Lecture Notes: Math Refresher[1]

Math Facts

The following results from calculus will be used over and over throughout the course. A more complete list of useful results from calculus is posted on the course website.

Let c be a constant, n be a natural number, $f(x)$ and $g(x)$ be functions defined on $\mathbb{R}$. Then,

$$\begin{aligned}
f(x) &= c \Rightarrow f'(x) = 0 \\
f(x) &= x^n \Rightarrow f'(x) = nx^{n-1} \\
(f(x) + g(x))' &= f'(x) + g'(x) \\
(f(x) * g(x))' &= f'(x)g(x) + f(x)g'(x) \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f(x)g'(x) + f'(x)g(x)}{g(x) * g(x)} \\
f\left(g(x)\right)' &= f'(g(x)) * g'(x)
\end{aligned}$$

In addition, we will make use of the following facts about the natural log. **We will only use the natural log in the this class and NEVER log base 10**. Therefore, $log(x)$ and $ln(x)$ are going to be used interchangeably and both will mean the natural log of x.

$$\begin{aligned}
log(xy) &= log(x) + log(y) \\
log\left(\frac{x}{y}\right) &= log(x) - log(y)
\end{aligned}$$

Also, if $f(x) = log(x)$ then $f'(x) = \frac{1}{x}$. Finally, recall the definition of a partial derivative. If $f(x, y)$ is a differentiable function of x and y, then to calculate $\frac{\partial f(x,y)}{\partial x}$, pretend y is a constant (which implies that f is a function only of x) and differntiate f with respect to x using the normal rules.

Solving Constrained Maximization Problems

In practice, most economic problems can be described as someone maximizing some sort of payoff subject to some constraints. The payoffs typically take the form of a utility or a profit function and the constraints are some sort of budget constraint.

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A typical problem might look like

$$\max_{x,y} u(x,y)$$

s.t.

$$\begin{aligned} p_x x + p_y y &= M \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solving these problems may be difficult, as we have to make sure that we satisfy all of the constraints while also maximizing the function. There are two approaches to solving problems like this: using the Lagrangian function and transforming the problem into an unconstrained one.

The Lagrangian

Using the Lagrangian method involves more steps, but is typically safer and can simplify the algebra when trying to solve for the optimal solution. To begin with, assign a lagrange multiplier to each of the constraints:

$$\max_{x,y} u(x,y)$$

s.t.

$$\begin{aligned} p_x x + p_y y &= M \qquad (\lambda_1) \\ x &\geq 0 \qquad (\lambda_2) \\ y &\geq 0 \qquad (\lambda_3) \end{aligned}$$

And construct the Lagrangian function:

$$\mathcal{L}(x,y,\lambda_1,\lambda_2,\lambda_3) = u(x,y) + \lambda_1(M - p_x x - p_y y) + \lambda_2 x + \lambda_3 y$$

The Lagrangian's first term is equal to the thing we're trying to maximize and subsequent terms are equal to the constraint (with all terms collected on one side of the equality or inequality sign), multiplied by their designated multiplier.

If we are sure that conditions exist such that some of the constraints won't bind at the optimal solution, we can eliminate these constraints. By "won't bind," we mean that the constraints will not actually impact the way in which the agent maximizes his objective function, i.e. don't restrict the agent's behavior. For many of the function we will use in this class, the non-negativity constraints on consumption or leisure (here, x and y) will not bind. If you are unsure, keep those constraints in your Lagrangian. If you are sure that the constraints do not restrict agent behavior, you may remove them. For what follows, I assume that the non-negativity constraints are not binding at the optimum. Now, my Lagrangian is

$$\mathcal{L}(x, y, \lambda_1) = u(x, y) + \lambda_1(M - p_x x - p_y y)$$

I differentiate this with respect to x, y, and λ_1, yielding 3 first order condtions (F.O.C.s):

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} - \lambda_1 p_x &= 0 \\ \frac{\partial u(x,y)}{\partial y} - \lambda_1 p_y &= 0 \\ M - p_x x - p_y y &= 0 \end{aligned}$$

Rearranging these equations, we get:

$$\frac{\partial u(x,y)}{\partial x} = \lambda_1 p_x \quad (1)$$

$$\frac{\partial u(x,y)}{\partial y} = \lambda_1 p_y \quad (2)$$

$$p_x x = M - p_y y \quad (3)$$

Combining (1) and (2) yields:

$$\begin{aligned} \frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} &= \frac{p_x}{p_y} \\ p_x x &= M - p_y y \end{aligned}$$

Finally, if a function is given for $u(x, y)$, you can use the values for $\frac{\partial u(x,y)}{\partial x}$ and $\frac{\partial u(x,y)}{\partial y}$ to solve for x in terms of y. This should then be plugged into the final constaint (4) to solve for x as a function of p_x, p_y, and M. We can then plug this solution back into the relationship between x and y to solve for y as a function of p_x, p_y, and M.

Practice Problem

$$\max_{x,y} x^\theta y^{1-\theta}$$

s.t.

$$\begin{aligned} p_x x + p_y y &= M \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Assume x is the *numeraire* good (meaning that all prices are in terms of x). This implies that $p_x = 1$. Show that the inequality constraints won't bind and solve for x and y in terms of p_y and M.

Make the Problem Unconstrained

Assume again that we can show that the inequality constraints don't bind at the optimum. In the problem above (without a specified function for $u(x, y)$), we know that

$$M = p_x x + p_y y$$

We can rearrange this equation to solve for y in terms of the other variables:

$$y = \frac{M - p_x x}{p_y}$$

We can now solve

$$\max_x u\left(x, \frac{M - p_x x}{p_y}\right)$$

Practice Problem

Again, consider the problem:

$$\max_{x,y} x^\theta y^{1-\theta}$$

s.t.

$$\begin{aligned} p_x x + p_y y &= M \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

And solve for x and y as a function of p_y and M, assuming that $p_x = 1$.

How Do We Know if the Inequality Constraints Bind?

There is a set of conditions that are typically satisfied for the utility and profit functions that we consider which are sufficient for showing that the inequality constraints will not bind. These are referred to as the Inada conditions. They are

$$\begin{aligned} \frac{\partial u(0, y)}{\partial x} &= \infty \\ \frac{\partial u(x, 0)}{\partial y} &= \infty \end{aligned}$$

Sufficiency means that if the condition is satisfied, it is necessarily true that the constraints won't bind. However, this does not imply that the condition ***must*** be satisfied in order for the constraint to be non-binding.

In order to check if the Inada conditions are satisfied, take partial derivatives of the function with respect to both x and y. Then, set the value of x to 0 in the expression for $\frac{\partial u(x,y)}{\partial x}$ and see if the value of $\frac{\partial u(0,y)}{\partial x} = \infty$. Do the same for the partial derivative with respect to y.

Practice

Consider the following representative consumer's problem:

$$\max_{c,l} u(c,l)$$

s.t.

$$\begin{aligned} c &= w(h-l) + \Pi - T \\ c &\geq 0 \\ h &\geq l \geq 0 \end{aligned}$$

where c is consumption, l is leisure, h is the total hours available in the day, Π is profits rebated to the households, and T is lump sum taxes. Consider the following utility functions:

$$u(c,l) = \alpha c + \beta l \quad (4)$$
$$u(c,l) = log(c) + \gamma log(l) \quad (5)$$

and answer the following questions for each

1. Are the inada conditions satisfied?
2. Given your answer to 1, choose the solution method that best fits the problem.
3. What is the numeraire?
4. Solve for c and l in terms of the other variables.