Calculus refresher

Disclaimer: I claim no original content on this document, which is mostly a summary-rewrite of what any standard college calculus book offers. (Here I’ve used *Calculus* by Dennis Zill.) I consider this as a brief refresher of the bare-bones calculus requirements we will be using during the course. No exercises are offered: any calculus book provides an insane amount of practice problems.

1. Rules of differentiation: Basics

In what follows, let \( f(x) \) be a one-variable function, and denote its derivative by \( f'(x) \). The standard differentiation rules are the following:

**Theorem 1.1** (The Power Rule, 1). Let \( n \) be a positive integer. Then

\[
\frac{d}{dx}[x^n] = nx^{n-1}.
\]

**Example 1.2.** The derivative of \( y = x^4 \) is given by

\[
\frac{dy}{dx} = 4x^{4-1} = 4x^3.
\]

**Theorem 1.3** (Derivative of a constant function). \( f(x) = k \) and \( k \) is a constant, then \( f'(x) = 0 \).

**Theorem 1.4** (Derivative of a constant multiple of a function). If \( c \) is any constant and \( f \) is a differentiable function, then

\[
\frac{d}{dx}[cf(x)] = cf'(x).
\]

**Example 1.5.** Following theorems 1.1 and 1.4, the derivative of \( y = 3x^5 \) is given by:

\[
\frac{dy}{dx} = 3 \cdot \frac{d}{dx}x^5 = 3(5x^4) = 15x^4.
\]

**Theorem 1.6** (The Sum Rule). Let \( f \) and \( g \) be two differentiable functions. Then

\[
\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).
\]

**Example 1.7.** From theorems 1.1 and 1.6, the derivative of \( y = x^4 + x^3 \) is:

\[
\frac{dy}{dx} = \frac{d}{dx}x^4 + \frac{d}{dx}x^3 = 4x^3 + 3x^2.
\]

**Theorem 1.8** (The Product Rule). If \( f \) and \( g \) are differentiable functions, then

\[
\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).
\]

**Example 1.9.** The differential of \( y = (x^3 - 2x^2 + 4)(8x^2 + 5x) \) is:

\[
\frac{dy}{dx} = (x^3 - 2x^2 + 4) \cdot \frac{d}{dx}(8x^2 + 5x) + (8x^2 + 5x) \cdot \frac{d}{dx}(x^3 - 2x^2 + 4)
\]

\[
= (x^3 - 2x^2 + 4)(16x + 5) + (8x^2 + 5x)(3x^2 - 4x).
\]

**Theorem 1.10** (The Quotient Rule). If \( f \) and \( g \) are differentiable functions, then

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.
\]
Example 1.11. The differential of \( y = \frac{3x^2 - 1}{2x^3 + 5x^2 + 7} \) is:

\[
\frac{dy}{dx} = \frac{(2x^3 + 5x^2 + 7) \cdot \frac{d}{dx}(3x^2 - 1) - (3x^2 - 1) \cdot \frac{d}{dx}(2x^3 + 5x^2 + 7)}{(2x^3 + 5x^2 + 7)^2}
\]

\[
= \frac{(2x^3 + 5x^2 + 7) \cdot (6x) - (3x^2 - 1) \cdot (6x^2 + 10x)}{(2x^3 + 5x^2 + 7)^2}
\]

\[
= \frac{-6x^4 + 6x^2 + 52x}{(2x^3 + 5x^2 + 7)^2}.
\]

Theorem 1.12 (The Power Rule, 2). If \( n \) is a positive integer, then

\[
\frac{d}{dx}[x^{-n}] = -nx^{-n-1}.
\]

Example 1.13. The derivative of \( y = 5x^3 - \frac{1}{x^4} \) is (after rewriting \( y \) as \( 5x^3 - x^{-4} \)):

\[
\frac{dy}{dx} = 5 \cdot 3x^2 - (-4)x^{-5} = 15x^2 + \frac{4}{x^5}.
\]

2. RULES OF DIFFERENTIATION: THE CHAIN RULE

Theorem 2.1 (The Power Rule for Functions). If \( n \) is an integer and \( g \) is a differentiable function, then

\[
\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x).
\]

Example 2.2. Let \( y = (2x^3 + 4x + 1)^4 \). If \( g(x) = (2x^3 + 4x + 1) \) and \( n = 4 \), then from Theorem 2.1 it follows that

\[
\frac{dy}{dx} = 4(2x^3 + 4x + 1)^3 \frac{d}{dx}(2x^3 + 4x + 1) = 4(2x^3 + 4x + 1)^3(6x^2 + 4).
\]

Theorem 2.3 (The Chain Rule). If \( y = f(u) \) is a differentiable function of \( u \) and \( u = g(x) \) is a differentiable function of \( x \), then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(g(x)) \cdot g'(x).
\]

Example 2.4. From the rules of differentiation of trigonometric functions, it is true that:

\[
\frac{d}{dx} \sin u = \cos u \frac{du}{dx}.
\]

Let \( y = (9x^3 + 1)^2 \sin 5x \). Then, the derivative of \( y \) can be obtained first by applying Theorem 1.8:

\[
\frac{dy}{dx} = (9x^3 + 1)^2 \frac{d}{dx} \sin 5x + \sin 5x \frac{d}{dx}(9x^3 + 1)^2,
\]

followed by applications of theorems 2.1 and 2.2:

\[
\frac{dy}{dx} = (9x^3 + 1)^2 \cdot 5 \cos 5x + \sin 5x \cdot (27x^2) = (9x^3 + 1)(45x^3 \cos 5x + 54x^2 \sin 5x + 5 \cos 5x).
\]
3. Higher-order derivatives

**Definition 3.1 (The Second Derivative).** The derivative $f'(x)$ is a function derived from a function $y = f(x)$. By differentiating the first derivative $f'(x)$, we obtain yet another function called the second derivative, denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$:

$$\frac{d}{dx} \left( \frac{dy}{dx} \right).$$

Alternative notation for the second derivative is given by: $f''(x)$, $y''$, $\frac{d^2y}{dx^2}$, and $D^2_y$.

**Example 3.2.** To calculate the second derivative of $y = x^3 - 2x^2$, first we obtain $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 3x^2 - 4x.$$

Then, the second derivative is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (3x^2 - 4x) = 6x - 4.$$

**Remark 3.3.** Higher-order derivatives are obtained analogously: the third derivative is the derivative of the second derivative; the fourth derivative is the derivative of the third derivative, and so on. A standard notation for the first $n$ derivatives is (among others) $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, . . . , $f^{(n)}(x)$.

**Example 3.4.** The first five derivatives of the polynomial $f(x) = 2x^4 - 6x^3 + 7x^2 + 5x - 10$ are given by:

- $f'(x) = 8x^3 - 18x^2 + 14x + 5$,
- $f''(x) = 24x^2 - 36x + 14$,
- $f'''(x) = 48x - 36$,
- $f^{(4)}(x) = 48$,
- $f^{(5)}(x) = 0$.

4. Extremals of functions

**Definition 4.1 (Absolute extremals).** Let $f$ be a real-valued function. The absolute extremals of $f$ are defined as follows:

(a) A number $f(\bar{c})$ is an absolute maximum if $f(x) \leq f(\bar{c})$ for every $x$ in the domain of $f$.

(b) A number $f(\bar{c})$ is an absolute minimum if $f(x) \geq f(\bar{c})$ for every $x$ in the domain of $f$.

**Theorem 4.2 (The Extreme Value Theorem).** A continuous function $f$ defined on a closed interval $[a, b]$ always has an absolute maximum and an absolute minimum on the interval.

**Definition 4.3 (Critical point).** A critical point of a function $f$ is a number $c$ in its domain for which $f'(c) = 0$.

**Example 4.4.** To find the critical points of $f(x) = x^3 - 15x + 6$, we obtain the first derivative $f'(x)$:

$$f'(x) = 3x^2 - 15 = 3(x + \sqrt{5})(x - \sqrt{5}).$$

Hence, the critical points are those numbers for which $f'(x) = 0$, namely, $-\sqrt{5}$ and $\sqrt{5}$.

**Theorem 4.5.** If $f$ is continuous on a closed interval $[a, b]$, then an absolute extremum occurs either at an endpoint of the interval or at a critical point in the open interval $(a, b)$.

**Remark 4.6.** The previous theorem says that any absolute extremum must occur in an endpoint or in a critical point. This is not the same as assuming that because we have a critical point, it must be an extremum! Second-order conditions should be checked to verify that we have a minimum or maximum.
Example 4.7. Let $f(x) = x^3 - 3x^2 - 24x + 2$ be defined on the intervals $[-3, 1]$ and $[-3, 8]$. To find the absolute extremals, first we obtain $f'(x)$:

$$f'(x) = 3x^2 - 6x - 24 = 3(x + 2)(x - 4),$$

and it follows that the critical points of the function are -2 and 4. Simple calculations show that for the interval $[-3, 1]$, the absolute maximum is $f(-2) = 30$, and the absolute minimum is the endpoint extremum $f(1) = -24$. For the interval $[-3, 8]$, the absolute minimum is $f(4) = -78$, while the absolute maximum is reached at the endpoint extremum $f(8) = 130$.

5. The natural logarithmic function

**Definition 5.1** (The Natural Logarithmic Function). The natural logarithmic function, denoted by $\ln x$, is defined by:

$$\ln x = \int_1^x \frac{dt}{t}$$

for all $x > 0$.

**Theorem 5.2** (The Derivative of the Natural Logarithm). The derivative of $\ln x$ is given by:

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

for all $x > 0$.

**Theorem 5.3** (Laws of the Natural Logarithm). Let $a$ and $b$ be positive real numbers and let $t$ be a rational number. Then:

(a) $\ln ab = \ln a + \ln b$.
(b) $\ln \frac{a}{b} = \ln a - \ln b$.
(c) $\ln a^t = t \ln a$.

**Example 5.4.** To obtain the derivative of $y = \ln(2x - 3)$, note that for $2x - 3 > 0$, we have from Theorem 7.2:

$$\frac{dy}{dx} = \frac{1}{2x - 3} \frac{d}{dx} (2x - 3) = \frac{2}{2x - 3}.$$

6. The exponential function

**Definition 6.1** (The Natural Exponential Function). The natural exponential function is defined by:

$$y = \exp x \quad \text{if and only if} \quad x = \ln y.$$ 

For any real number $x$, $e^x = \exp x$.

**Theorem 6.2** (Laws of Exponents). Let $r$ and $s$ be any real numbers and let $t$ be a rational number. Then:

(a) $e^0 = 1$.
(b) $e^1 = e$.
(c) $e^r e^s = e^{r+s}$.
(d) $e^{-r} = e^{-s}$.
(e) $(e^r)^t = e^{rt}$.
(f) $e^{-r} = \frac{1}{e^r}$.

1 Usually the symbol $\ln x$ is pronounced “ell-en of $x$”
**Theorem 6.3** (The derivative of the exponential function). The derivative of $y = \exp x$ is given by:

$$\frac{d}{dx}[\exp x] = \exp x.$$  \hspace{1cm} (12)

Using the Chain Rule, we can generalize the above to

$$\frac{d}{dx}[\exp u] = \exp u \frac{du}{dx},$$  \hspace{1cm} (13)

where $u = g(x)$ is a differentiable function.

**Example 6.4.** The derivative of $y = \exp 4x$ is given by:

$$\frac{dy}{dx} = \exp 4x \cdot \frac{d}{dx}(4x) = \exp 4x \cdot 4 = 4\exp 4x.$$

## 7. Partial differentiation

**Remark 7.1** (Partial differentiation). Let $z = f(x, y)$. To compute $\partial z/\partial x$, use the laws of ordinary differentiation while treating $y$ as a constant. To compute $\partial z/\partial y$, use the laws of ordinary differentiation while treating $x$ as a constant.

**Remark 7.2.** If $z = f(x, y)$, alternative notation for partial derivatives is $\partial z/\partial x = \partial f/\partial x = z_x = f_x$, and similarly, $\partial z/\partial y = \partial f/\partial y = z_y = f_y$.

**Example 7.3.** Let $z = 4x^3y^2 - 4x^2 + y^6 + 1$. The partial derivatives of $z$ with respect to $x$ and $y$ are given by:

$$\frac{\partial z}{\partial x} = 12x^2y^2 - 8x,$$

$$\frac{\partial z}{\partial y} = 8x^3y + 6y^5.$$

**Theorem 7.4** (Equality of Mixed Partial s). Let $f$ be a function of two variables. If $f_x, f_y, f_{xy},$ and $f_{yx}$ are continuous on an open region $R$, then $f_{xy} = f_{yx}$ at each point of $R$.

**Example 7.5.** Consider the partial derivatives obtained in Example 7.3. The continuity conditions of Theorem 7.4 are satisfied in this case, so it should be that $f_{xy} = f_{yx}$; this is so since

$$\frac{\partial^2 z}{\partial x \partial y} = 24x^2y,$$

$$\frac{\partial^2 z}{\partial y \partial x} = 24x^2y.$$