

An Introduction to Cross Sections

1. Definition of cross section for scattering or reactions

Let $N_{incident}$ = number of incident (beam) particles

N_{events} = number of events (beam-target interactions)

$$n = \text{target atoms per unit volume} = \frac{\rho N_{Avogadro}}{A}$$

A = mass number of target (assuming a single pure isotope)

ρ = mass density of target (g/cm^3)

x = thickness of target (cm)

ρx = areal density of target (g/cm^2)

$$nx = \text{areal number density (atoms/cm}^2) = \frac{\rho x N_{Avogadro}}{A}$$

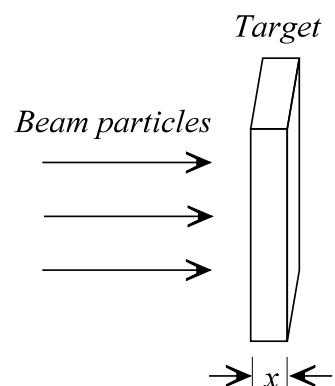


Figure 1

If we assume that (a) the probability of interaction depends on the properties of the beam and target particles, and (b) the target is “thin”, so that only a small fraction of beam particles actually interact, then the following scaling rules must apply:

- (1) The number of interactions is proportional to the number of incident particles.
- (2) The number of interactions is proportional to the thickness x of the target.
- (3) The number of interactions is proportional to the density ρ of the target.

If rules (2) and (3) are not obvious, note that the number of atoms which fall within a given distance of a traversing beam particle is proportional to the product $nx = \text{constant} \times \rho x$.

Thus we can write

$$N_{events} = \text{Constant} \times N_{incident} \times nx \times \sigma \quad (1.1)$$

where σ is a “strength” parameter which depends only on the properties of the beam and target particle and the particular type of event which is being observed. Setting the constant to 1, we obtain

$$N_{events} = N_{incident} nx \left(\frac{\text{atoms}}{\text{cm}^2} \right) \sigma = N_{incident} \frac{\rho (\text{g/cm}^3) x (\text{cm}) N_{Avogadro} (\text{atoms/mole})}{A (\text{g/mole})} \sigma . \quad (1.2)$$

The strength parameter σ thus has dimensions of area (or area per atom), so we give it the name “cross section”. Note that we have not identified any physical area represented by the quantity σ . In quantum physics, no such identification is possible. The cross section for a particular process is **defined** by Equation (1.2):

$$\sigma (\text{cm}^2) \equiv \frac{N_{events}}{N_{incident} nx (\text{atoms/cm}^2)} = \frac{N_{events} A}{N_{incident} \rho x N_{Avogadro}} . \quad (1.3)$$

An equivalent definition, used mainly by theorists, is

$$\sigma (\text{cm}^2) = \frac{N_{\text{events}}}{(\text{beam particles per unit area}) \times (\text{target particles})} . \quad (1.4)$$

This definition would be useful if the flux of beam particles were uniform over the full width of the beam, which is seldom the case, while Eq. (1.3) presupposes that the target density and thickness are uniform over the target area hit by the beam, a much easier condition to achieve.

Another way of defining the cross section is in terms of a quantity called the **luminosity** \mathcal{L} , where

$$\text{Events per unit time} = \mathcal{L} \sigma . \quad (1.5)$$

Since the cross section has dimensions of area, the luminosity has dimensions ($\text{area}^{-1} \text{time}^{-1}$). Luminosity is mainly used for describing colliding-beam experiments (see Povh Eq. (4.11)), but also applies to fixed-target experiments like those described above. Povh Eq. (4.10) says

$$\mathcal{L} = \frac{\text{incident particles}}{\text{area} \cdot \text{time}} \cdot \text{target particles} = \frac{\text{incident particles}}{\text{time}} \cdot \frac{\text{target particles}}{\text{area}} . \quad (1.6)$$

Taken together, Eqs. (1.5) and (1.6) can be seen to be equivalent to (1.3) and (1.4).

2. Types of cross section

If N_{events} in Eq. (1.3) refers to *any* kind of interaction of the beam with the target, then σ is called the **total cross section**. It is also possible to restrict N_{events} to events of a particular type, in which case we define **partial cross sections** such as σ_{elastic} , $\sigma_{\text{inelastic}}$, σ_{fission} , $\sigma_{\pi \text{ production}}$, ...

It is also possible to restrict N_{events} to the case in which an outgoing particle goes into a particular range of angles in space, e.g. into a particular detector of finite size. In this case, we define the **differential cross section** $\frac{d\sigma}{d\Omega}$, such that

$$N_{\text{events into solid angle } \Delta\Omega} = N_{\text{incident}} n_x \frac{d\sigma}{d\Omega} \Delta\Omega . \quad (2.1)$$

The quantity $d\Omega = \sin \theta d\theta d\phi$ is called an infinitesimal element of **solid angle**, measured in steradians (sr). The differential cross section is typically a function of angles θ and ϕ . Eq. (2.1) assumes that $\frac{d\sigma}{d\Omega}$ does not vary appreciably over the angles subtended by of the detector; if it does, then the right-hand side must contain an integral over the angular acceptance.

The term “total cross section” can be ambiguous in meaning. Sometimes it refers to a sum over all processes, as defined in the first paragraph of this section, but one can also speak of the **total cross section for a particular process**, which means the integral over all angles of the differential cross section for that process. Be alert!

3. Approximate classical calculation of Coulomb (Rutherford) scattering

Sometimes (e.g. the Bohr model of H atom) an invalid classical calculation accidentally gives a correct result. This turns out to be true for the Coulomb scattering of spinless charged particles at low energies, as in Rutherford's early experiments on the scattering of α particles from heavy nuclei.

Consider the elastic scattering of two point particles, where the beam particle has charge ze , mass m , and incident velocity v , and the target particle has charge Ze and mass $M \gg m$ (and thus can be considered to remain at rest after the scattering.) If we treat this as a central-force problem in classical mechanics, we know that the actual trajectory is a hyperbola. This calculation of the exact differential scattering cross section is worked out in many classical mechanics texts (see also Williams Sec. 1.2). It gives a result which is identical to the result derived using the Born approximation in non-relativistic quantum mechanics (see Povh Section 5.2), which it turns out is *also* an invalid derivation for entirely different reasons.

In this note I want to work out a simple **approximate** classical derivation using only very elementary physics ideas. To simplify, I will consider only the case of scattering through small angles ($\theta \ll 1$ rad), but will generalize the usual calculation to allow relativistic velocities.

Figure 2 defines the variables in the problem. A particle of charge ze , mass m and momentum $\mathbf{p} = m\mathbf{v}$ is incident on a target particle of charge Ze and mass M . The **impact parameter** b is defined as the minimum distance between the incident particle and the target particle if there were no deflection.

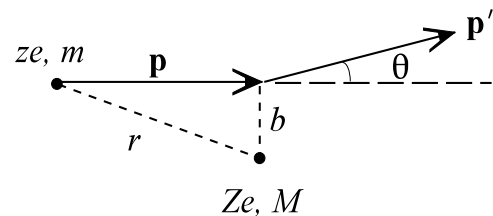


Figure 2: Definitions of variables

Assuming the target particle remains at rest, the Coulomb force on the beam particle is given by

$$F = \frac{\alpha \hbar c z Z}{r^2} = \frac{\alpha z Z}{r^2} \quad (\text{natural units}) \quad (3.1)$$

This force imparts an impulse (momentum transfer) to the beam particle which, for small deflection angles, is nearly perpendicular to the incident momentum \mathbf{p} . This impulse is dominated by the region of closest approach, where $r \approx b$, and this region is of length $\Delta x \approx 2b$, corresponding to a time interval $\Delta t \approx 2b/v$:

$$\Delta p_{\perp} = \bar{F}_{\perp} \Delta t \approx \frac{z Z \alpha}{b^2} \frac{2b}{v} \approx \frac{2z Z \alpha}{bv} \quad (\text{non-relativistic}). \quad (3.2)$$

Eq. (3.2) has been calculated assuming non-relativistic velocities. It can be made relativistically correct by including two effects of the Lorentz transformation:

1) In the frame of the moving particle, the electric field of the target particle in the transverse direction is increased by a factor $\gamma = \sqrt{1 - (v/c)^2}$.

2) The electric field is compressed by the same factor γ into a smaller region along the direction of particle motion, so that the effective interaction time is $\Delta t \approx 2b/v\gamma$ instead of $\Delta t \approx 2b/v$. Thus Eq. (3.2) can be rewritten

$$\Delta p_{\perp} = \bar{F}_{\perp} \Delta t \approx \frac{\alpha z Z [\gamma]}{b^2} \frac{2b}{v[\gamma]} \approx \frac{2zZ\alpha}{bv} \quad (\text{approximate, relativistic}). \quad (3.3)$$

The two factors of γ cancel out, so that the result is identical to (3.2).

The scattering angle can now be calculated as

$$\theta \approx \frac{\Delta p_{\perp}}{p} \approx \frac{2zZ\alpha}{bv} \frac{1}{p} = \frac{2zZ\alpha}{pvb} \quad (\text{approximate, relativistic}), \quad (3.4)$$

which agrees with the exact non-relativistic classical calculation (using $2E_k = mv^2 = pv$)

$$\tan\left(\frac{\theta}{2}\right) = \frac{zZ\alpha}{2bE_k} = \frac{zZ\alpha}{pvb} \quad (\text{exact, non-relativistic}). \quad (3.5)$$

(Note that, for small angles, $\tan\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$.)

We now want to calculate the differential probability of scattering by an angle θ , which (at least in the classical picture) is related to the impact parameter b by Eq. (3.4). Since the impact parameter is far too small to observe directly, we assume that the incident particles are distributed randomly across a thin target which contains nx nuclei per cm^2 . For each nucleus, the probability of scattering by an angle between θ and $\theta+d\theta$ is equal to the probability of the incident particle having an impact parameter between b and $b+db$, and is given by the expression

$$\frac{dP}{d\theta} d\theta = \frac{dP}{db} db = 2\pi b db \cdot nx = (\text{area per nucleus})(\text{nuclei}/\text{cm}^2). \quad (3.6)$$

Using (3.4) we can write

$$\frac{dP}{d\theta} = \frac{dP}{db} \frac{db}{d\theta} = 2\pi b nx \frac{db}{d\theta} = 2\pi nx \frac{2zZ\alpha}{pv\theta} \frac{2zZ\alpha}{pv\theta^2} = 2\pi nx \left(\frac{2zZ\alpha}{pv}\right)^2 \frac{1}{\theta^3}. \quad (3.7)$$

It is traditional to express scattering results in terms of a **differential cross section**

$$\frac{d\sigma}{d\Omega} = \frac{\text{events into solid angle } d\Omega \text{ at } (\theta, \phi)}{d\Omega \cdot (\text{incident particles}) \cdot (\text{target nuclei}/\text{cm}^2)}, \quad (3.8)$$

where $d\Omega = \sin\theta d\theta d\phi$ is an element of **solid angle**, measured in units of steradians (sr).

When there is no dependence on the azimuthal angle ϕ (as in the present case), we can integrate over azimuth and write $d\Omega = 2\pi \sin\theta d\theta$. Then

$$nx \frac{d\sigma}{d\Omega} d\Omega = nx \frac{d\sigma}{d\Omega} 2\pi \sin\theta d\theta = \frac{dP}{d\theta} d\theta. \quad (3.9)$$

In small-angle approximation ($\sin\theta \approx \theta$), Eqs. (3.7) and (3.9) combine to give

$$\frac{d\sigma}{d\Omega} = \frac{1}{nx 2\pi \theta} \frac{dP}{d\theta} d\theta \approx \left(\frac{2zZ\alpha}{pv}\right)^2 \frac{1}{\theta^4} \quad (\text{approximate, relativistic}). \quad (3.10)$$

For comparison, the exact classical non-relativistic calculation (see Povh (5.16)) gives

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{zZ\alpha}{4E_k} \right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \quad (\text{exact, non-relativistic}) \\ &= \left(\frac{zZ\alpha}{2pv} \right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \quad (\text{exact, relativistic or non-relativistic}) \end{aligned} \quad (3.11)$$

which agrees with (10) in the small-angle limit.

Since the derivation was carried out in natural (energy) units, Eq. (3.10) has dimensions of $\text{MeV}^{-2} \text{sr}^{-1}$. To convert to units of area, we must multiply by $(\hbar c)^2 = (197.33 \text{ MeV}\cdot\text{fm})^2$ to express the differential cross section in units of $\text{fm}^2/\text{sr} = 10^{-26} \text{ cm}^2/\text{sr} = 10 \text{ mb}/\text{sr}$, the latter using the annoying but traditional unit of 1 barn ($1 \text{ b} = 10^{-28} \text{ m}^2 = 10^{-24} \text{ cm}^2 = 100 \text{ fm}^2$).

The above calculations are independent of the relative signs of the two charges. (In the exact calculation, the trajectories corresponding to attractive and repulsive scattering forces are both hyperbolas with one focus in common.) Thus the result which Rutherford derived for α -particle scattering from nuclei can be used (with some caution and some important modifications – see Povh Sec. 5.3) as the basis for calculating the scattering of electrons from nuclei.

The classical calculation should be valid when the two charged particles are very far apart, but when the separation becomes comparable with the DeBroglie wavelength, then the uncertainty principle tells us that the uncertainty in momentum is larger than the momentum itself, and concepts of “impact parameter” and “particle orbit” are no longer meaningful. The fact that the classical calculation described here works for Rutherford’s large scattering angles (hence small impact parameters) is a happy accident. An even happier accident is that the same result can be derived using non-relativistic quantum scattering theory (see Povh Sec. 5.2), and this result can be generalized to relativistic velocities and to finite nuclear size. The only legitimate way to do the calculation is in relativistic quantum mechanics based on the Dirac equation. In such a calculation of electron scattering, the electron spin is an essential ingredient, leading to a result (Mott cross section, Povh Sec. 5.3) which differs substantially from the Rutherford formula but in a well-behaved way which preserves many results derived in the non-relativistic formalism.