the results are provided in Table 10. It is worth noting that only under Option (5), we have $q_H^* = \frac{1+z}{4}\theta_H = \frac{\theta_H + \theta_L}{4}$, which implies the quality of product H is less than the ideal quality for the H consumers ($\frac{1}{2}\theta_H$). This is because under Option (5), (q_H, q_H) is used to target the HH, HL, LH segments, and q_H^* has to take into account the preferences of both H and L types of consumers.

We first compare among Options $(4-6)$ to see if any option is dominated, before we proceed to compare across Options $(1-6)$. To facilitate presentation of the results, we define the following function:

$$f_2(\alpha) \triangleq \frac{8\alpha - 14\alpha^2 + 11\alpha^3 - 3\alpha^4 + 2\sqrt{2}\alpha(1-\alpha)^2\sqrt{(2-\alpha)(1-\alpha)}}{8 - 16\alpha + 14\alpha^2 - 5\alpha^3 + \alpha^4}$$

$f_2(\alpha)$ is strictly increasing in $\alpha \in [0,1]$, with $f_2(0) = 0$ and $f_2(1) = 1$. It turns out Option (4) is weakly dominated. The detailed results are as follows: (i) when $z \geq f_2(\alpha)$, Option (5) with $q_L^* > 0$ is optimal; (ii) when $\frac{3\alpha-2}{2-\alpha} < z \leq f_2(\alpha)$, Option (6) with $q_L^* > 0$ is optimal ; (iii) when $z \leq \frac{3\alpha-2}{2-\alpha}$, Option (4) or (6) with

Table 9 Profit functions and optimal prices for bundling Option $(4-6)$ (vertical differentiation)

Option	Profit function	Price Constraints	p^*_{HH}	p^*_{LL}
(4)	$\alpha^2(p_{HH}-2q_H^2)$ $+(1-\alpha^2)(p_{LL}-2q_L^2)$	$2\theta_H q_H - p_{HH} \geq 2\theta_H q_L - p_{LL}$ $2\theta_H q_H - p_{HH} \geq 0$ $\theta_H q_L + \theta_L q_L - p_{LL} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_L + \theta_L q_L - p_{LL} \geq 0$ $2\theta_L q_L - p_{LL} > 2\theta_L q_H - p_{HH}$ $2\theta_L q_L - p_{LL} \geq 0$	$2(\theta_H q_H + \theta_L q_L - \theta_H q_L)$	$2\theta_L q_L$
(5)	$\alpha(2-\alpha)(p_{HH}-2q_H^2)$ $+(1-\alpha)^2(p_{LL}-2q_L^2)$	$2\theta_H q_H - p_{HH} \geq 2\theta_H q_L - p_{LL}$ $2\theta_H q_H - p_{HH} \geq 0$ $\theta_H q_H + \theta_L q_H - p_{HH} \geq \theta_H q_L + \theta_L q_L - p_{LL}$ $\theta_H q_H + \theta_L q_H - p_{HH} \geq 0$ $2\theta_L q_L - p_{LL} > 2\theta_L q_H - p_{HH}$ $2\theta_L q_L - p_{LL} \geq 0$	$(\theta_H+\theta_L)q_H - (\theta_H-\theta_L)q_L$	$2\theta_L q_L$
(6)	$\alpha^2(p_{HH}-2q_H^2)$ $+2\alpha(1-\alpha)(p_{LL}-2q_L^2)$	$2\theta_H q_H - p_{HH} \geq 2\theta_H q_L - p_{LL}$ $2\theta_H q_H - p_{HH} \geq 0$ $\theta_H q_L + \theta_L q_L - p_{LL} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_L + \theta_L q_L - p_{LL} \geq 0$	$2\theta_H q_H + \theta_L q_L - \theta_H q_L$	$(\theta_H+\theta_L)q_L$

Table 10 Optimal quality and profit for bundling Option $(4-6)$ (vertical differentiation)

Option (#)	Cases (within option)	Optimal Quality q^*_H	Optimal Quality q^*_L	Total Profit Π^*	Optimality Condition (across options 4 - 6)
(4)	$z>\alpha^2$	$\frac{1}{2}\theta_H$	$\frac{z-\alpha^2}{2(1-\alpha^2)}\theta_H$	$\frac{\alpha^2(1-2z)+z^2}{2(1-\alpha^2)}\theta_H^2$	-
	$z\leq\alpha^2$	$\frac{1}{2}\theta_H$	0	$\frac{\alpha^2}{2}\theta_H^2$	$z\leq\frac{3\alpha-2}{2-\alpha}$ $(*)$
(5)	$z>\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$	$\frac{1+z}{4}\theta_H$	$(\frac{1+z}{4}-\frac{1-z}{4(1-\alpha)^2})\theta_H$	$\frac{4z^2+\alpha(2-\alpha)(1-2z-3z^2)}{8(1-\alpha)^2}\theta_H^2$	$z\geq f_2(\alpha)$
	$z\leq\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$	$\frac{1}{2}\theta_H$	0	$\frac{\alpha(2-\alpha)(1+z)^2}{8}\theta_H^2$	-
(6)	$z>\frac{3\alpha-2}{2-\alpha}$	$\frac{1}{2}\theta_H$	$\frac{(2-\alpha)z-(3\alpha-2)}{8(1-\alpha)}\theta_H$	$\frac{\alpha(2-\alpha)[(2-\alpha)(1+z^2)+2(2-3\alpha)z]}{16(1-\alpha)}\theta_H^2$	$\frac{3\alpha-2}{2-\alpha}<z\leq f_2(\alpha)$
	$z\leq\frac{3\alpha-2}{2-\alpha}$	$\frac{1}{2}\theta_H$	0	$\frac{\alpha^2}{2}\theta_H^2$	$z\leq\frac{3\alpha-2}{2-\alpha}$ $(*)$

$q^*_L=0$ is optimal. For Option (4) or (6) with $q^*_L=0$, it becomes mathematically equivalent to Option (3). We provide an illustration of the optimal conditions for Options $(4-6)$ in Figure **??**.

Optimal bundling option. We next compare across Options $(1-6)$ to identify the overall optimal option for the bundling strategy. To facilitate presentation, we further define:

$$f_3(\alpha) \triangleq \frac{2\alpha-3\alpha^2+3\alpha^3-\alpha^4+2(1-\alpha)^2\sqrt{\alpha(1-\alpha)(2-\alpha)}}{4-10\alpha+11\alpha^2-5\alpha^3+\alpha^4}.$$

$f_3(\alpha)$ is also increasing for $\alpha\in[0,1]$. Moreover, $f_3(\alpha)<f_1(\alpha)$ if and only if $\frac{2}{3}<\alpha<1$.

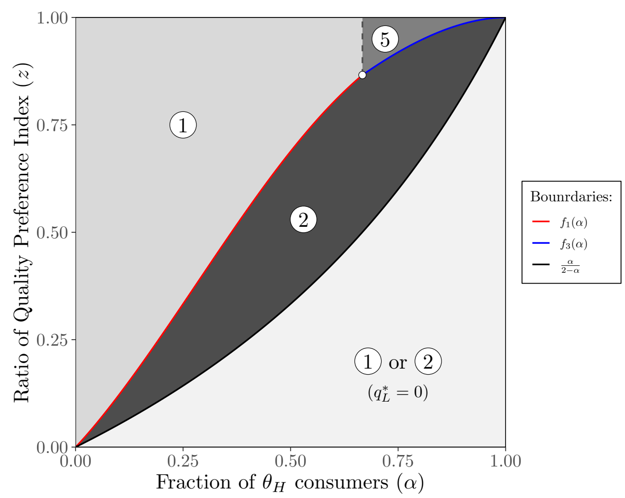
The optimal results are characterized as follows: (i) when $z\leq\frac{\alpha}{2-\alpha}$, Option (1) or (2) with $q^*_L=0$ is optimal; (ii) when $\frac{\alpha}{2-\alpha}<z\leq\min\{f_1(\alpha),f_3(\alpha)\}$, Option (2) with $q^*_L>0$ is optimal; (iii) when $\alpha\leq\frac{2}{3}$ & $z\geq f_1(\alpha)$, Option (1) with $q^*_L>0$ is optimal; (iv) when $\alpha\geq\frac{2}{3}$ & $z\geq f_3(\alpha)$, Option (5) with $q^*_L>0$ is optimal. We summarize the results in Table 11 and provide an illustration of the optimal conditions in Figure 8.

The results indicate that only Option (1), (2), (5) can be optimal when compared across $(1-6)$, and the others are dominated. When the L consumers' willingness to pay for quality is significantly lower than that

Table 11 The optimal option and corresponding qualities for the bundling strategy (vertical differentiation)

Option	Optimal Quality		Total Profit	Optimality Condition
(#)	q_H^*	q_L^*	Π^*	(Across all options)
(1) or (2)	$\frac{\theta_H}{2}$	0	$\frac{\alpha\theta_H^2}{2}$	$z \le \frac{\alpha}{2-\alpha}$
(2)	$\frac{\theta_H}{2}$	$\frac{2\theta_L-\alpha(\theta_H+\theta_L)}{4(1-\alpha)}$	$\frac{\alpha(2-\alpha)\left[(2-\alpha)(\theta_H^2+\theta_L^2)-2\alpha\theta_H\theta_L\right]}{8(1-\alpha)}$	$\frac{\alpha}{2-\alpha} < z \le \min\{f_1(\alpha), f_3(\alpha)\}$
(1)	$\frac{\theta_H}{2}$	$\frac{\theta_L-\alpha\theta_H}{2(1-\alpha)}$	$\frac{\alpha\theta_H(\theta_H-2\theta_L)+\theta_L^2}{2(1-\alpha)}$	$\alpha \le \frac{2}{3}$ & $z \ge f_1(\alpha)$
(5)	$\frac{\theta_H+\theta_L}{4}$	$\frac{\theta_H+\theta_L}{4} - \frac{\theta_H-\theta_L}{4(1-\alpha)^2}$	$\frac{4\theta_L^2+\alpha(2-\alpha)(\theta_H^2-2\theta_H\theta_L-3\theta_L^2)}{8(1-\alpha)^2}$	$\alpha \ge \frac{2}{3}$ & $z \ge f_3(\alpha)$

Figure 8 The region of the optimal options for the bundling strategy (vertical differentiation)



of the H consumers (i.e., $z \le \frac{\alpha}{2-\alpha}$), the firm should abandon the L consumers (set $q_L^* = 0$) and offer Option (1) or (2). These bundling options effectively reduce to the component strategy targeting the H consumers. When the L consumers' willingness to pay gets higher but is still moderately lower than that of the H consumers (i.e., $\frac{\alpha}{2-\alpha} < z \le \min\{f_1(\alpha), f_3(\alpha)\}$), the firm should abandon the LL segment and target the rest with Option (2). When the L consumers' willingness to pay gets even higher till it is identical to that of the H consumers (i.e., $z > \min\{f_1(\alpha), f_3(\alpha)\}$), the firm's bundling choice depends on the size of the H consumers: (i) if $\alpha < \frac{2}{3}$, it should separate the HH, HL, LH, LL segments and target each with a unique bundle using Option (1); (ii) otherwise, the firm should pool the HH, HL, LH segments together and target them with (q_H^*, q_H^*), and target LL with (q_L^*, q_L^*) using Option (5). One plausible explanation for the last case here is that when the H and L consumers' willingness to pay are similar and α is large (hence the HH segment is large), the firm will benefit greatly by reducing the hierarchy level of bundles and hence reducing the cannibalization effect, which brings the most gain from the HH segment (Recall that as we set the optimal bundle prices, the price of a bundle is set such that the targeted segment is indifferent between buying it and the lower quality alternative. This allows the targeted segment to keep some surplus to themselves. As the firm increases the hierarchy level of bundles, the HH consumer segment can keep more and more surplus to

Table 12 The optimal strategy between component and bundling (vertical differentiation)

Optimal Strategy	Vertical Differentiation?	Product Quality		Total Profit	Optimality Condition
		q_H^*	q_L^*	Π^*	
Component	No	$\frac{\theta_H}{2}$	–	$\frac{\alpha\theta_H^2}{2}$	$z \le \frac{\alpha}{2-\alpha}$
Component	Yes	$\frac{\theta_H}{2}$	$\frac{\theta_L-\alpha\theta_H}{2(1-\alpha)}$	$\frac{\alpha\theta_H(\theta_H-2\theta_L)+\theta_L^2}{2(1-\alpha)}$	$\alpha \le \frac{2}{3}$ & $z \ge f_1(\alpha)$
Bundling (2)	Yes	$\frac{\theta_H}{2}$	$\frac{2\theta_L-\alpha(\theta_H+\theta_L)}{4(1-\alpha)}$	$\frac{\alpha(2-\alpha)\left[(2-\alpha)(\theta_H^2+\theta_L^2)-2\alpha\theta_H\theta_L\right]}{8(1-\alpha)}$	$\frac{\alpha}{2-\alpha} \le z \le \min\{f_1(\alpha), f_3(\alpha)\}$
Bundling (5)	Yes	$\frac{\theta_H+\theta_L}{4}$	$\frac{\theta_H+\theta_L}{4}-\frac{\theta_H-\theta_L}{4(1-\alpha)^2}$	$\frac{4\theta_L^2+\alpha(2-\alpha)(\theta_H^2-2\theta_H\theta_L-3\theta_L^2)}{8(1-\alpha)^2}$	$\alpha \ge \frac{2}{3}$ & $z \ge f_3(\alpha)$

themselves because the surplus stacks. As a result, reducing the hierarchy may reduce the surplus the HH consumers can keep to themselves, which benefits the firm greatly).

3.3. Component vs. Bundling Strategy

In order to identify whether component or bundling strategy provides more value to the firm in the presence of vertical differentiation, we can compare the firm's profit under the two strategies. In Section 3.2, we have noted the equivalence between the bundling Option (1) and the component strategy where the optimal decisions coincide. As a result, the comparison between the component and bundling strategy is embedded in the analysis of the bundling strategy. What we will do here is, when a bundling strategy is equivalent to some component strategy, we explicitly note it as a component strategy, due to its simplicity in practical implementation and being more natural to think of. We provide the summary in the following proposition.

PROPOSITION 2. *The optimal strategy (component vs bundling) of the firm in the presence of vertical differentiation is summarized in Table 12.*

The results suggest that only Option (2) and (5) of the the bundling strategy can provide additional value to the firm when compared against the component strategy. These bundling options outperform the component strategy under market conditions that are exemplified by the grey area in Figure 8. When we compare the optimal bundling region in the presence of vertical differentiation (Figure 8) with that of the single quality (Figure 4), we observe that the bundling region is enlarged. This implies that as the firm increases its product variety (through vertical differentiation), the bundling strategy is able to provide higher profit than the component strategy over a wider range of market conditions. The exact choice of the bundling strategy, though, needs to be carefully curated to the market conditions.

4. Extension: Precise Targeting (via Bundling)

In Section 3, we studied the case where two quality levels are offered in a product type. One might wonder if there is any benefit to further differentiate a product type and offer more qualities. Under the component strategy, offering more than two quality levels does not provide additional benefit, because the H and L consumers will buy at most two different products and offering a third (or fourth) can only improve consumer

Table 13 Optimal quality and profit for bundling Option (7 − 9) (Precise Targeting)

Option (#)	Optimal Quality				Total Profit	Optimality Condition
	q_H^*	q_h^*	q_l^*	q_L^*	Π^*	
(7)	$\frac{\theta_H}{2}$	$\frac{\theta_H}{2}$	$\frac{(2-\alpha)\theta_L-\alpha\theta_H}{4(1-\alpha)}$	$\frac{\theta_H+\theta_L}{4}-\frac{\theta_H-\theta_L}{4(1-\alpha)^2}$	$\frac{\alpha(2-\alpha)^2\theta_H^2-2\alpha(2-\alpha)^2\theta_H\theta_L+(\alpha^3-4\alpha+4)\theta_L^2}{8(1-\alpha)^2}$	$f_4(\alpha)\le z\le 1$
(8)	$\frac{\theta_H}{2}$	$\frac{\theta_H}{2}$	$\frac{(2-\alpha)\theta_L-\alpha\theta_H}{4(1-\alpha)}$	−	$\frac{\alpha(2-\alpha)^2\theta_H^2-2\alpha^2(2-\alpha)\theta_H\theta_L+\alpha(2-\alpha)^2\theta_L^2}{8(1-\alpha)}$	$\frac{\alpha}{2-\alpha}<z\le f_4(\alpha)$
(7) or (8)	$\frac{\theta_H}{2}$	$\frac{\theta_H}{2}$	0	−	$\frac{\alpha\theta_H^2}{2}$	$0\le z\le\frac{\alpha}{2-\alpha}$

surplus (i.e., a larger set of alternatives increases consumers' reserved utility). However, under the bundling strategy, offering more than two qualities may still be beneficial. The increased product differentiation may allow the firm to more precisely target the consumer segments via a bundling strategy.

We study the counterparts of Option $(1-3)$ that was introduced in Section 3.2, where we replace (q_H, q_L) and (q_L, q_H) by (q_h, q_l) and (q_l, q_h), with $q_H \ge q_h \ge q_l \ge q_L$. We define the following options:

(a) Option (7): target the HH, HL, LH, LL segments by (q_H, q_H), (q_h, q_l), (q_l, q_h), (q_L, q_L), respectively;

(b) Option (8): target the HH, HL, LH segments by (q_H, q_H), (q_h, q_l), (q_l, q_h), respectively;

(c) Option (9): target the HH segment by (q_H, q_H);

Since we allow the qualities to be equal, the above options capture the counterparts of Option $(4-6)$ as well.

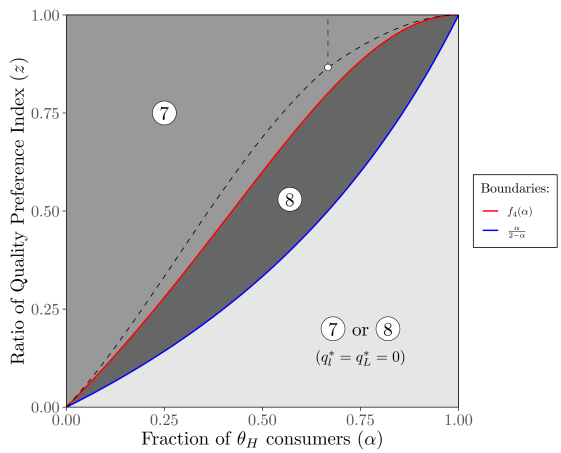
We analyze Option $(7-9)$ similarly to how we analyze Option $(1-3)$. We characterize the optimal price and quality decisions for each option first, and then compare across the options. Since Option (7) includes the component strategy as a special case, the study of the bundling strategy includes the comparison between component and bundling strategies as well. The details of the analysis are relegated to Appendix A.3. Below we summarize the key results for the precise targeting setting.

PROPOSITION 3. *The optimal bundling option and the corresponding product qualities for the precise targeting setting is summarized in Table 13. Within the table,* $f_4(\alpha) \triangleq \frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$.

The results suggest that there are three cases for the optimal option of the precise targeting setting: (i) when $f_4(\alpha) \le z \le 1$, Option (7) is optimal with $q_H^* = q_h^* > q_l^* > q_L^* > 0$; (ii) when $\frac{\alpha}{2-\alpha} < z \le f_4(\alpha)$, Option (8) is optimal with $q_H^* = q_h^* > q_l^* > 0$; (iii) when $0 \le z \le \frac{\alpha}{2-\alpha}$, Option (7) or (8) is optimal with $q_H^* = q_h^* > q_l^* = 0$. An illustration of the optimal conditions is provided in Figure 9.

We can draw the following managerial insights from the study of this extended model. First, increasing product differentiation can provide further value to the firm under the bundling strategy. As shown for Option (7), the firm should offer three quality levels within each product type (q_H^*, q_l^*, q_L^*), and the increased differentiation is aimed for the L consumers. When we compare the quality offerings to the L consumers under Option (7) with those under Option (1), we find that $q_{l(7)}^* > q_{L(1)}^* > q_{L(7)}^*$. This indicates the L consumers in the segments HL and LH enjoy a higher quality under Option (7) than under Option (1), and the L consumers in the segment LL enjoy a lower quality under Option (7) than under Option (1). Second, as

Figure 9 The region of the optimal options for the bundling strategy (Precise Targeting)



the firm further increases the level of vertical differentiation, the bundling strategy is able to provide higher profit than the component strategy over a broader range of market conditions, which reassures our findings in the vertical differentiation analysis and is illustrated by comparing the bundling region in Figure 9 (the two darkest areas) with the bundling region in Figure 8 (the two darkest areas). Note that Option (8) and Option (2) share the same product qualities and profits in optimality and are therefore equivalent, when we compare the optimal region for them between vertical differentiation and precise targeting, we find that this bundling strategy becomes less favorable to the firm as the level of vertical differentiation is increased. With increased differentiation, Option (8) (or (2)) loses attractiveness to (7) under some market conditions.

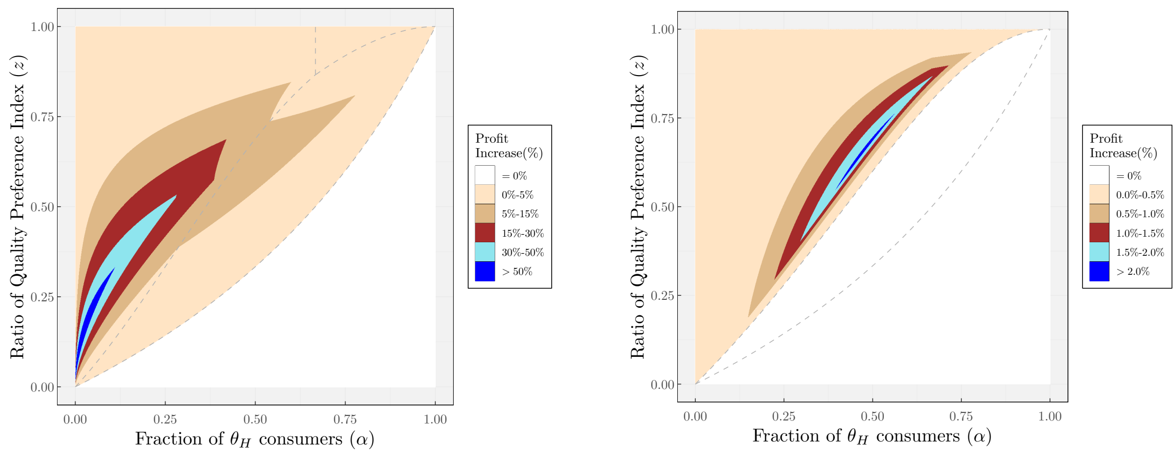
We conclude the section with a comparison of the firm's profit under the single quality, vertical differentiation, and precise targeting settings. We denote the three cases by SQ, VD, and PT, respectively. Let Π be the optimal profit of the firm. We have the following result.

PROPOSITION 4. $\Pi^{(\mathrm{SQ})} \le \Pi^{(\mathrm{VD})} \le \Pi^{(\mathrm{PT})}$.

The result is not surprising as the lower differentiation model can always be regarded as a special case of the higher differentiation model, and hence its optimal result is sub-optimal for the bigger model. Below through some numerical experiments, we show the magnitude of profit improvement over the models.

In the numerical experiments, we fix $\theta_H = 1$ and vary $\theta_L \in [0, 1]$. We report the percentage increase of firm profit from single quality to vertical differentiation to precise targeting. In Figure 10(a), we show the profit increase from single quality (SQ) to vertical differentiation (VD). We observe that the firm's optimal profit can increase by more than 30% under a small range of market conditions, and by 5% - 30% under a much wider range of market conditions. In Figure 10(b), we show the profit increase from vertical differentiation (VD) to precise targeting (PT). We observe that the profit increases at a much slower rate, which indicates there is a diminishing rate of return for the firm to increase the level of product differentiation. If there is a

Figure 10 Profit Increase from Single Quality (SQ) to Vertical Differentiation (VD) to Precise Targeting (PT)



(a) From SQ to VD

(b) From VD to PT

moderate or significant fixed cost for increasing product differentiation, as considered in Zou et al. (2020), the firm may be advised not to maximize its product differentiation.

5. Conclusion

In this paper, we study the product bundling problem in the presence of vertical differentiation for firms that seek to maximize profit when selling multiple types of products. We investigate increased differentiation levels from single quality to two qualities to more than two qualities, and for each we identify the optimal bundling design as well as the conditions under which the bundling strategy outperforms the component strategy. To the best of our knowledge, our work is among the first to study both product bundling and vertical differentiation, and we identify conditions under which the two strategies should be jointly deployed.

We study a setting where a firm sells two product types, with consumers having a high and low willingness to pay for quality in each type. Denoting consumers with a high and low willingness to pay by H and L, and based on a consumer's willingness to pay in two product types, the consumer market can be segmented into four groups: HH, HL, LH, and LL. In the single quality case, we show that the bundling strategy that targets the HH, HL, LH segments with qualities $(\frac{\theta_H+\theta_L}{4}, \frac{\theta_H+\theta_L}{4})$ is the only strategy that outperforms the component strategy under some market conditions. In the vertical differentiation case (a maximum of two qualities), we find that two bundling strategies may outperform the component strategy under some market conditions: (i) targeting the HH, HL, LH segments separately with three distinct bundles: (q_H^*, q_H^*), (q_H^*, q_L^*), (q_L^*, q_H^*), and abandoning the LL segment; (ii) targeting the HH, HL, LH segments together with one bundle (q_H^*, q_H^*), and targeting the LL segment with another (q_L^*, q_L^*). Moving to the precise targeting case (a maximum of three qualities or more), we find that two modified bundling strategies may outperform the component strategy: (1) targeting the HH, HL, LH, LL segments separately with four distinct bundles (requiring three quality levels in each product type); (ii) targeting the HH, HL, LH segments with three distinct bundles and abandoning the LL segment (requiring two quality levels in each product type). Our results also show that as the firm increases the level of vertical differentiation, the bundling strategy is able to provide higher profit than the component strategy under a broader range of market conditions. Therefore the bundling strategy becomes more favorable as the level of vertical differentiation increases.

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Appendix A: Proofs

A.1. Single Quality

A.1.1. Pure Component. We denote the profit function of the firm under this strategy by Π^{sc}. Under this strategy, the optimal profit of the firm when targeting θ_H consumers only is $\frac{1}{2}\alpha\theta_H^2$ and the optimal profit when targeting all consumers is $\frac{1}{2}\theta_L^2$. We have $\frac{1}{2}\alpha\theta_H^2 \geq \frac{1}{2}\theta_L^2$ if and only if $\alpha \geq z^2$, where $z = \frac{\theta_L}{\theta_H}$.

A.1.2. Pure Bundling. We denote the profit function of the firm under this strategy by Π^{sb}. Under this strategy, the firm has four targeting options. Let Π_j^{sb} denote the profit of the firm under option $j \in \{1,2,3,4\}$, where the Π_j^{sb}'s are presented in Table 3. The optimal quality decision for each option can be obtained by taking the first order of the profit function. The quality decisions and the firm's profit functions (obtained at the quality decisions) for the options are provided in Table 4. In order to derive the optimal option, we first compare the options pairwise:

(i) $\Pi_1^{sb} \geq \Pi_2^{sb} \iff 0 \leq z \leq \sqrt{-1+2\alpha}$.

(ii) $\Pi_1^{sb} \geq \Pi_3^{sb} \iff 0 \leq z \leq -1+2\sqrt{\frac{\alpha}{2-\alpha}}$.

(iii) $\Pi_1^{sb} \geq \Pi_4^{sb} \iff 0 \leq z \leq \alpha$.

(iv) $\Pi_2^{sb} \geq \Pi_3^{sb} \iff 0 \leq z \leq \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}$.

(v) $\Pi_2^{sb} \geq \Pi_4^{sb} \iff 0 \leq z \leq \sqrt{\frac{\alpha}{2-\alpha}}$.

(vi) $\Pi_3^{sb} \geq \Pi_4^{sb} \iff 0 \leq z \leq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$.

With these results, we can characterize the optimal option as follows (summarized in Table 4).

(1) Option 1 is optimal if $0 \leq z \leq \min\{\sqrt{2\alpha-1}, 2\sqrt{\frac{\alpha}{2-\alpha}}-1, \alpha\}$. In fact, we have $\alpha \geq \sqrt{2\alpha-1}$ for $0 \leq \alpha \leq 1$. Hence the optimality condition reduces to $0 \leq z \leq \min\{\sqrt{2\alpha-1}, 2\sqrt{\frac{\alpha}{2-\alpha}}-1\}$.

(2) Option 2 is optimal if $\sqrt{2\alpha-1} \leq z \leq \min\{\frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}, \sqrt{\frac{\alpha}{2-\alpha}}\}$. In fact, we have $\frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha} \leq \sqrt{\frac{\alpha}{2-\alpha}}$ for $0 \leq \alpha \leq 1$. Hence the optimality condition reduces to $\sqrt{2\alpha-1} \leq z \leq \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}$.

(3) Option 3 is optimal if $\max\{2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}\} \leq z \leq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$.

(4) Option 4 is optimal if $z \geq \max\{\alpha, \sqrt{\frac{\alpha}{2-\alpha}}, \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}\}$. In face, we have $\frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4} \geq \alpha \geq \sqrt{\frac{\alpha}{2-\alpha}}$ for $0 \leq \alpha \leq 1$. Hence the optimality condition reduces to $z \geq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$.

A.1.3. Pure Component vs Pure Bundling. We denote the optimal profit of the firm for the single quality case by Π^s, where $\Pi^s = \max\{\Pi^{sc}, \Pi^{sb}\}$. We compare Π^{sc} and Π^{sb} below:

(1) when $0 \leq z \leq \min\{\sqrt{2\alpha-1}, 2\sqrt{\frac{\alpha}{2-\alpha}}-1\}$: in this case, $\Pi^{sc} = \frac{1}{2}\alpha\theta_H^2$, $\Pi^{sb} = \frac{1}{2}\alpha^2\theta_H^2$. We have $\Pi^{sc} \geq \Pi^{sb}$.

(2) when $\sqrt{2\alpha-1} \leq z \leq \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}$: in this case, $\Pi^{sc} = \frac{1}{2}\alpha\theta_H^2$, $\Pi^{sb} = \frac{1}{4}\alpha(\theta_H^2+\theta_L^2)$. We have $\Pi^{sc} \geq \Pi^{sb}$.

(3) when $\max\{2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}\} \leq z \leq \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$: we consider two sub-cases:

(3.a) when $z \leq \sqrt{\alpha}$: in this case, $\Pi^{sc} = \frac{1}{2}\alpha\theta_H^2$, $\Pi^{sb} = \frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2$. Therefore, $\Pi^{sc} - \Pi^{sb} = \frac{1}{2}\alpha\theta_H^2 - \frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2 = \frac{1}{8}\alpha\theta_H^2\left[4-(2-\alpha)(1+z)^2\right]$. Let $f(z) = 4-(2-\alpha)(1+z)^2$. We have $f(z) \geq 0 \iff 0 \leq z \leq \frac{2}{\sqrt{2-\alpha}}-1$. As a result, we have

(i) $\Pi^{sc} \geq \Pi^{sb}$ when $\max(2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}) \leq z \leq \min(\frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}, \sqrt{\alpha}, \frac{2}{\sqrt{2-\alpha}}-1)$.

(ii) $\Pi^{sc} \leq \Pi^{sb}$ when $\max(2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}, \frac{2}{\sqrt{2-\alpha}}-1) \leq z \leq \min(\frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}, \sqrt{\alpha})$.

(3.b) when $z > \sqrt{\alpha}$: in this case, $\Pi^{sc} = \frac{1}{2}\theta_L^2$, $\Pi^{sb} = \frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2$. $\Pi^{sc} - \Pi^{sb} = \frac{1}{2}\theta_L^2 - \frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2 = \frac{1}{8}\theta_H^2\left[4z^2 - \alpha(2-\alpha)(1+z)^2\right]$. Let $f(z) = 4z^2 - \alpha(2-\alpha)(1+z)^2$. We have $f(z) \ge 0 \iff \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4} \le z \le 1$. As the condition of case (3) requires $z \le \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$, we have

(i) $\Pi^{sc} \le \Pi^{sb}$ when $\max(2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}, \sqrt{\alpha}) < z \le \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$.

(4) when $z > \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$: we consider two subcases:

(4.a) if $z \le \sqrt{\alpha}$: in this case, $\Pi^{sc} = \frac{1}{2}\alpha\theta_H^2$, $\Pi^{sb} = \frac{1}{2}\theta_L^2$. We have $\Pi^{sc} \ge \Pi^{sb}$.

(4.b) if $z > \sqrt{\alpha}$: in this case, $\Pi^{sc} = \frac{1}{2}\theta_L^2$, $\Pi^{sb} = \frac{1}{2}\theta_L^2$. We have $\Pi^{sc} = \Pi^{sb}$.

To summarize the results in the above four cases, we can show and apply the following claim (proof is omitted):

CLAIM 1. *For $\alpha \in [0,1]$, we have*

(a) $\min(\frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}, \sqrt{\alpha}, \frac{2}{\sqrt{2-\alpha}}-1) = \min(\frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}, \frac{2}{\sqrt{2-\alpha}}-1)$.

(b) $\max(2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}, \frac{2}{\sqrt{2-\alpha}}-1) = \frac{2}{\sqrt{2-\alpha}}-1$.

(c) $\max(2\sqrt{\frac{\alpha}{2-\alpha}}-1, \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha}, \sqrt{\alpha}) = \sqrt{\alpha}$.

By applying the above claim, we can summarize the results into the following three cases:

(i) $\Pi^s = \Pi^{sc} = \frac{1}{2}\alpha\theta_H^2$ when $z \le \min(\sqrt{\alpha}, \frac{2}{\sqrt{2-\alpha}}-1)$.

(ii) $\Pi^s = \Pi^{sb} = \frac{1}{8}\alpha(2-\alpha)(\theta_L+\theta_H)^2$ when $\frac{2}{\sqrt{2-\alpha}}-1 < z \le \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4}$.

(iii) $\Pi^s = \Pi^{sc} = \frac{1}{2}\theta_L^2$ when $z > \max(\sqrt{\alpha}, \frac{\alpha(2-\alpha)+2\sqrt{\alpha(2-\alpha)}}{\alpha^2-2\alpha+4})$.

A.2. Vertical Differentiation

A.2.1. Pure Component. We denote the profit function of the firm under this strategy by Π^{vc}. For given qualities q_H and q_L, the firm seeks to maximize the total profit by setting prices p_H and p_L, such that the θ_H consumers buys product H and the θ_L consumers buys product L. The optimal prices are presented in the following claim.

CLAIM 2. *The optimal prices for products H and L are $p_H = \theta_H q_H - (\theta_H - \theta_L)q_L$ and $p_L = \theta_L q_L$.*

Proof of Claim 2. The profit-maximization problem of the firm is expressed by the following linear program:

$$\Pi^{vc} = \max_{p_H, p_L} \quad 2[\alpha(p_H - q_H^2) + (1-\alpha)(p_L - q_L^2)] \qquad \text{(VC-Price)}$$

$$\text{s.t.} \quad \theta_H q_H - p_H \ge \theta_H q_L - p_L \qquad \text{(1a)}$$

$$\theta_H q_H - p_H \ge 0 \qquad \text{(1b)}$$

$$\theta_L q_L - p_L > \theta_L q_H - p_H \qquad \text{(1c)}$$

$$\theta_L q_L - p_L \ge 0 \qquad \text{(1d)}$$

We denote the optimal prices by p_H^* and p_L^*. We show the following results must hold. Constraint (1b) is satisfied if constraints (1a) and (1d) are satisfied. Therefore, we are going to drop constraint (1b) in our proof below.

(1) $p_L^* = \theta_L q_L$. We prove by contradiction. Suppose $p_L^* \ne \theta_L q_L$, by constraint (1d), we must have $\theta_L q_L - p_L^* > 0$. Let $\Delta_1 \triangleq \theta_L q_L - p_L^* > 0$. We construct a feasible solution $(\hat{p}_H, \hat{p}_L)$ that violates the optimality of (p_H^*, p_L^*). Let $\hat{p}_H = p_H^* + \Delta_1$, $\hat{p}_L = p_L^* + \Delta_1$. First, we can verify $(\hat{p}_H, \hat{p}_L)$ satisfies all the constraints of Problem (VC-Price). It is straightforward to verify that constraints (1a), (1c), (1d) are satisfied by $(\hat{p}_H, \hat{p}_L)$. Second, we can show that $(\hat{p}_H, \hat{p}_L)$ achieves a higher objective value than (p_H^*, p_L^*). This is trivial. Hence we proved $p_L^* = \theta_L q_L$.

(2) $p_H^* = \theta_H q_H - (\theta_H - \theta_L) q_L$. We prove by contradiction as well. Suppose $p_H^* \neq \theta_H q_H - (\theta_H - \theta_L) q_L$, which is equivalent to $\theta_H q_H - p_H^* \neq \theta_H q_L - \theta_L q_L$. By constraint (1a) and plugging in $p_L^* = \theta_L q_L$, we have $p_H^* < \theta_H q_H - (\theta_H - \theta_L) q_L$. Let $\Delta_2 \triangleq \theta_H q_H - (\theta_H - \theta_L) q_L - p_H^* > 0$. We construct a feasible solution $(\hat{p}_H, \hat{p}_L)$ that violates the optimality of (p_H^*, p_L^*). We let $\hat{p}_H = p_H^* + \Delta_2$, $\hat{p}_L = p_L^*$. First, we can verify $(\hat{p}_H, \hat{p}_L)$ is a feasible solution to Problem (VC-Price). Constraint (1a) is satisfied because

$$\theta_H q_H - \hat{p}_H = \theta_H q_H - (p_H^* + \Delta_2) = (\theta_H - \theta_L) q_L = \theta_H q_L - \hat{p}_L$$

Constraints (1c) and (1d) are satisfied trivially.

Second, we can show $(\hat{p}_H, \hat{p}_L)$ achieves a higher objective value than (p_H^*, p_L^*). This is trivial and omitted. Hence we proved $p_H^* = \theta_H q_H - (\theta_H - \theta_L) q_L$.

□

Plugging in the optimal price solution in Claim 2, the profit function of the firm becomes

$$2[\alpha(\theta_H q_H - (\theta_H - \theta_L) q_L - q_H^2) + (1-\alpha)(\theta_L q_L - q_L^2)]$$

This function is concave in q_H and q_L. Taking the first order derivative gives us the unconstrained optimal solution for quality: $q_H^* = \frac{\theta_H}{2}$, $q_L^* = \frac{\theta_L - \alpha\theta_H}{2(1-\alpha)}$. Note that $q_L^* > 0$ if and only if $\alpha < \frac{\theta_L}{\theta_H}$. Therefore when $\alpha \geq \frac{\theta_L}{\theta_H} (= z)$, the quality of product L should be $q_L^* = 0$, which is equivalent to not offering product L. These results are summarized in Table 6.

A.2.2. Pure Bundling. We denote the profit function of the firm under this strategy by Π^{vb}. Under this strategy, the firm has six targeting options. The profit functions of the firm for Options $(1-3)$ are provided in Table 7, and the profit functions for Options $(4-6)$ are provided in Table 9. We denote the profit of option j by Π_j^{vd}.

To obtain the optimal prices of the options, we solve the corresponding linear program for each option by applying a similar approach described in the proof of Claim 2. We omit the proofs and show the optimal price results in Table 7 and 9. The optimal quality decisions of the options can then be solved by taking the first order derivative of the updated profit function (by plugging in the optimal prices). Similar to the pure component case (Section A.2.1), the solution of q_L^* should be forced to 0 when it becomes negative. The optimal quality results and the updated profit functions for the six options are provided in Table 8 and 10.

We then compare across the options to identify the optimal option for pure bundling. We do this in two steps.

(1) First, we compare the options within $(1-3)$ and $(4-6)$ separately, which is detailed below.

(1.1) **The optimal option among** $(1-3)$. We compare the profit functions of the firm for the three options (see the functions in Table 8) by the following cases:

(1.1.a) When $z \leq \frac{\alpha}{2-\alpha}$, we have $\Pi_1^{vb} = \Pi_2^{vb} > \Pi_3^{vb}$.

(1.1.b) When $\frac{\alpha}{2-\alpha} < z \leq \alpha$, we have $\Pi_2^{vb} > \Pi_1^{vb} > \Pi_3^{vb}$.

(1.1.c) When $\alpha < z$, we have $\Pi_1^{vb} > \Pi_3^{vb}$ and $\Pi_2^{vb} > \Pi_3^{vb}$. To compare Π_1^{vb} and Π_2^{vb}, we compute

$$\begin{aligned}\Pi_1^{vb} - \Pi_2^{vb} &= \frac{\alpha(1-2z)+z^2}{2(1-\alpha)}\theta_H^2 - \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)-2\alpha z\right]}{8(1-\alpha)}\theta_H^2 \\ &= \frac{\theta_H^2}{8(1-\alpha)}[z^2(4-4\alpha+4\alpha^2-\alpha^3)+z(-8\alpha+4\alpha^2-2\alpha^3)+4\alpha^2-\alpha^3]\end{aligned}$$

Let $f(z) \triangleq z^2(4-4\alpha+4\alpha^2-\alpha^3)+z(-8\alpha+4\alpha^2-2\alpha^3)+4\alpha^2-\alpha^3$. First, we note that $(4-4\alpha+4\alpha^2-\alpha^3)>0$ for $0<\alpha<1$. Second, the two roots of $f(z)=0$ are

$$z_0 = \frac{4\alpha-2\alpha^2+\alpha^3 \pm 2\alpha(1-\alpha)\sqrt{\alpha}}{4-4\alpha+4\alpha^2-\alpha^3}$$

The smaller root is negative and the larger root falls in between 0 and 1. As a result, we have $\Pi_1^{vb} \geq \Pi_2^{vb}$ if and only if $z \geq \frac{4\alpha-2\alpha^2+\alpha^3+2\alpha(1-\alpha)\sqrt{\alpha}}{4-4\alpha+4\alpha^2-\alpha^3}$. Define $f_1(\alpha) \triangleq \frac{4\alpha-2\alpha^2+\alpha^3+2\alpha(1-\alpha)\sqrt{\alpha}}{4-4\alpha+4\alpha^2-\alpha^3}$. We can verify that $f_1(\alpha)>\alpha$ always holds for $0<\alpha<1$. Therefore $\Pi_1^{vb} \geq \Pi_2^{vb}$ when $z \geq f_1(\alpha)$, and $\Pi_1^{vb} < \Pi_2^{vb}$ when $\alpha < z < f_1(\alpha)$.

Summary for Options $(1-3)$. (1) when $z \geq f_1(\alpha)$, $\Pi_1^{vb} \geq \Pi_2^{vb} > \Pi_3^{vb}$ (for Π_1^{vb}, $q_L^* > 0$); (2) when $\frac{\alpha}{2-\alpha} < z < f_1(\alpha)$, $\Pi_2^{vb} > \Pi_1^{vb} > \Pi_3^{vb}$ (for Π_2^{vb}, $q_L^*>0$); (3) when $z \leq \frac{\alpha}{2-\alpha}$, we have $\Pi_1^{vb} = \Pi_2^{vb} > \Pi_3^{vb}$ (for Π_1^{vb} and Π_2^{vb}, $q_L^*=0$).

(1.2) **The optimal option among** $(4-6)$. We compare the profit functions of the firm for the three options (see the functions in Table 10) by the following cases (note that $\frac{3\alpha-2}{2-\alpha} \leq \alpha^2 \leq \frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$):

(1.2.a) When $z \leq \frac{3\alpha-2}{2-\alpha}$, we have $\Pi_4^{vb} = \Pi_6^{vb} \geq \Pi_5^{vb}$. In this case, $\Pi_4^{vb} = \Pi_6^{vb} = \frac{\alpha^2}{2}\theta_H^2$, $\Pi_5^{vb} = \frac{\alpha(2-\alpha)(1+z)^2}{8}\theta_H^2$, and

$$\Pi_4^{vb} - \Pi_5^{vb} = \frac{\alpha}{8}\theta_H^2[4\alpha-(2-\alpha)(1+z)^2]$$

Let $f(z)=4\alpha-(2-\alpha)(1+z)^2$. The two roots of $f(z)$ are $z_0=-1\pm 2\sqrt{\frac{\alpha}{2-\alpha}}$. Moreover, we have $\frac{3\alpha-2}{2-\alpha} \leq -1+2\sqrt{\frac{\alpha}{2-\alpha}}$ for all $0 \leq \alpha \leq 1$. Therefore $\Pi_4^{vb}-\Pi_5^{vb} \geq 0$ when $z \leq \frac{3\alpha-2}{2-\alpha}$.

(1.2.b) When $\frac{3\alpha-2}{2-\alpha} < z \leq \alpha^2$, we have $\Pi_6^{vb} \geq \Pi_5^{vb}$ and $\Pi_6^{vb} \geq \Pi_4^{vb}$. In this case, $\Pi_4^{vb} = \frac{\alpha^2}{2}\theta_H^2$, $\Pi_5^{vb} = \frac{\alpha(2-\alpha)(1+z)^2}{8}\theta_H^2$, and $\Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$. We have $\Pi_6^{vb}-\Pi_4^{vb} = \theta_H^2\frac{\alpha}{16(1-\alpha)}(-2+z(-2+\alpha)+3\alpha)^2 \geq 0$ and $\Pi_6^{vb}-\Pi_5^{vb} = \theta_H^2\frac{\alpha^2(2-\alpha)}{16(1-\alpha)}(1-z)^2 \geq 0$.

(1.2.c) When $\alpha^2 < z \leq \frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$, we have $\Pi_6^{vb} \geq \Pi_5^{vb} > \Pi_4^{vb}$. In this case, $\Pi_4^{vb} = \frac{\alpha^2(1-2z)+z^2}{2(1-\alpha^2)}\theta_H^2$, $\Pi_5^{vb} = \frac{\alpha(2-\alpha)(1+z)^2}{8}\theta_H^2$, and $\Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$. Therefore $\Pi_6^{vb}-\Pi_5^{vb} = \theta_H^2\frac{\alpha^2(2-\alpha)}{16(1-\alpha)}(1-z)^2 \geq 0$ and $\Pi_5^{vb}-\Pi_4^{vb} = \frac{\theta_H^2}{8(1-\alpha^2)}[(-4+2\alpha-\alpha^2-2\alpha^3+\alpha^4)z^2+2\alpha(2+3\alpha-2\alpha^2+\alpha^3)z+\alpha(2-5\alpha-2\alpha^2+\alpha^3)]$.

Let $f(z) \triangleq (-4+2\alpha-\alpha^2-2\alpha^3+\alpha^4)z^2+2\alpha(2+3\alpha-2\alpha^2+\alpha^3)z+\alpha(2-5\alpha-2\alpha^2+\alpha^3)$. First we have $-4+2\alpha-\alpha^2-2\alpha^3+\alpha^4<0$ for $0 \leq \alpha \leq 1$. Second, we have $f(\alpha^2)=\alpha(1-\alpha)^4(2+3\alpha+2\alpha^2+\alpha^3)>0$ and $f(\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}) = \frac{4\alpha(1-\alpha)^3(2-3\alpha+3\alpha^2)}{(2-2\alpha+\alpha^2)^2}>0$. Therefore, $\Pi_5^{vb}-\Pi_4^{vb}>0$ for $\alpha^2 < z \leq \frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}$.

(1.2.d) When $\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2} < z \leq 1$, we have $\Pi_5^{vb} > \Pi_4^{vb}$ but the relationship between Π_5^{vb} and Π_6^{vb} is undertermined. In this case, $\Pi_4^{vb} = \frac{\alpha^2(1-2z)+z^2}{2(1-\alpha^2)}\theta_H^2$, $\Pi_5^{vb} = \frac{4z^2+\alpha(2-\alpha)(1-2z-3z^2)}{8(1-\alpha)^2}\theta_H^2$, and $\Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$. Therefore $\Pi_5^{vb}-\Pi_4^{vb} = \frac{\theta_H^2}{8(1-\alpha)^2(1+\alpha)}\left[2\alpha(1-z)^2+3z^2(1-\alpha)^2(1+\alpha)\right] \geq 0$ and $\Pi_5^{vb}-\Pi_6^{vb} = \frac{\theta_H^2}{16(1-\alpha)^2}\left[(8-16\alpha+14\alpha^2-5\alpha^3+\alpha^4)z^2+2\alpha(-8+14\alpha-11\alpha^2+3\alpha^3)z+\alpha^2(6-5\alpha+\alpha^2)\right]$.

Let $f(z) \triangleq (8-16\alpha+14\alpha^2-5\alpha^3+\alpha^4)z^2+2\alpha(-8+14\alpha-11\alpha^2+3\alpha^3)z+\alpha^2(6-5\alpha+\alpha^2)$. First, we have $(8-16\alpha+14\alpha^2-5\alpha^3+\alpha^4)>0$ for $0 \leq \alpha \leq 1$. Second, we have $f(1)=8(1-\alpha)^4 \geq 0$ and $f(\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}) = -\frac{4(2-\alpha)(1-\alpha)^5\alpha^2}{(2-2\alpha+\alpha^2)^2} \leq 0$. Next, let z_0 be the root of $f(z_0)$ where $z_0 \in (\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2}, 1)$, which is $z_0 = \frac{8\alpha-14\alpha^2+11\alpha^3-3\alpha^4+2\sqrt{2}\alpha(1-\alpha)^2\sqrt{(2-\alpha)(1-\alpha)}}{8-16\alpha+14\alpha^2-5\alpha^3+\alpha^4}$. Define $f_2(\alpha)=z_0$. Therefore, when $f_2(\alpha) \leq z \leq 1$, $\Pi_5^{vb} \geq \Pi_6^{vb}$; when $\frac{1-(1-\alpha)^2}{1+(1-\alpha)^2} < z < f_2(\alpha)$, $\Pi_5^{vb} < \Pi_6^{vb}$.

Summary for Options $(4-6)$. (1) when $f_2(\alpha) \le z \le 1$, $\Pi_5^{vb} \ge \max(\Pi_4^{vb}, \Pi_6^{vb})$ (for Π_5^{vb}, $q_L^* > 0$); (2) when $\frac{3\alpha-2}{2-\alpha} < z < f_2(\alpha)$, $\Pi_6^{vb} \ge \max(\Pi_4^{vb}, \Pi_5^{vb})$ (for Π_6^{vb}, $q_L^* > 0$); (3) when $z \le \frac{3\alpha-2}{2-\alpha}$, we have $\Pi_4^{vb} = \Pi_6^{vb} > \Pi_5^{vb}$ (for Π_4^{vb} and Π_6^{vb}, $q_L^* = 0$).

(2) **The optimal option among** $(1-6)$: we next compare the optimal option from (1-3) and (4-6), whose details are presented above. We denote the optimal profit among Options $(1-3)$ and $(4-6)$ by Π_{123}^{vb} and Π_{456}^{vb}, respectively, and among all options by Π^{vb}, i.e., $\Pi^{vb} = \max(\Pi_{123}^{vb}, \Pi_{456}^{vb})$. To find Π^{vb}, we discuss the following cases (we note that $\frac{3\alpha-2}{2-\alpha} \le \frac{\alpha}{2-\alpha} \le \min\{f_1(\alpha), f_2(\alpha)\}$).

(2.a) When $0 \le z \le \frac{3\alpha-2}{2-\alpha}$, we claim $\Pi_{123}^{vb} \ge \Pi_{456}^{vb}$, In this case, $\Pi_{123}^{vb} = \Pi_1^{vb} (= \Pi_2^{vb}) = \frac{\alpha}{2}\theta_H^2$ and $\Pi_{456}^{vb} = \Pi_4^{vb} (= \Pi_6^{vb}) = \frac{\alpha^2}{2}\theta_H^2$. Hence $\Pi_1^{vb} \ge \Pi_4^{vb}$.

(2.b) When $\frac{3\alpha-2}{2-\alpha} < z \le \frac{\alpha}{2-\alpha}$, we claim $\Pi_{123}^{vb} \ge \Pi_{456}^{vb}$, In this case, $\Pi_{123}^{vb} = \Pi_1^{vb} (= \Pi_2^{vb}) = \frac{\alpha}{2}\theta_H^2$, $\Pi_{456}^{vb} = \Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$.

$$\Pi_1^{vb} - \Pi_6^{vb} = \frac{\alpha\theta_H^2}{16(1-\alpha)}\left[-(2-\alpha)^2 z^2 - 2(3\alpha^2 - 8\alpha + 4)z - (\alpha^2 + 4\alpha - 4)\right]$$

Let $f(z) \triangleq -(2-\alpha)^2 z^2 - 2(3\alpha^2 - 8\alpha + 4)z - (\alpha^2 + 4\alpha - 4)$. We have $f(\frac{3\alpha-2}{2-\alpha}) = 8(1-\alpha)^2 \ge 0$ and $f(\frac{\alpha}{2-\alpha}) = 4(1-\alpha)^2 \ge 0$. Therefore $\Pi_1^{vb} \ge \Pi_6^{vb}$ for $\frac{3\alpha-2}{2-\alpha} < z \le \frac{\alpha}{2-\alpha}$.

(2.c) When $\frac{\alpha}{2-\alpha} < z \le \min\{f_1(\alpha), f_2(\alpha)\}$, we claim $\Pi_{123}^{vb} \ge \Pi_{456}^{vb}$, In this case, $\Pi_{123}^{vb} = \Pi_2^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)-2\alpha z\right]}{8(1-\alpha)}\theta_H^2$ and $\Pi_{456}^{vb} = \Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$. Based on this, we have $\Pi_2^{vb} - \Pi_6^{vb} = \frac{\alpha(2-\alpha)^2(1-z)^2}{16(1-\alpha)}\theta_H^2 \ge 0$.

(2.d) When $\min\{f_1(\alpha), f_2(\alpha)\} < z \le \max\{f_1(\alpha), f_2(\alpha)\}$, we discuss two sub-cases. First we note that $f_1(\alpha) \le f_2(\alpha) \iff \alpha \le \widetilde{\alpha}$ for some $\widetilde{\alpha} \in [0.4, 0.5]$. Hence we discuss two cases where (1) $f_1(\alpha) \le f_2(\alpha)$ and (2) $f_1(\alpha) \ge f_2(\alpha)$.

(2.d.1) $f_1(\alpha) \le z \le f_2(\alpha)$. In this case, we have $\Pi_{123}^{vb} = \Pi_1^{vb} = \frac{\alpha(1-2z)+z^2}{2(1-\alpha)}\theta_H^2$ and $\Pi_{456}^{vb} = \Pi_6^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)+2(2-3\alpha)z\right]}{16(1-\alpha)}\theta_H^2$. We claim that $\Pi_1^{vb} \ge \Pi_2^{vb} \ge \Pi_6^{vb}$ holds when $f_1(\alpha) \le z \le f_2(\alpha)$. The former inequality holds by Case (1.1.c) and the latter holds by Case (2.c).

(2.d.2) $f_2(\alpha) \le z \le f_1(\alpha)$. In this case, we have $\Pi_{123}^{vb} = \Pi_2^{vb} = \frac{\alpha(2-\alpha)\left[(2-\alpha)(1+z^2)-2\alpha z\right]}{8(1-\alpha)}\theta_H^2$ and $\Pi_{456}^{vb} = \Pi_5^{vb} = \frac{4z^2+\alpha(2-\alpha)(1-2z-3z^2)}{8(1-\alpha)^2}\theta_H^2$. Based on this, we have

$$\Pi_2^{vb} - \Pi_5^{vb} = \frac{\theta_H^2}{8(1-\alpha)^2}[-z^2(4-10\alpha+11\alpha^2-5\alpha^3+\alpha^4) - 2z\alpha(-2+3\alpha-3\alpha^2+\alpha^3) - \alpha(-2+7\alpha-5\alpha^2+\alpha^3)]$$

Let $f(z) \triangleq -z^2(4-10\alpha+11\alpha^2-5\alpha^3+\alpha^4) - 2z\alpha(-2+3\alpha-3\alpha^2+\alpha^3) - \alpha(-2+7\alpha-5\alpha^2+\alpha^3)$. First, we note that $4-10\alpha+11\alpha^2-5\alpha^3+\alpha^4 > 0$ for $0 \le \alpha \le 1$. Second, we denote the two roots of $f(z)$ by z_0, z_1, where

$$z_0, z_1 = \frac{2\alpha - 3\alpha^2 + 3\alpha^3 - \alpha^4 \pm 2(1-\alpha)^2\sqrt{\alpha(1-\alpha)(2-\alpha)}}{4-10\alpha+11\alpha^2-5\alpha^3+\alpha^4}$$

with $z_0 < z_1$. It holds that $z_0 < f_2(\alpha) < z_1$ for all $0 < \alpha < 1$, and $z_1 < f_1(\alpha) \iff \frac{2}{3} < \alpha < 1$. Define $f_3(\alpha) = z_1$. Hence, for (i) $\alpha \le \frac{2}{3}$ & $f_2(\alpha) \le z \le f_1(\alpha)$, $\Pi_2^{vb} \ge \Pi_5^{vb}$; (ii) $\alpha \ge \frac{2}{3}$ & $f_2(\alpha) \le z \le f_3(\alpha)$, $\Pi_2^{vb} \ge \Pi_5^{vb}$; (i) $\alpha \ge \frac{2}{3}$ & $f_3(\alpha) \le z \le f_1(\alpha)$, $\Pi_2^{vb} \le \Pi_5^{vb}$.

(2.e) When $\max\{f_1(\alpha), f_2(\alpha)\} < z \le 1$. In this case, we have $\Pi_{123}^{vb} = \Pi_1^{vb} = \frac{\alpha(1-2z)+z^2}{2(1-\alpha)}\theta_H^2$ and $\Pi_{456}^{vb} = \Pi_5^{vb} = \frac{4z^2+\alpha(2-\alpha)(1-2z-3z^2)}{8(1-\alpha)^2}\theta_H^2$. $\Pi_1^{vb} - \Pi_5^{vb} = \frac{\alpha(1-z)^2(2-3\alpha)}{8(1-\alpha)^2}\theta_H^2$. Hence, $\Pi_1^{vb} \ge \Pi_5^{vb} \iff \alpha \le \frac{2}{3}$.

Summarizing (2.a) - (2.e), we have (1) $0 \le z \le \frac{\alpha}{2-\alpha}$, $\Pi^{vb} = \Pi_1^{vb}$ (with $q_L^* = 0$); (2) $\frac{\alpha}{2-\alpha} < z \le \min\{f_1(\alpha), f_3(\alpha)\}$, $\Pi^{vb} = \Pi_2^{vb}$ (with $q_L^* > 0$); (3) $\alpha \le \frac{2}{3}$ & $z \ge f_1(\alpha)$, $\Pi^{vb} = \Pi_1^{vb}$ (with $q_L^* > 0$); (4) $\alpha \ge \frac{2}{3}$ & $z \ge f_3(\alpha)$, $\Pi^{vb} = \Pi_5^{vb}$ (with $q_L^* > 0$). We provide an illustration of the relationship among f_1, f_2, f_3, which help us combine the cases.

A.2.3. Pure Component vs Pure Bundling. Since Option 1 of pure bundling yields an optimal solution that is equivalent to the optimal solution of pure component, the comparison across pure component and pure bundling is contained in the pure bundling analysis.

A.3. Precise Targeting (via Bundling)

We provide the profit function and the price constraints for Option (7 − 9) in Table 14. Fixing quality levels with $q_H \ge q_h \ge q_l \ge q_L$, we solve the optimal price decisions for the options and provide these results in the table as well. Note that Option (9) is identical to Option (3), so we omit the analysis for it here.

Table 14 Profit function and optimal prices for bundling Option (7 − 9) (Precise Targeting)

Option	Profit function	Price Constraints	p_{HH}^*	p_{hl}^*	p_{LL}^*
(7)	$\alpha^2(p_{HH} - 2q_H^2)$ $+2\alpha(1-\alpha)(p_{hl} - q_h^2 - q_l^2)$ $+(1-\alpha)^2(p_{LL} - 2q_L^2)$	$2\theta_H q_H - p_{HH} \ge \theta_H q_h + \theta_H q_l - p_{hl}$ $2\theta_H q_H - p_{HH} \ge 2\theta_H q_L - p_{LL}$ $2\theta_H q_H - p_{HH} \ge 0$ $\theta_H q_h + \theta_L q_l - p_{hl} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_h + \theta_L q_l - p_{hl} \ge \theta_H q_L + \theta_L q_L - p_{LL}$ $\theta_H q_h + \theta_L q_l - p_{hl} \ge 0$ $2\theta_L q_L - p_{LL} > 2\theta_L q_H - p_{HH}$ $2\theta_L q_L - p_{LL} > \theta_L q_h + \theta_L q_l - p_{hl}$ $2\theta_L q_L - p_{LL} \ge 0$	$2\theta_H q_H - (\theta_H - \theta_L)(q_l + q_L)$	$\theta_H(q_h - q_L) + \theta_L(q_l + q_L)$	$2\theta_L q_L$
(8)	$\alpha^2(p_{HH} - 2q_H^2)$ $+2\alpha(1-\alpha)(p_{hl} - q_h^2 - q_l^2)$	$2\theta_H q_H - p_{HH} \ge \theta_H q_h + \theta_H q_l - p_{hl}$ $2\theta_H q_H - p_{HH} \ge 0$ $\theta_H q_h + \theta_L q_l - p_{hl} > \theta_H q_H + \theta_L q_H - p_{HH}$ $\theta_H q_h + \theta_L q_l - p_{hl} \ge 0$	$2\theta_H q_H - (\theta_H - \theta_L) q_l$	$\theta_H q_h + \theta_L q_l$	–
(9)	Option (9) = Option (3)				

Plugging in the optimal prices, we can show that the profit function of each option is concave in the quality variables. We take the derivative to obtain the unconstrained option solution to the quality decision. Then we can verify that the optimal solution satisfies the ordering of quality variables and we just need to make sure they are all non-negative. The non-negative constraints, together with the profit comparison across the options, gives the non-dominated cases for the options that are provided in Table 13. The details are omitted.