Value of Operational Flexibility in Co-Production Systems with Yield and Demand Uncertainty

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We study the production decisions of a firm that operates in co-production system (single input but multiple simultaneous outputs) with random yield and demand. The firm uses its outputs to meet multiple end-market demands with different quality requirements. Outputs serving a market segment may be a result of a blending process of different quality products of the co-production system. After the realization of yield and before the realization of demand, the firm has the option to upgrade the quality of outputs in order to better position itself to meet market demands. With the goal of maximizing expected profits, we formulate a three stage stochastic programming problem, to investigate the value of intermediate upgrading flexibility. Using a stylized model of two products and two markets with different quality requirements, we characterize the optimal decisions and show that the quality upgrading policy is of a single-threshold type. Although upgrading is costly, it creates value for the firm through reducing total production cost and better managing the yield uncertainty. Positive correlation between demands of the end-markets decreases the value of upgrading. The more general model formulation offers the blueprint of a stochastic programming model that can be solved for realistic applications.

Keywords: Stochastic programming, operational flexibility, random yield, product substitution, newsvendor demand.

1. Introduction

Operational flexibility has been demonstrated to have tremendous value to supply chain management. In a frequently used form of operational flexibility, its execution requires that multiple(one) resources are used hedge demand or supply uncertainties in one(multiple) market segment(s). The key in this form of flexibility is that firms have the capability to re-allocate or re-distribute resources(products, capacities or services) after any uncertainty (demand or supply) has been resolved, in the most effective way to match supply and demand. Most of the value of this flexibility comes from the risk pooling of the resources in handling a diverse set of demand or supply risks. Examples of such flexibility include substitution(upgrading/downgrading of products or services) and transshipment of products among retailers or distribution channels in a network system. We are going to present a new application of this type of flexibility that involves costly upgrading of the quality of the resources prior to demand uncertainty resolution in multi-segment markets with varied quality requirements.

We consider a production/market environment with both yield and demand uncertainty. Yield uncertainty is a common phenomenon in many production and service environments. Random yield can emerge in many forms and measures, ranging from simple fraction defectives to broader classification schemes based on some quantitative or qualitative evaluation resulting in a finite number of quality levels. We study a co-production system where outputs are differentiated by

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quality levels. A co-production process is one that yields multiple outputs from a single input. Outputs are used to meet demands in a multi-segment market, where each market imposes certain quality requirement on the outputs that serve them. An output serving a market segment may be the blended result of some of the original outputs of the co-production process.

Prior to decide how to blend the outputs of the co-production process to meet each market segment demand, the firm has the opportunity to upgrade the quality levels of the original products through refining processes. These processes happen after the realization of the yield of the co-production process and before the realization of market segment demands. We consider transfers in the direction of increasing inventories of higher quality level outputs and refer to this option as "upgrading flexibility". This practice is common in agriculture and chemical industries, where post-production refining processes can improve quality levels of the crude products produced in the co-production process(crushing, milling, etc). Although appealing in many respects, the upgrading process is costly and needs to take place prior to the selling season when the uncertainty of demand realizes. Therefore, a balance has to be made between the early production decision and the subsequent upgrading decision in order to match supply and demand effectively.

We take the perspective of a profit maximizing firm operating in the described environment. We outline a stylized three-stage stochastic model to study the structure of the optimal decisions in this production and quality upgrading process of a co-production system, in the presence of both yield and demand uncertainties. We are particularly interested in understanding the value of upgrading flexibility in the co-production environment, and when it is best exercised.

1.1 *Motivating Example*

Our motivating example is the growth and production of vegetable seeds in the agriculture industry. A typical process comprises of three main stages. (1) growing season, which can take between 6 to 24 months (depending on the seed type), when the firm decides the amount of seeds to grow (2) After the harvest season and quality assessment, seeds are rated by quality levels. At this stage, the firm has the opportunity to engage in upgrading processes, to increase the proportion of higher quality seeds through refining of lower quality seeds inventory. The quality upgrading flexibility occurs only before demand uncertainty is resolved. The two most important quality measures of seeds are purity and germination. There are many complex processes to separate and enhance the purity of seeds, and other processes/treatment to enhance the germination of seeds. All these processes incur significant cost in order to improve the quality of seeds. (3) With the start of selling season, the firm satisfies different markets' demand according to quality requirements. High-end markets (like North America) demand can be solely satisfied from higher quality product(higher purity and germination), whereas the demand of low-end markets (like India, Bangladesh, Malaysia) can be fulfilled through a blend of low and high quality seeds.

Our proposed modeling framework and problem formulation apply to other industries as well. For instance, in the semiconductor industry, a single input of Silicon Dioxide can be fabricated to silicon wafers of different quality grades. A frequently heard term 'ingot', is an intermediate outcome of the first manufacturing process which is referred to as crystal growth. It is then sliced into substrates(wafers) before going through final polishing. Though current manufacturing processes can control size(diameter), thickness and other geometric properties, they can not perfectly control cleanness, flatness, and certain circuit properties of wafers, which are key quality measures of them. However, there are recurrent/post polishing processes which are used to improve wafer qualities, though they usually incur additional cost. For example, an Epi layer, which is an additional layer of purity Silicon, can be deposited on the top of a polished wafer substrate to improve its quality. Finally, wafers of different qualities can be used to meet customers' demands with varied quality requirements. Another example could be found in the steel industry, where steel beams of strong/weak strength are outputs of a co-production process and additional refinement processes can improve the weak strength steel beams, before they are used to meet customer demands with varied quality requirements.

2. Related Literature

Relevant literature to this study can be divided into two streams: (1) co-production systems with yield uncertainty and substitution; (2) value of operational flexibility.

In the co-production literature, a common feature of models is full downward substitution, i.e. satisfying demand of lower quality specifications from the inventory of higher quality products. Joint inventory and allocation decisions in different settings is the main goal of these papers. The early work of Bitran and Dasu (1992), Bitran and Leong (1992), and Bitran and Gilbert (1994) proposed heuristics and approximation methods for the N-product finite horizon problem with deterministic demand and random yield. Motivated by semi-conductor industry, Hsu and Bassok (1999) studied a two-stage stochastic model with N outputs in the presence of both demand and yield uncertainty. Tomlin and Wang (2008) considers joint decisions of quantity, down-conversion (converting high quality to low quality), pricing and allocation in a customer utility maximization framework. In a beef supply chain, Boyabatl et al. (2011) analyzed a co-production system with two output quality levels (boxed beef and ground beef), to study the impact of market conditions on optimal sourcing decisions. While in most of these papers a unit demand of market with lower quality expectations can be fully substituted by higher quality inventory, in our model, we require a blend of low and high quality inventories to meet quality requirements for the lower-end market. Furthermore, upgrading of quality of available outputs is possible prior to demand realization.

Operational flexibility provides firms with a hedge against uncertainties. There is a vast literature on the value of operational flexibility in its various forms: e.g. postponement (Van Mieghem and Dada 1999), component commonality (Van Mieghem 2004) , substitution (Bassok et al. 1999), and responsive pricing (Tomlin and Wang 2008).

Closest to our work in modeling operational flexibility is the recent work of Dong et al. (2014), in which they analyzed early procurement, production and later upgrading decisions of an oil refinery facing price uncertainty in both input and output markets, with a co-production system producing two quality levels of outputs that we used to meet demand in two end-markets. They investigated two types of operational flexibilities: range flexibility (the ability to process crude oil of diverse quality), and conversion flexibility (the ability to convert low-quality crude oil to high-quality crude oil). In our paper, we only consider conversion flexibility of a low quality output to a high quality output. While Dong et al. (2014) considered an environment without any yield uncertainty and with end-market price uncertainty, we are interested in co-production systems with yield uncertainty and end-markets with preset fixed prices but subject to random demand.

Contributions: While there is a rich body of literature on production planning under joint demand and yield uncertainty, very few have focused on co-production systems with yield uncertainty and upgrading flexibility of the output quality after yield realization but still in the presence of demand uncertainty. Our main results are provided by a model with two products and two end-markets. This stylized model offers useful managerial insights on the nature of upgrading policies and for what environments upgrading flexibility pays off. Moreover, we formulate the more general problem with n products and m demand segments(markets), and discuss how it can be solved. The prevailing full downward substitution serves as a special case in our general model.

We have organized our remaining paper in 4 sections. In Section 3, we present notations and our stylized two product, two market segment model. In Section 4, we present the solutions and insights of the previously introduced stylized model. Next in Section 5 we extend the stylized model to a general formulation. Finally we conclude our paper with Section 6.

3. Model

In this section, we provide a three-stage stochastic optimization formulation for a single-period upgrading problem with uncertain yield and demand. We analyze a two product, two market model and explain the main trade-offs and results of our model.

To be more specific, we assume that the firm is selling two products in two markets. The end markets are high-end(H) and low-end(L), and the products are high quality(H) and low quality(L). As in the agribusiness example, North America is a high-end market, and India/Indonesia/Malaysia/other countries are low-end markets. The firm is a price taker in both end markets with exogenous prices for high and low end markets denoted as p_H and p_L respectively. The prices are dependent on the end markets which impose certain requirements on product qualities. In other words, the prices are market specific, not product specific. Given these prices, end markets demands are random. The joint distribution of end market demands is characterized by a density function $f(d_H, d_L)$, where d_H, d_L is the demand in each market. To supply the markets, the firm makes production and quality upgrading decisions in the presence of demand uncertainty, and market allocation decisions after demand is realized. The detailed decision process is characterized as follows:

Stage 1: Production. The firm decides on the quantity Q of the single input to be produced at an exogenous unit production cost. The yield is uncertain, with αQ realized as high quality and $(1-\alpha)Q$ as low quality product.

Stage 2: Upgrading. After observing the availability of each quality grade (y_H, y_L), the firm has the option to upgrade (transfer) inventories from lower to higher quality grade at an exogenous unit upgrading cost. After the upgrading, the firm has inventory levels of (q_H, q_L) of the two products .

Stage 3: Allocation. With end market demand realized, the firm satisfies demand from available inventory. The quantity of product j allocated to market k is x_{jk} where $j \in \{H, L\}$ and $k \in \{H, L\}$. The allocation decision is subject to the usual product supply and end market demand constraints as well as quality requirement constraints imposed by each market. We will discuss these quality constraints.

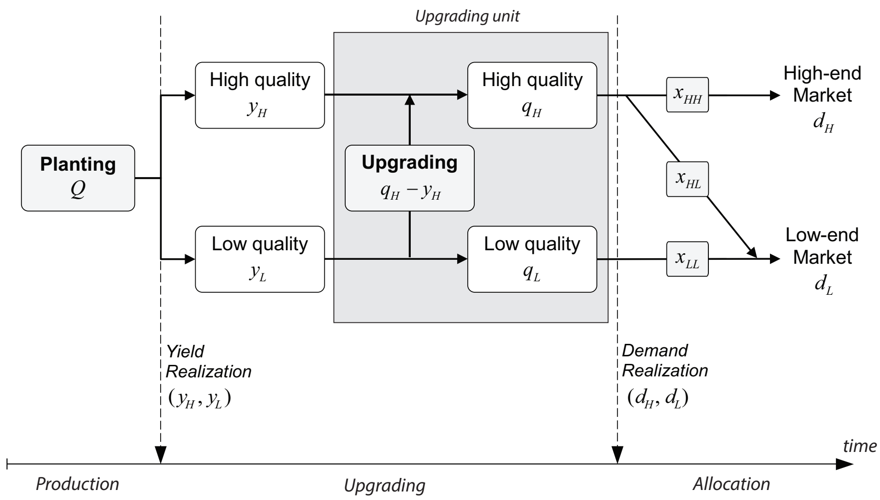

Figure 1. Simple model diagram

3.1 *Quality Constraints*

The markets have heterogeneous requirement on product qualities. For instance, in the agribusiness industry North America market has a higher requirement on seed quality than Asian/South American countries.

In our model, we normalize the high quality level $g_H = 1$, and the low quality level $g_L = 0$. The high end market only accepts high quality product, while the low end market expects that the average quality of offered products is above a quality threshold γ. If $x_{jk}, j \in \{H, L\}, k \in \{H, L\}$ represents the quantity of product j allocated to market k, then $x_{HL} \geq \gamma(x_{HL} + x_{LL})$, or

$$\gamma x_{LL} - (1 - \gamma)x_{HL} \leq 0$$

3.2 *Mathematical Formulation*

Based on the model specification above, we formulate a three stage mathematical programing. Our notational convention is that when we compress subscripts of a variable, we denote a vector of the corresponding variable.

In Stage three, the firm has to decide on the allocation of the two products to the two end markets. x_{HH} and x_{HL} are the allocation of the high quality product to markets H and L respectively, while x_{LL} is the only possible allocation of the low quality product to the low-end market. The optimization problem in this stage, referred to as the "Allocation" Problem (**AP**) is:

$$\text{(AP)} \qquad r(q,d) = \max_{x_{HH}, x_{HL}, x_{LL}} \; p_H x_{HH} + p_L(x_{HL} + x_{LL}) \tag{1}$$

$$\text{subject to:} \quad x_{HH} \leq d_H \tag{1a}$$

$$x_{HL} + x_{LL} \leq d_L \tag{1b}$$

$$x_{HH} + x_{HL} \leq q_H \tag{1c}$$

$$x_{LL} \leq q_L \tag{1d}$$

$$\gamma x_{LL} - (1 - \gamma)x_{HL} \leq 0 \tag{1e}$$

$$x_{HH}, x_{HL}, x_{LL} \geq 0 \tag{1f}$$

The revenue function $r(q, d)$ has as its state variables the vector of available quantities of the two product q and the vector of realized demand d. Constraints (1a) and (1b) are demand constraints on allocated quantities in each market for meeting realized demand. Constraints (1c) and (1d) are supply constraints on available quantities of the two products. Constraint (1e) captures the quality requirement imposed by blending or mixing the quantities of the two quality products in an effort to meet the low end market quality requirement.

In stage two, after the co-production process had its realized yield of high and low quality products, the firm decides on how much of the low quality product to upgrade via refined processing to bring up to the high quality level. The resulting optimization problem referred to as "Upgrading" problem (**UP**) is:

$$\text{(UP)} \qquad \Pi(y) = \max_{q_H, q_L} \; \mathbb{E}_d\left[r(q,d)\right] - c_u(q_H - y_H) \tag{2}$$

$$\text{subject to:} \quad q_H + q_L = y_H + y_L, \tag{2a}$$

$$q_H \geq y_H, \tag{2b}$$

$$q_L \geq 0. \tag{2c}$$

The objective function $\Pi(y)$ has an state variable the vector of available quantities of the two products after the production stage, including any leftover units from previous periods, and is the expected profit from the upgrading decision, which reflects the subsequent allocation decision in stage three and the costs of upgrading. The upgrading constraints (2a) and (2b) reflect that only low quality product can be upgraded and the total quantity of the two quality products is preserved under upgrading.

Finally, in the first stage our firm makes a production decision on the total quantity of input in the co-production process, Q. The objective function $J(Q)$ will account for all expected profits and costs of the upgrading and allocation stages, $\mathbb{E}_\alpha\left[\Pi(\alpha Q,(1-\alpha)Q)\right]$, and the relevant production cost of the co-production system, cQ. Thus, our first stage "Production" problem (**PP**) is:

$$\text{(PP)} \qquad \max_{Q\geq 0} J(Q) = \mathbb{E}_\alpha\left[\Pi(\alpha Q,(1-\alpha)Q)\right] - cQ \tag{3}$$

In order to calculate the impact of upgrading flexibility, we define the *value of upgrading* (VoU) as the difference between optimal expected profits with upgrading $J_U(Q^*_U)$, and without upgrading $J_O(Q^*_O)$:

$$VoU = J_U(Q^*_U) - J_O(Q^*_O) \tag{4}$$

where $J_O(Q^*_O)$ is solved by imposing $q = y$(no upgrading).

4. Analytical Results

In the following subsections, we first characterize optimal allocation decision, then the optimal upgrading decision, and finally the optimal production decision.

4.1 *Optimal Allocation Decision*

The optimal allocation decision is determined by the realization of the demand, relative availability of the two products, as well as relative profitability of products due to substitution scheme in place. We assume that market prices (p_H, p_L) favor the high quality product; i.e., $\gamma p_H \geq p_L$. As a result, the firm prioritizes demand of high-end market.

Figure 2 illustrates optimal allocation decision. Two graphs indicate different cases based on relative availability of two products. When available inventory of high quality is low (left), leftover of low-quality product is non-negative. This is due to the quality requirement of low demand class and low availability of high quality product. When there is ample supply of high quality product (right), there are instances with no leftover inventory of either products (region $\Omega_1^{(2)}$).

Table 1. Optimal allocation and shadow prices

Region	$(x^*_{HH}, x^*_{HL} + x^*_{LL})$	$\lambda(q,d)$
Ω_0	(d_H, d_L)	$(0,0)$
Ω_1	$(d_H, q_H + q_L - d_H)$	(p_L, p_L)
Ω_2	$(d_H, \frac{q_H - d_H}{\gamma})$	$(\frac{p_L}{\rho_L}, 0)$
Ω_3	$(q_H, 0)$	$(p_H, 0)$

It is clear from table 1 that the marginal value of low quality product is non-zero in region $\Omega_1^{(2)}$. The difference in the area corresponding to each region depends on relative availability of products q_H and q_L. Next result is about the property of expected value of optimal allocation profit, which is analogous to results of multidimensional newsvendor model.

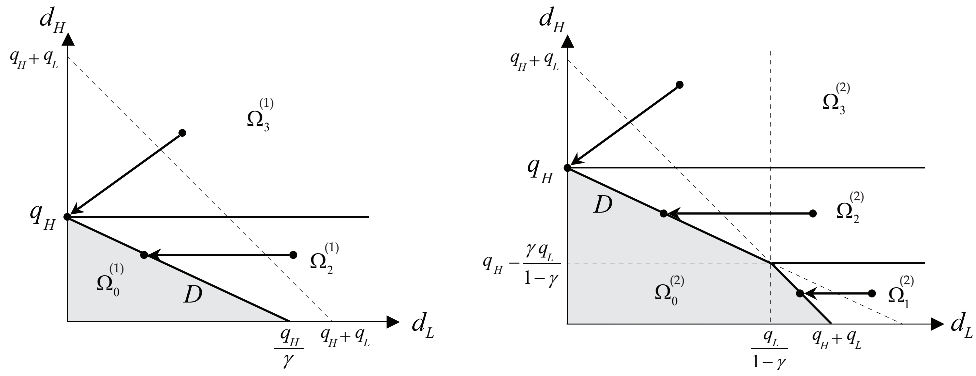

Figure 2. Left: $q_H \leq \gamma(q_H + q_L)$, Right: $q_H > \gamma(q_H + q_L)$, line $\mathbf{D}: d_H = q_H - \gamma d_L$

Lemma 1. *Let d be a continuous random variable that is finite with probability 1. Then the expected profit function $\mathbb{E}_d r(q, d)$ is differentiable in q and the gradient is equal to expected shadow price $\nabla \mathbb{E}_d r(q, d) = \mathbb{E}_d \lambda(q, d)$, where $\lambda(q, d)$ is the shadow price of q.*

The proof follows directly from Harrison and Van Mieghem (1999) and is omitted.

4.2 *Optimal Upgrading Decision*

At the upgrading stage, the firm adjusts the inventories of the two products by refined processing of some of the lower quality product while anticipating demand. The processing is costly and results in increasing the inventory of high quality product. Figure 3 illustrates the decision space of the upgrading problem for a given realized inventory y_H and y_L of the high and low quality products respectively. We assume that $y_H \geq \alpha Q$ and $y_L \geq (1-\alpha)Q$ with the interpretation that $(y_H - \alpha Q)$ and $(y_L - (1-\alpha)Q)$ represent initial inventories of the two products before production in this period, as a result of previous procurement or production decisions(e.g., in the case of extension to a multi-period problem).

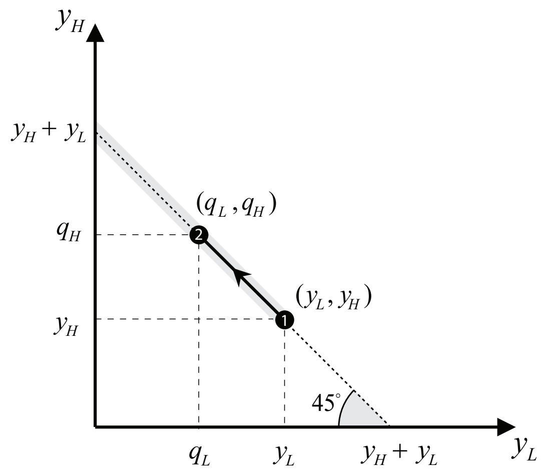

Figure 3. Upgrading decision space

The following result characterizes the optimal upgrading policy.

Proposition 1. *Given total production quantity Q, there exists a unique threshold $q^t \in [0, Q]$ such that for realized inventory y, the optimal post-upgrading inventory of high quality product is*

$q_H^*(y) = \max\{q^t, y_H\}$. *Furthermore,* $q^t(Q)$ *is non-decreasing in* Q *and there exist thresholds* $\underline{Q}$ *and* $\overline{Q}$, *with* $\underline{Q} \leq \overline{Q}$, *such that* $q^t(Q) = Q$ *when* $Q \leq \underline{Q}$ *and* $q^t(Q)$ *is constant when* $Q \geq \overline{Q}$.

This result states that optimal upgrading decision is a single-threshold policy; i.e., if available high quality product quantity y_H is less than $q^t(Q)$, the firm should upgrade $(q^t(Q) - y_H)$ units of its low quality product to reach the optimal target level $q^t(Q)$, otherwise, no action is required. Figure 4 illustrates the optimal upgrading policy.

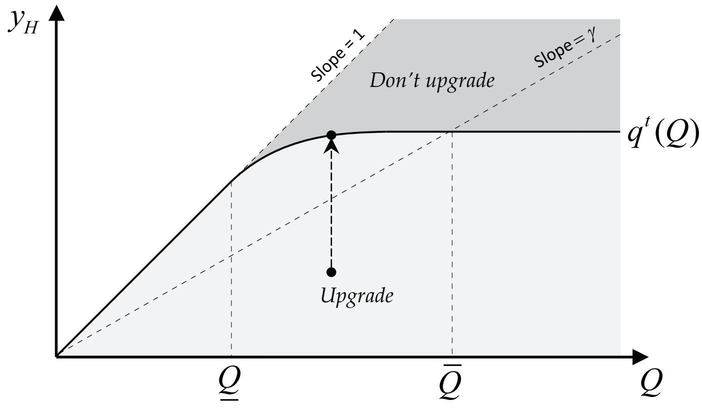

Figure 4. Optimal Upgrading Policy

4.3 *Optimal Production Decision*

In this section, we prove that the production stage objective function is concave in Q and an optimal Q exists. We are particularly interested in studying the properties of optimal expected profit of the upgrading problem and compare with the case of no-upgrading as a benchmark. We use J_U and J_O to refer to profit functions of upgrading model and benchmark, respectively:

$$\begin{aligned} J_U(Q) = &\quad -cQ + \mathbb{E}_\alpha\left[\Pi(\alpha Q, (1-\alpha)Q)\right], \text{ and} \\ J_O(Q) = &\quad -cQ + \mathbb{E}_\alpha\left[\mathbb{E}_d\left[r(\alpha Q, (1-\alpha)Q, d)\right]\right] \end{aligned}$$

Proposition 2 below summaries the key results on the firm's production decision(part 1 of the Proposition), and our results of the effects of upgrading flexibility on such decisions and on the objective value of the firm:

Proposition 2. *For the production problem, we have the following results:*

(1) The optimal allocation profit $\mathbb{E}_d r(q, d)$ *is concave in* q, *the optimal upgrading profit* $\Pi(y)$ *is concave in* y, *and the optimal firm value* $J(Q)$ *is concave in* Q.

(2) For any $Q \geq 0$, $J_U(Q) \geq J_O(Q)$ *and therefore,* $J_U^* \geq J_O^*$. *Moreover,* $Q_U^* \leq Q_O^*$.

This result states that upgrading flexibility always creates value for the firm, and yields a lower production level compared to the non-upgrading model. In other words, the upgrading flexibility enables the firm to produce at lower production level, and by deploying that flexibility the firm is able to achieve higher expected profit.

4.4 *Sensitivity Analysis*

The purpose of this section is to explore the significance of value of upgrading (VoU). We study the impact of production and upgrading costs, as well as selling price.

4.4.1 Impact of Purchase and Upgrading cost

Production and upgrading costs directly influence the value of upgrading. The following proposition characterizes the impact of these cost parameters.

Proposition 3. *VoU increases in production cost c and decreases in upgrading cost c_u. Furthermore, optimal order quantity Q_U^* decreases in c and increases in c_u.*

With production cost increasing, optimal expected profits of both upgrading and benchmark case decrease. However, the decrease in profit is higher when no upgrading takes place. On the other hand, when the cost of upgrading increases, the firm reacts by increasing production quantity and reducing upgrading quantity, which overall decreases the total expected profit, resulting in a decline in the value of upgrading.

4.4.2 Impact of Prices

End-market sales prices are important factors that affect optimal production, upgrading and allocations decisions. The next result is about the impact of change in high end market selling price on the value of upgrading and production decision.

Proposition 4. *Optimal expected profits J_U^* and J_O^* increase in p_H. Furthermore, increase in p_H increases order quantities in both cases.*

With higher profit potential in the high-end market, the firm further increases engagement in production and upgrading activities.

4.4.3 Impact of Demand Correlation and Cost of Upgrading

In our numerical observations, we have observed decrease in value of upgrading and increase in optimal order quantity as demand correlation increases. As upgrading increases the inventory of high quality product, which can be used to meet demand of both markets, it allows risk pooling of demand risks. Therefore, it is expected that upgrading flexibility will have risk pooling features.

Our numerical results study the behavior of various optimal profit functions against demand correlation and cost of upgrading. Demand distribution assumed in this experiment is a bivariate normal with the following parameters $\mu_H = 15$, $\mu_L = 10$ and $\sigma_H^2 = 9$, $\sigma_L^2 = 4$, $\sigma_{HL} = 6\rho$. We fix $c = 3$, $\gamma = 0.7$. Yield distribution is $Beta(3, 1)$. As expected, our results show that expected profits decrease in correlation coefficient ρ. Finally we establish that an increase in the cost of upgrading will decrease the optimal upgrading profit as well as value of upgrading. Even with highest cost of upgrading at $c_u = 4$, value of upgrading on average accounts for 15% of the optimal expected profit without upgrading.

4.4.4 Impact of Yield Uncertainty

Uncertainty in yield is a key factor in optimal upgrading decision. In general, we expect the value of upgrading to increase when the distribution of yield favors lower rates of the high quality product α. The following sensitivity result will help illustrate how the optimal expected profits depend on the characteristic of yield distribution. We assume a parametric family of Beta distributions with scalar parameter θ such that $\alpha(\theta)$ follows $Beta(t(1-\theta)+1, t\theta+1)$, where $t > 0$ and $\theta \in [0, 1]$. Figure 6 illustrates the effect of θ on the shape of distribution. We use the properties of stochastic order for Beta distribution proved by Adell et al. (1993).(See Shaked and Shanthikumar (2007) for more stochastic order results)

Proposition 5. *Let yield rate $\alpha(\theta)$ follow beta distribution with scale parameter θ. Optimal expected profits J_U^* and J_O^* are decreasing in θ and VoU is increasing in θ.*

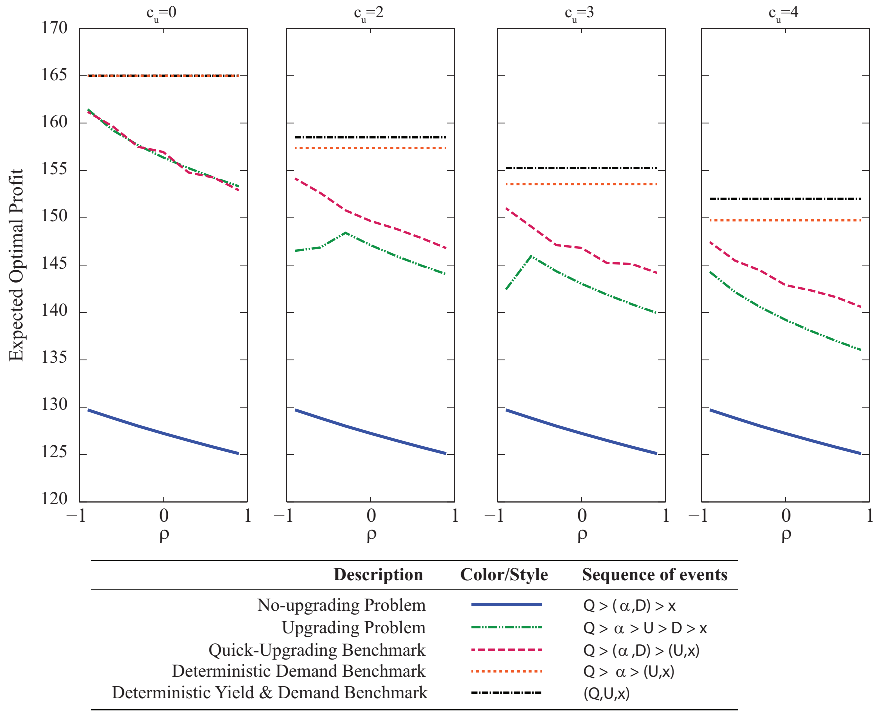

Figure 5. Effect of demand correlation ρ and upgrading cost c_u

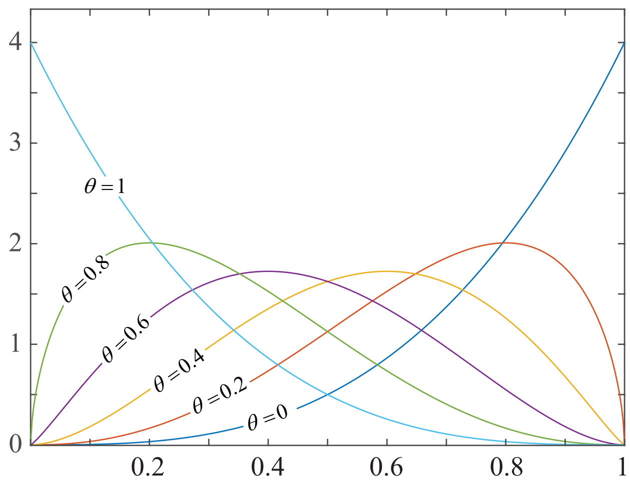

Figure 6. Family of Beta distributions with $t = 3$ and scale parameter θ

Note that as θ increases, average yield decreases and yield variance first increases and then decreases. One can interpret increases in θ as a transformation of the yield distribution towards lower yield rates α. Intuitively, with worse yield rate, the optimal expected profit in both cases decrease. Furthermore, when there is no upgrading flexibility, profit declines faster, which results in an increase in the value of upgrading. Therefore, upgrading is of more value to the firm when distribution of yield is associated with higher values of θ.

In our numeric analysis we used a bivariate normal distribution with $\mu_H = 15$, $\mu_L = 10$ and $\sigma_H^2 = 9$, $\sigma_L^2 = 4$, $\sigma_{HL} = 0$. Yield rate $\alpha(\theta)$ follows the parametric beta distribution as previously introduced. First observation is the dramatic decrease in the optimal expected profit in case of no-upgrading, for both price values, as yield distribution shifts toward lower mean and positive skewness. Even though optimal upgrading profit is also decreasing, the overall effect leads to in-

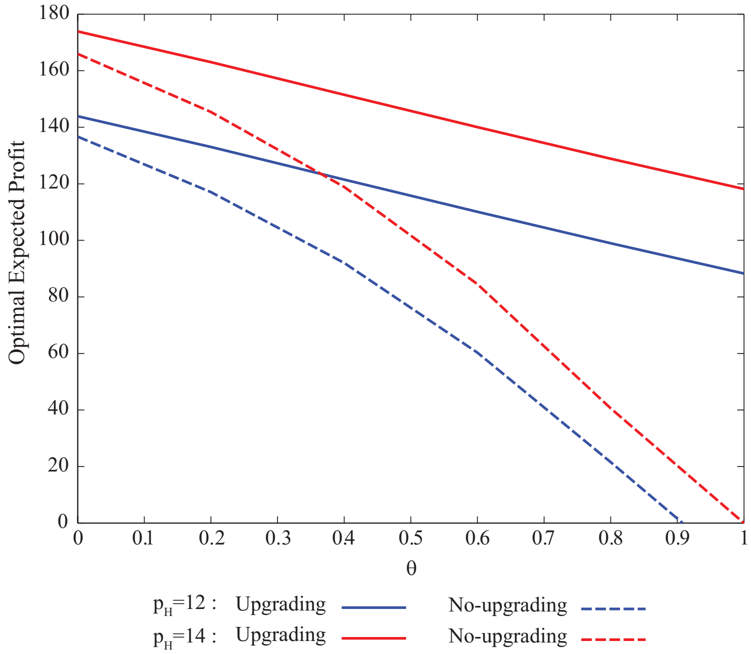

Figure 7. Effect of yield parameter θ and high quality sale price p_H

crease in the value of upgrading. That is, upgrading is more valuable when yield distribution results in lower yield rates for the high quality product. The value of upgrading accounts for 7% ($\theta = 0$)to almost 100% ($\theta = 1$).

5. Formulation of a General Model

In a general model there can be n markets and m products. Yield rate is denoted by the random vector $\alpha = (\alpha_1, ...\alpha_n)$, where $\sum_i \alpha_i = 1$. In the beginning of the second stage, each product i has $y_i = \alpha_i Q$ available inventory, with specified quality grade $g_i \in [0, 1]$. Without loss of generality, we assume $0 \leq g_n < ... < g_1 \leq 1$. The firm can transfer u_{ij} units of product from lower quality g_i to higher quality grade g_j at unit cost c_{ij}, to arrive at post-upgrade inventory $q = (q_1, ..., q_n)$. In the beginning of the selling season and after realization of demand, firm satisfies demand of m classes from on-hand inventory q. Demand is a random vector $d = (d_1, ..., d_m)$ with joint probability density function $f(d)$. We denote the set of products as P and set of markets as C.

Before preceding with the model formulation, we discuss appropriate foundations of quality constraints in each end-market as encountered in various applications.

5.1 *Formulation of Quality Constraints*

The end-markets have heterogeneous requirement on product qualities. As noted before, the quality level of product j as g_j. The end-markets quality requirements can be of two forms: 1) the average quality of supplied products to market k must meet a certain threshold γ_k; 2) the fraction of supply to market k which meet a quality level γ_k has to be higher than a percentile ρ_k. Mathematically the two forms can be expressed as:

(1): Average quality requirement(γ_k)

$$\frac{\sum_j g_j x_{jk}}{\sum_j x_{jk}} \geq \gamma_k \tag{5a}$$

(2): Fractional quality requirement(γ_k, ρ_k)

$$\frac{\sum_j 1\{g_j \geq \gamma_k\} x_{jk}}{\sum_j x_{jk}} \geq \rho_k \tag{5b}$$

The fractional quality requirement encompass a wider range of situations than the average quality requirement, and therefore our model will be focusing on this class.

We would like to point out that both of the above quality requirement constraints can be captured in the following transformed constraint:

$$\sum_{j \in P} (1 - a_{jk}) x_{jk} \leq 0 \tag{6}$$

where a_{jk} is appropriately defined depending on the end-market quality requirement used, when using the "average quality requirement" definition as in (5a),

$$a_{jk} = \frac{g_j}{\gamma_k} \tag{6a}$$

and when using the "Fractional quality requirement" definition as in (5b),

$$a_{jk} = \frac{I_{\{g_j \geq \gamma_k\}}}{\rho_k} \tag{6b}$$

5.2 *Stochastic Program Formulation*

As for the simpler two product, two market problem, we proceed to formulate the firm's decisions as a three stage stochastic optimization problem.

Allocation Decision. In this stage, firm maximizes the sales revenue from allocation of available inventory q to demand classes d, according to quality requirement of each demand class. The allocation decision(**AP**) can be formulated as a linear program:

$$\text{(AP)} \qquad r(q,d) := \max_{\substack{x_{jk} \\ j \in P,\, k \in C}} \sum_{j \in P} \sum_{k \in C} p_k x_{jk} \tag{7}$$

$$\text{subject to:} \quad \sum_{j \in P} x_{jk} \leq d_k, \qquad \forall k \in C, \tag{7a}$$

$$\sum_{k \in C} x_{jk} \leq q_j, \qquad \forall j \in P, \tag{7b}$$

$$\sum_{j \in P} (1 - a_{jk}) x_{jk} \leq 0, \qquad \forall k \in C, \tag{7c}$$

$$x_{jk} \geq 0, \qquad \forall j \in P,\ \forall k \in C. \tag{7d}$$

where (7a) and (7b) are demand and capacity constraints, respectively. The quality constraint (7c) is discussed as above.

Upgrading Decision. At the second stage, the firm decides on the optimal target inventory level q after using quality upgrading processes. In other words, with $y = (y_1, ..., y_n)$ on hand inventories of the various products, and exogenous upgrading costs c_{ij} in upgrading product of quality i to product of quality j, the firm seeks a strategy of product upgrading that maximizes the expected value of sales minus upgrading costs. The "Upgrading" problem(**UP**) in this stage can be formulated as follows:

$$\text{(UP)} \qquad \Pi(y) = \max_{\substack{u_{ij} \\ i,j\in P,\ i>j}} \quad \mathbb{E}_d\left[r(q,d)\right] - \sum_{j=1}^{n-1}\sum_{i>j} c_{ij}u_{ij} \tag{8}$$

$$\text{subject to:} \quad q_i = y_i + \sum_{j>i} u_{ji} - \sum_{j<i} u_{ij}, \qquad \forall i \in P, \tag{8a}$$

$$\sum_{j<i} u_{ij} \leq y_i, \qquad \forall i = \{1, ..., n\}, \tag{8b}$$

$$u_{ij} \geq 0, \qquad \forall i,j \in P. \tag{8c}$$

Though we don't put any constraint on possibly downgrading, i.e., $u_{ij} \neq 0$ for $i \geq j$, it will be trivial to see that such decisions would never occur if $c_{ij} \geq 0$ for $i \geq j$. Under this condition, any downgrading decision would be dominated by a leave-alone decision.

Production Decision. While facing yield uncertainty, the firm decides the total quantity of the single input Q in the co-production system at unit production cost c. The "Production" problem(**PP**) at this stage is as follows:

$$\text{(PP)} \qquad \max_{Q\geq 0} J(Q) := \mathbb{E}_\alpha[\Pi(\alpha Q)] - cQ \tag{9}$$

With this general formulation of the three stage optimization problem, the following result can be stated:

Proposition 6. *The optimal allocation profit $r(q,d)$ is concave in q, the optimal upgrading profit $\Pi(y)$ is concave in y and the optimal firm value $J(Q)$ is concave in Q.*

One important implication of Proposition 6 is that an optimal production quantity Q^* exists. However, solving the general formulation of the problem for multi-product, multi-market application may be computationally challenging. We suggest exploiting the concavity of the objective functions in the stochastic variables as follows.

By considering a deterministic demand equal to expected value of d, we obtain an upper-bound on the solution of the expected profit. Also, since $\Pi(.)$ is concave, replacing α by the deterministic yield rate at equal expected value, yields an upper-bound on the optimal expected profit. In other words, the solution of a large deterministic linear program would provide reasonable bound on expected profits and suggested values on input quantities to the co-production system that can be further evaluated on their actual profit performance.

6. Concluding Remarks and Future Directions

In this paper we studied the operational decisions of a co-production system of a single input and multiple output products under both yield and demand uncertainty. Products are differentiated by quality levels and the yield rate of the co-production process for each quality level product is random. After the yield uncertainty is realized, the firm has an intermediate opportunity to engage in a costly upgrading process to adjust the inventories of products of different quality levels. End-market demand is segmented into classes with heterogeneous quality requirements. Blending of products of different quality levels may be used to meet quality requirements of different segments. We formulated the problem of deciding the quantity of the early input to the co-production process, the quality upgrading decision after yield realization, and the optimal blending decision for meeting the multi-segment end-market demand as a three-stage stochastic model, and characterized optimal decisions at each stage.

Our most insightful results are presented for a highly stylized model of a two products, two demand segments case, with the high-quality product serving as a sole source for meeting high-end market demand and a blend of high and low quality products as a source for meeting low-end market demand. We show that if prices favor the high quality product, the blending scheme follows a greedy pattern that prioritizes high-end market demand. Furthermore, the quality upgrading policy is of a simple single threshold type; i.e., if the realized high quality yield is lower than a threshold, the firm needs to upgrade and otherwise no upgrading is needed.

Although upgrading is costly, we show that the total expected profit is always higher with upgrading. Moreover, the total input quantity to the co-production process is lower when upgrading is available than it's not. Multiple factors of the decision environment can affect the magnitude of the value of the upgrading flexibility. We show that increases in co-production costs leads to higher value of upgrading. On the other hand, higher costs of upgrading decreases the flexibility value. Furthermore, higher demand correlation leads to lower total expected profits, and also decreases the value of upgrading flexibility.

The specific characteristics of the yield ratio distribution play an important role in affecting the value of upgrading as well. We offer insights on the effects of changes in the distribution of high quality yield ratio. We show that the value of upgrading flexibility is increasing in the scale parameter of a Beta distribution which represents the yield ratio of the high quality output.

Future research can focus on providing more effective heuristic policies for the general model when the number of products and market segments limits the applicability of a stochastic programming approach. In the general model of the problem, we show that optimal decisions of the three stage problem exist but it is hard to characterize the exact policy. A good heuristic policy may provide great value for practical purposes.

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Appendix A. Proofs

Proof of Proposition 1 Let $\pi(q,y) = \mathbb{E}_D[r(q,d)] - c_u(q_H - y_H)$. We know from proposition 6 that $\pi(.,y)$ is concave, and by lemma 1 differentiable in everywhere. Fix Q and define $A(Q) = \{q|q_H + q_L = Q\}$. $A(Q)$ is closed and compact. Therefore, there exists a unique maximizer $q^t(Q)$ of $\pi(q,y)$ over $A(Q)$ such that $q_H^* = q^t$ and $Q_L^* = Q - q^t$. For the constrained problem with one way transfer possibility, the optimal values are adjusted so that $q_H^*(y) = max\{q^t, y_H\}$. To characterize $q^t(Q)$, we use the directional derivative of $\pi(q,y)$ in the direction $d(q_H, q_L) = (+1,-1)d\epsilon$, where $d\epsilon > 0$. Using the result of lemma 1 and shadow vectors in table 1, we have:

$$\sqrt{2}d\pi = \mathbb{E}\lambda(q,d).(+1,-1)d\epsilon - c_u d\epsilon$$
$$\sqrt{2}\frac{d\pi}{d\epsilon} = \frac{p_L}{\gamma}\mathbb{P}_2 + p_H\mathbb{P}_3 - c_u$$

where $\mathbb{P}_i$ is shorthand for $Pr\left(\Omega_i(q)\right)$. Let $\mathbb{F}(q_H, Q) = \frac{p_L}{\gamma}\mathbb{P}_2 + p_H\mathbb{P}_3 - c_u$. We have $\mathbb{F}(0,0) = p_H - c_u > 0$. Consider partial derivative of $\mathbb{F}(q_H, Q)$ w.r.t. q_H:

$$\frac{\partial \mathbb{F}}{\partial q_H} = -\left(p_H - \frac{p_L}{\gamma}\right)\int_0^\infty f(q_H, l)dl - \frac{p_L}{1-\gamma}\int_{\frac{Q-q_H}{1-\gamma}}^\infty f(\frac{(q_H - \gamma Q)^+}{1-\gamma}, l)dl$$
$$- \frac{p_L}{\gamma^2}\int_{\left(\frac{q_H-\gamma Q}{1-\gamma}\right)^+}^{q_H} f(h, \frac{q_H - h}{\gamma})dh < 0 \tag{A1}$$

Fix $q_L = 0$. By $\mathbb{F}$ decreasing in q_H, there exists $\overline{q}_H > 0$ such that $\mathbb{F}(\overline{q}_H, \overline{q}_H) = 0$. Hence, $\underline{Q} = \overline{q}_H$. Fix any $Q < \underline{Q}$. Since $\mathbb{F}$ is decreasing in q_H and $\mathbb{F}(Q,Q) > 0$, we have $q^t(Q) = Q$. Fix any $Q > \underline{Q}$. $\mathbb{F} = p_H - c_u > 0$ at $q_H = 0$ and negative when $q_H = Q$, hence, there exist a unique $q^t(Q)$ such that $\mathbb{F}(q^t(Q), Q) = 0$. Next we show that $q^t(Q)$ is non-decreasing in Q. For $Q < \underline{Q}$, we have $q^t(Q) = Q$. For $Q > \underline{Q}$, we use the implicit function theorem:

$$\frac{\partial q^t(Q)}{\partial Q} = -\frac{\frac{\partial \mathbb{F}(q_H,Q)}{\partial Q}}{\frac{\partial \mathbb{F}(q_H,Q)}{\partial q_H}} \tag{A2}$$

where

$$\frac{\partial \mathbb{F}(q_H, Q)}{\partial Q} = \frac{p_L}{1-\gamma}\int_{\frac{Q-q_H}{1-\gamma}}^\infty f(\frac{(q_H - \gamma Q)^+}{1-\gamma}, l)dl \geq 0. \tag{A3}$$

By (A1)-(A3), we conclude that $q^t(Q)$ is non-decreasing in Q. Finally, note that when $q_H \leq \gamma Q$ in (A3), we get $\frac{\partial q^t(Q)}{\partial Q} = 0$. Let $\overline{Q}$ satisfy $q^t(\overline{Q}) = \gamma\overline{Q}$. $\overline{Q}$ exists and is unique by $q^t(Q)$ non-decreasing. Therefore, for any $Q \geq \overline{Q}$, $q^t(Q)$ is constant. □

Proof of proposition 2

(1) Proposition 6 proves a general case and the results can be directly borrowed.
(2) Let $Q = 0$. Then $J_U(Q) = J_O(Q) = 0$. Fix $Q > 0$. The difference between $J_U(Q)$ and $J_O(Q)$ can be written as follows:

$$\begin{aligned}
J_U(Q) - J_O(Q) =& \mathbb{E}_{\alpha<\frac{q^t(Q)}{Q}}\left[\mathbb{E}_d\left[r(q^t(Q), Q - q^t(Q), d)\right] - \mathbb{E}_d\left[r(\alpha Q, (1-\alpha Q), d)\right] - c_u(q^t(Q) - \alpha Q)\right] \\
=& \mathbb{E}_{\alpha<\frac{q^t(Q)}{Q}}\left[\frac{\mathbb{E}_d\left[r(q^t(Q), Q - q^t(Q), d)\right] - c_u(q^t(Q) - \alpha Q) - \mathbb{E}_d\left[r(\alpha Q, (1-\alpha Q), d)\right]}{q^t(Q) - \alpha Q}\right]
\end{aligned}$$

The argument inside the expectation is the directional derivative of $\pi(., d)$ in the direction $\overrightarrow{v} = (+1, -1)'(q^t(Q) - \alpha Q)$. Recall from the discussion of optimal upgrading policy, that the directional derivative is non-negative when $y_H = \alpha Q < q^t(Q)$. Therefore, the difference is non-negative. Next, we show that optimal production decision is always smaller with upgrading. Let Q^*_U be a maximizer of $J_U(Q)$. Therefore, Q^*_U satisfies the first order condition:

$$\begin{aligned}
\frac{\partial J_U(Q)}{\partial Q} =& -c + \int_0^{\frac{q^t(Q)}{Q}} \left[\frac{\partial \mathbb{E}_d r}{\partial q_H} \times \frac{\partial q^t(Q)}{\partial Q} + \frac{\partial \mathbb{E}_d r}{\partial q_L} \times (1 - \frac{\partial q^t(Q)}{\partial Q}) - c_u\left(\frac{\partial q^t(Q)}{\partial Q} - \alpha\right)\right] g(\alpha)d\alpha \\
&+ \int_{\frac{q^t(Q)}{Q}}^1 \left(\frac{\partial}{\partial Q}\mathbb{E}_d\left[r(\alpha Q, (1-\alpha)Q, d)\right]\right) g(\alpha)d\alpha \\
=& -c + \int_0^{\frac{q^t(Q)}{Q}} \left(\frac{\partial q^t(Q)}{\partial Q} \times \mathbb{F}(q^t(Q), Q) + p_L \mathbb{P}_1 + \alpha c_u\right) g(\alpha)d\alpha \\
&+ \int_{\frac{q^t(Q)}{Q}}^1 \left(\frac{\partial}{\partial Q}\mathbb{E}_d\left[r(\alpha Q, (1-\alpha)Q, d)\right]\right) g(\alpha)d\alpha \\
=& -c + \int_0^{\frac{q^t(Q)}{Q}} \left(p_L \mathbb{P}_1 + \frac{\alpha p_L}{\gamma}\mathbb{P}_2 + \alpha p_H \mathbb{P}_3\right) g(\alpha)d\alpha + \int_{\frac{q^t(Q)}{Q}}^1 \left(\frac{\partial}{\partial Q}\mathbb{E}_d\left[r(\alpha Q, (1-\alpha)Q, d)\right]\right) g(\alpha)d\alpha
\end{aligned}$$

Now, we evaluate the derivative of $J_O(Q)$ at Q^*_U. We have:

$$\begin{aligned}
\frac{\partial J_O(Q)}{\partial Q} =& -c + \int_0^1 \left(\frac{\partial}{\partial Q}\mathbb{E}_d\left[r(\alpha Q, (1-\alpha)Q, d)\right]\right) g(\alpha)d\alpha \\
=& -c + \int_0^1 \left(p_L \mathbb{P}_1 + \frac{\alpha p_L}{\gamma}\mathbb{P}_2 + \alpha p_H \mathbb{P}_3\right) g(\alpha)d\alpha
\end{aligned}$$

We ignore the common components in both derivatives and only analyze the difference. Note that $\mathbb{P}_i$ is a function of (q_H, Q). We need to compare the value of $\frac{\partial \mathbb{E}r}{\partial Q} = p_L\mathbb{P}_1 + \frac{\alpha p_L}{\gamma}\mathbb{P}_2 + \alpha p_H \mathbb{P}_3$ at $(\alpha Q^*_U, Q^*_U)$ versus $(q^t(Q^*_U), Q^*_U)$. It is easy to show that $\frac{\partial^2 \mathbb{E}r}{\partial Q \partial q_H} < 0$. Since within the bounds of the integral; i.e. $\alpha \in \left(0, \frac{q^t(Q)}{Q}\right)$ we have $\alpha Q^*_U \leq q^t(Q^*_U)$, therefore, $\frac{\partial J_O(Q)}{\partial Q}|_{Q=Q^*_U} > 0$. Therefore, $J_O(Q)$ is non-decreasing at $Q = Q^*_U$. By concavity of $J_O(.)$, we conclude $Q^*_O > Q^*_U$.

□

Proof of proposition 3 We begin by analyzing the effect of production cost c. Observe that $\frac{\partial J(Q)}{\partial c} = -Q$. Hence, $\frac{\partial VoU}{\partial c} = -Q^*_U + Q^*_O$ and by proposition 2 we conclude that VoU increases in

c. Next, we use the implicit function theorem to find the impact of c on Q_U^*.

$$\frac{\partial Q_U^*}{\partial c} = -\frac{\frac{\partial^2 J(Q,c)}{\partial Q \partial c}}{\frac{\partial^2 J(Q,c)}{\partial Q^2}} = -\frac{-1}{\frac{\partial^2 J(Q,c)}{\partial Q^2}} < 0$$

The inequality holds by concavity of J in Q. Hence, Q_U^* and Q_O^* both decrease in c. Now, consider the effect of upgrading cost c_u. Clearly the benchmark problem and it's optimal solution is not affected by a change in c_u. For the upgrading problem, we have $\frac{\partial J_U(Q,c_u)}{\partial c_u}|_{Q=Q_U^*} = \mathbb{E}_{\alpha<\frac{q^t(Q)}{Q}}\left[-q^t(Q_U^*) + \alpha Q\right] < 0$. Hence, J_U^* and therefore VoU are decreasing in c_u. To show that Q_U^* is increasing in c_u, we use implicit function theorem:

$$\frac{\partial Q_U^*}{\partial c_u} = -\frac{\frac{\partial^2 J(Q,c_u)}{\partial Q \partial c_u}}{\frac{\partial^2 J(Q,c_u)}{\partial Q^2}} = -\frac{\mathbb{E}[\alpha|\alpha < \frac{q^t(Q)}{Q}]}{\frac{\partial^2 J(Q,c)}{\partial Q^2}} > 0$$

□

Proof of proposition 5 We begin by showing that $\Pi(y(\alpha, Q))$ is concave increasing in α. First, note that $\Pi(y)$ is increasing in y, therefore, Π is increasing in α. Concavity of Π in α is shown below:

$$\begin{aligned}\Pi(\alpha^\beta Q, (1-\alpha^\beta)Q) =& \Pi(\beta\alpha^1 Q + \overline{\beta}\alpha^2 Q, \beta(1-\alpha^1)Q, \overline{\beta}(1-\alpha^2)Q), \\ &\geq \beta\Pi(\alpha^1 Q, (1-\alpha^1)Q) + (1-\beta)\Pi(\alpha^2 Q, (1-\alpha^2)Q).\end{aligned} \tag{A4}$$

Next, we use the following result by Adell et al. (1993); If $X(\theta)$ follows $Beta(t(1-\theta)+1, t\theta+1)$, and for $t > 0$ and $\theta \in (0,1)$, then $\{\alpha(\theta), \theta \in (0,1)\} \in SDCX$; i.e. if $\theta_2 > \theta_1$, then $\alpha(\theta_2) \geq_{dcx} \alpha(\theta_1)$. As a result, $\mathbb{E}[f(\alpha(\theta_2))] \geq \mathbb{E}[f(\alpha(\theta_1))]$ for any convex decreasing f. The result follows by letting $f = -\Pi$. □

Proof of Proposition 6 Concavity of $r(q,d)$ in q follows from basic principles of linear programming. We include the proof here for convenience. Let $X = \{x_{jk} | j \in P, k \in C\}$. Let X^1 and X^2 be optimal solutions of the maximization problem (7) for arbitrary $q^1, q^2 \in \mathbb{R}_+^n$, respectively. Let $q^\beta = \beta q^1 + \overline{\beta} q^2$, where $\overline{\beta} = 1 - \beta$ and $\beta \in [0,1]$. By linearity of constraints (7b)-(7d), $X^\beta = \beta X^1 + \overline{\beta} X^2$ is a feasible solution for q^β. Therefore,

$$\begin{aligned}r(q^\beta, d) &\geq \sum_{j\in P}\sum_{k\in C} p_k x_{jk}^\beta, \\ &= \beta \sum_{j\in P}\sum_{k\in C} p_k x_{jk}^1 + \overline{\beta}\sum_{j\in P}\sum_{k\in C} p_k x_{jk}^2, \\ &= \beta r(q^1, d) + \overline{\beta} r(q^2, d).\end{aligned}$$

Next, we show concavity of $\Pi(y)$. By linearity of q in (u,y) in (8a), concavity of $r(.,d)$ and preservation of concavity under expectation, we have $\mathbb{E}_d[r(q(u,y),d)]$ is jointly concave in (u,y). Concavity of $\pi(q(u,y))$ follows directly from linearity of second term in objective function. Now, let $C = \{(u,y) | u \geq 0,\ u \in U(y)\}$ be the feasible space of the upgrading problem, where $U(y) = \left\{u | \sum_{j<i} u_{ij} \leq y_i, \forall i = 2, ..., n\right\}$. It is straightforward to show that $U(y)$ is non-empty and C is convex. Let $y^1, y^2 \in \mathbb{R}_+^n$ be arbitrary. Let $\delta > 0$. There exists $u^t \in U(y^t)$ such that

$\pi(q(u^t, y^t)) \geq \Pi(y^t) - \delta$ for $t = 1, 2$. Let $\theta \in [0, 1]$ and $\overline{\theta} = 1 - \theta$. We have:

$$\begin{aligned}
\theta\Pi(y^1) + \overline{\theta}\Pi(y^2) &\leq \theta\pi(q(u^1, y^1)) + \overline{\theta}\pi(q(u^2, y^2)) + \delta, \\
&\leq \pi(\theta q(u^1, y^1) + \overline{\theta} q(u^2, y^2)) + \delta, \\
&= \pi\left(q(\theta u^1 + \overline{\theta} u^2, \theta y^1 + \overline{\theta} y^2)\right) + \delta, \\
&\leq \Pi(\theta y^1 + \overline{\theta} y^2) + \delta
\end{aligned}$$

where the first inequality follows by assumption, second inequality is by concavity of $\pi(.)$, third inequality follows form linearity of $q(u, y)$ and last inequality holds by definition of $\Pi(.)$. Since δ is arbitrary, the inequality must hold for $\delta = 0$. Otherwise, a contradiction can be reached. Hence, $\Pi(y)$ is concave in y. Finally, we need to show that $J(.)$ is concave. Fix α, let $Q^1, Q^2 \in \mathbb{R}_+$ be arbitrary and $\theta, \overline{\theta} \in [0, 1]$ as defined above. We have:

$$\begin{aligned}
\Pi\left(\alpha(\theta Q^1 + \overline{\theta} Q^2)\right) - c(\theta Q^1 + \overline{\theta} Q^2) &\geq \theta\Pi(\alpha Q^1) + \overline{\theta}\Pi(\alpha Q^2) - c(\theta Q^1 + \overline{\theta} Q^2), \\
&= \theta\left(\Pi(\alpha Q^1) - cQ^1\right) + \overline{\theta}\left(\Pi(\alpha Q^2) - cQ^2\right).
\end{aligned}$$

Taking expectation over α from both sides will give the result; that is, $J(Q)$ is concave in Q. $\square$