# Calculus refresher

**Disclaimer:** I claim no original content on this document, which is mostly a summary-rewrite of what any standard college calculus book offers. (Here I've used *Calculus* by Dennis Zill.) I consider this as a brief refresher of the bare-bones calculus requirements we will be using during the course. No exercises are offered: any calculus book provides an insane amount of practice problems.

## 1. Rules of differentiation: Basics

In what follows, let $f(x)$ be a one-variable function, and denote its derivative by $f'(x)$. The standard differentiation rules are the following:

**Theorem 1.1** (The Power Rule, 1)**.** *Let $n$ be a positive integer. Then*

$$\frac{d}{dx}[x^n] = nx^{n-1}. \tag{1}$$

**Example 1.2.** The derivative of $y = x^4$ is given by

$$\frac{dy}{dx} = 4x^{4-1} = 4x^3.$$

**Theorem 1.3** (Derivative of a constant function)**.** *If $f(x) = k$ and $k$ is a constant, then $f'(x) = 0$.*

**Theorem 1.4** (Derivative of a constant multiple of a function)**.** *If $c$ is any constant and $f$ is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = cf'(x). \tag{2}$$

**Example 1.5.** Following theorems 1.1 and 1.4, the derivative of $y = 3x^5$ is given by:

$$\frac{dy}{dx} = 3 \cdot \frac{d}{dx}x^5 = 3(5x^4) = 15x^4.$$

**Theorem 1.6** (The Sum Rule)**.** *Let $f$ and $g$ be two differentiable functions. Then*

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x). \tag{3}$$

**Example 1.7.** From theorems 1.1. and 1.6, the derivative of $y = x^4 + x^3$ is:

$$\frac{dy}{dx} = \frac{d}{dx}x^4 + \frac{d}{dx}x^3 = 4x^3 + 3x^2.$$

**Theorem 1.8** (The Product Rule)**.** *If $f$ and $g$ are differentiable functions, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x). \tag{4}$$

**Example 1.9.** The differential of $y = (x^3 - 2x^2 + 4)(8x^2 + 5x)$ is:

$$\begin{aligned}\frac{dy}{dx} &= (x^3 - 2x^2 + 4) \cdot \frac{d}{dx}(8x^2 + 5x) + (8x^2 + 5x) \cdot \frac{d}{dx}(x^3 - 2x^2 + 4) \\ &= (x^3 - 2x^2 + 4)(16x + 5) + (8x^2 + 5x)(3x^2 - 4x).\end{aligned}$$

**Theorem 1.10** (The Quotient Rule)**.** *If $f$ and $g$ are differentiable functions, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}. \tag{5}$$

**Example 1.11.** The differential of $y = \frac{3x^2-1}{2x^3+5x^2+7}$ is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x^3+5x^2+7)\cdot\frac{d}{dx}(3x^2-1)-(3x^2-1)\cdot\frac{d}{dx}(2x^3+5x^2+7)}{(2x^3+5x^2+7)^2} \\ &= \frac{(2x^3+5x^2+7)\cdot(6x)-(3x^2-1)\cdot(6x^2+10x)}{(2x^3+5x^2+7)^2} \\ &= \frac{-6x^4+6x^2+52x}{(2x^3+5x^2+7)^2}.\end{aligned}$$

**Theorem 1.12** (The Power Rule, 2)**.** *If $n$ is a positive integer, then*

$$\frac{d}{dx}[x^{-n}] = -nx^{-n-1}. \tag{6}$$

**Example 1.13.** The derivative of $y = 5x^3 - \frac{1}{x^4}$ is (after rewriting $y$ as $5x^3 - x^{-4}$):

$$\frac{dy}{dx} = 5\cdot 3x^2 - (-4)x^{-5} = 15x^2 + \frac{4}{x^5}.$$

## 2. Rules of differentiation: The Chain Rule

**Theorem 2.1** (The Power Rule for Functions)**.** *If $n$ is an integer and $g$ is a differentiable function, then*

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x). \tag{7}$$

**Example 2.2.** Let $y = (2x^3+4x+1)^4$. If $g(x) = (2x^3+4x+1)$ and $n = 4$, then from Theorem 2.1 it follows that

$$\frac{dy}{dx} = 4(2x^3+4x+1)^3\frac{d}{dx}(2x^3+4x+1) = 4(2x^3+4x+1)^3(6x^2+4).$$

**Theorem 2.3** (The Chain Rule)**.** *If $y = f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then*

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(g(x))\cdot g'(x). \tag{8}$$

**Example 2.4.** From the rules of differentiation of trigonometric functions, it is true that:

$$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}.$$

Let $y = (9x^3+1)^2\sin 5x$. Then, the derivative of $y$ can be obtained first by applying Theorem 1.8:

$$\frac{dy}{dx} = (9x^3+1)^2\frac{d}{dx}\sin 5x + \sin 5x\frac{d}{dx}(9x^3+1)^2,$$

followed by applications of theorems 2.1 and 2.2:

$$\begin{aligned}\frac{dy}{dx} &= (9x^3+1)^2\cdot 5\cos 5x + \sin 5x\cdot 2(9x^3+1)\cdot(27x^2) \\ &= (9x^3+1)(45x^3\cos 5x + 54x^2\sin 5x + 5\cos 5x).\end{aligned}$$

## 3. Higher-order derivatives

**Definition 3.1** (The Second Derivative)**.** The derivative $f'(x)$ is a function derived from a function $y = f(x)$. By differentiating the first derivative $f'(x)$, we obtain yet another function called the *second derivative*, denoted by $f''(x)$. In terms of the operation symbol $\frac{d}{dx}$:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Alternative notation for the second derivative is given by: $f''(x)$, $y''$, $\frac{d^2y}{dx^2}$, and $D_x^2 y$.

**Example 3.2.** To calculate the second derivative of $y = x^3 - 2x^2$, first we obtain $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 3x^2 - 4x.$$

Then, the second derivative is:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x) = 6x - 4.$$

**Remark 3.3.** Higher-order derivatives are obtained analogously: the *third derivative* is the derivative of the second derivative; the *fourth derivative* is the derivative of the third derivative, and so on. A standard notation for the first $n$ derivatives is (among others) $f'(x), f''(x), f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x)$.

**Example 3.4.** The first five derivatives of the polynomial $f(x) = 2x^4 - 6x^3 + 7x^2 + 5x - 10$ are given by:

$$\begin{aligned}
f'(x) &= 8x^3 - 18x^2 + 14x + 5,\\
f''(x) &= 24x^2 - 36x + 14,\\
f'''(x) &= 48x - 36,\\
f^{(4)}(x) &= 48,\\
f^{(5)}(x) &= 0.
\end{aligned}$$

## 4. Extremals of functions

**Definition 4.1** (Absolute extremals)**.** Let $f$ be a real-valued function. The absolute extremals of $f$ are defined as follows:

(a) A number $f(\bar{c})$ is an *absolute maximum* if $f(x) \leq f(\bar{c})$ for every $x$ in the domain of $f$.
(b) A number $f(\bar{c})$ is an *absolute minimum* if $f(x) \geq f(\bar{c})$ for every $x$ in the domain of $f$.

**Theorem 4.2** (The Extreme Value Theorem)**.** *A continuous function $f$ defined on a closed interval $[a, b]$ always has an absolute maximum and an absolute minimum on the interval.*

**Definition 4.3** (Critical point)**.** A *critical point* of a function $f$ is a number $c$ in its domain for which $f'(c) = 0$.

**Example 4.4.** To find the critical points of $f(x) = x^3 - 15x + 6$, we obtain the first derivative $f'(x)$:

$$f'(x) = 3x^2 - 15 = 3(x + \sqrt{5})(x - \sqrt{5}).$$

Hence, the critical points are those numbers for which $f'(x) = 0$, namely, $-\sqrt{5}$ and $\sqrt{5}$.

**Theorem 4.5.** *If $f$ is continuous on a closed interval $[a, b]$, then an absolute extremum occurs either at an endpoint of the interval or at a critical point in the open interval $(a, b)$.*

**Remark 4.6.** The previous theorem says that any absolute extremum *must* occur in an endpoint or in a critical point. This is *not* the same as assuming that because we have a critical point, it must be an extremum! Second-order conditions should be checked to verify that we have a minimum or maximum.

**Example 4.7.** Let $f(x) = x^3 - 3x^2 - 24x + 2$ be defined on the intervals $[-3, 1]$ and $[-3, 8]$. To find the absolute extremals, first we obtain $f'(x)$:

$$f'(x) = 3x^2 - 6x - 24 = 3(x + 2)(x - 4),$$

and it follows that the critical points of the function are -2 and 4. Simple calculations show that for the interval $[-3, 1]$, the absolute maximum is $f(-2) = 30$, and the absolute minimum is the endpoint extremum $f(1) = -24$. For the interval $[-3, 8]$, the absolute minimum is $f(4) = -78$, while the absolute maximum is reached at the endpoint extremum $f(8) = 130$.

## 5. The natural logarithmic function

**Definition 5.1** (The Natural Logarithmic Function)**.** The *natural logarithmic function*, denoted by $\ln x$,[1] is defined by:

$$\ln x = \int_1^x \frac{dt}{t} \tag{9}$$

for all $x > 0$.

**Theorem 5.2** (The Derivative of the Natural Logarithm)**.** *The derivative of* $\ln x$ *is given by:*

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \tag{10}$$

*for all* $x > 0$.

**Theorem 5.3** (Laws of the Natural Logarithm)**.** *Let* $a$ *and* $b$ *be positive real numbers and let* $t$ *be a rational number. Then:*

(a) $\ln ab = \ln a + \ln b$.
(b) $\ln \frac{a}{b} = \ln a - \ln b$.
(c) $\ln a^t = t \ln a$.

**Example 5.4.** To obtain the derivative of $y = \ln(2x - 3)$, note that for $2x - 3 > 0$, we have from Theorem 7.2:

$$\frac{dy}{dx} = \frac{1}{2x - 3}\frac{d}{dx}(2x - 3) = \frac{2}{2x - 3}.$$

## 6. The exponential function

**Definition 6.1** (The Natural Exponential Function)**.** The *natural exponential function* is defined by:

$$y = \exp x \quad \text{if and only if} \quad x = \ln y. \tag{11}$$

For any real number $x$, $e^x = \exp x$.

**Theorem 6.2** (Laws of Exponents)**.** *Let* $r$ *and* $s$ *be any real numbers and let* $t$ *be a rational number. Then:*

(a) $e^0 = 1$.
(b) $e^1 = e$.
(c) $e^r e^s = e^{r+s}$.
(d) $\frac{e^r}{e^s} = e^{r-s}$.
(e) $(e^r)^t = e^{rt}$.
(f) $e^{-r} = \frac{1}{e^r}$.

[1] Usually the symbol $\ln x$ is pronounced "ell-en of $x$"

**Theorem 6.3** (The derivative of the exponential function)**.** *The derivative of $y = \exp x$ is given by:*

$$\frac{d}{dx}[e^x] = e^x. \tag{12}$$

*Using the Chain Rule, we can generalize the above to*

$$\frac{d}{dx}[e^u] = e^u \frac{du}{dx}, \tag{13}$$

*where $u = g(x)$ is a differentiable function.*

**Example 6.4.** The derivative of $y = e^{4x}$ is given by:

$$\begin{aligned} \frac{dy}{dx} &= e^{4x} \cdot \frac{d}{dx}(4x) \\ &= e^{4x}(4) \\ &= 4e^{4x} \end{aligned}$$

## 7. Partial differentiation

**Remark 7.1** (Partial differentiation)**.** Let $z = f(x, y)$. To compute $\partial z/\partial x$, use the laws of ordinary differentiation while treating $y$ as a constant. To compute $\partial z/\partial y$, use the laws of ordinary differentiation while treating $x$ as a constant.

**Remark 7.2.** If $z = f(x, y)$, alternative notation for partial derivatives is $\partial z/\partial x = \partial f/\partial x = z_x = f_x$, and similarly, $\partial z/\partial y = \partial f/\partial y = z_y = f_y$.

**Example 7.3.** Let $z = 4x^3y^2 - 4x^2 + y^6 + 1$. The partial derivatives of $z$ with respect to $x$ and $y$ are given by:

$$\begin{aligned} \frac{\partial z}{\partial x} &= 12x^2y^2 - 8x \\ \frac{\partial z}{\partial y} &= 8x^3y + 6y^5. \end{aligned}$$

**Theorem 7.4** (Equality of Mixed Partials)**.** *Let $f$ be a function of two variables. If $f_x, f_y, f_{xy}$, and $f_{yx}$ are continuous on an open region $R$, then $f_{xy} = f_{yx}$ at each point of $R$.*

**Example 7.5.** Consider the partial derivatives obtained in Example 7.3. The continuity conditions of Theorem 7.4 are satisfied in this case, so it should be that $f_{xy} = f_{yx}$; this is so since

$$\begin{aligned} \frac{\partial^2 z}{\partial x\,\partial y} &= 24x^2y \\ \frac{\partial^2 z}{\partial y\,\partial x} &= 24x^2y. \end{aligned}$$