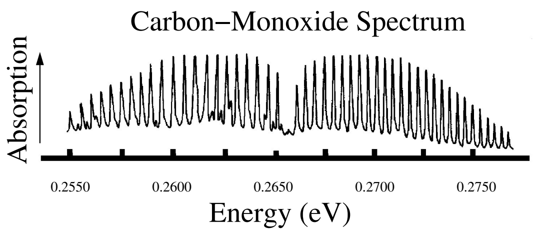
Physics 309 Final

I pledge that I have given nor received unauthorized assistance during the completion of this work.

Signature: ______________________________

Questions (3 pts. apiece) Answer questions in complete, well-written sentences WITHIN the spaces provided. For multiple-choice questions circle the correct answer.

1. Recall the vibration-rotation spectrum of carbon monoxide shown in the figure. There is a high-energy 'lobe' with peaks separated by constant energy $\hbar^2/I$ and a similar low-energy lobe. The lobes are separated by a gap at 0.2660 eV. What transitions create the separate lobes?



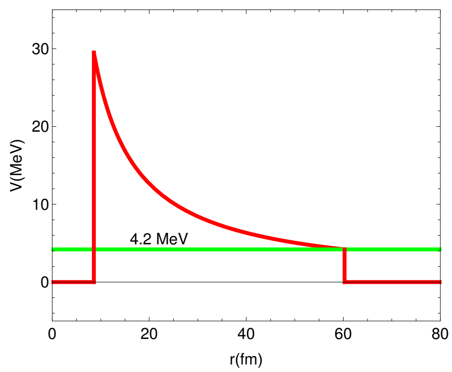
2. Consider the one-dimensional, time-dependent Schroedinger equation (SE) with a potential that depends only on x. What are the steps to solve the SE using the separation of variables method? Don't actually perform the steps, but describe them.

3. The Schroedinger equation in spherical coordinates is shown below. What is the operator form of the square of the angular momentum L^2? Explain your reasoning.

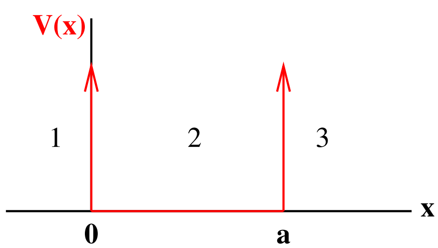
$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi+V(r)\psi=E\psi$$

Do not write below this line.

4. The figure below shows the energy diagram we used to calculate alpha decay. What would happen to the calculation of the lifetime if the nuclear radius (*i.e.* $R = 1.4A^{1/3}$) is decreased? Explain.



5. When we solved the particle-in-a-box (infinitely deep potential well, see the figure) we required the wave function at the potential well's boundaries to be continuous, *e.g.* $\phi_1(x = 0) = \phi_2(x = 0)$ and $\phi_2(x = a) = \phi_3(x = a)$ where a is the width of the well and the subscript refers to the different regions labeled in the figure. Why? How did we justify this requirement?



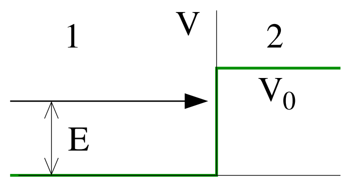
6. Why do we express the wave function in terms of energy eigenstates?

7. List one experimental result that led to the development of quantum mechanics. Why was that result important?

Do not write below this line.

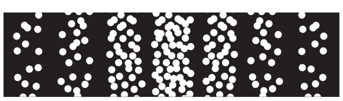
8. A particle of energy $E < V_0$ is incident on a step potential of height V_0 as shown in the figure. Let $k_1 = \sqrt{2mE}/\hbar$ and $k_2 = \sqrt{2m(E - V_0)}/\hbar$ where the subscripts refer to the region labeled in the figure. Circle your choice for the transmission coefficient from the list below. Explain your choice.

(a)	1	(d)	$4k_1^2/(k_1^2 + k_2^2)$
(b)	0	(e)	k_1/k_2
(c)	k_1^2/k_2^2	f	g



9. The Heisenberg Uncertainty Principle in one dimension $\Delta x \Delta p \geq \hbar/2$ relates uncertainty in momentum Δp and position Δx. Why these two quantities?

10. The figure is a simulation of the electrons detected behind two closely spaced slits. Each bright dot represents one electron. How will this pattern change if the left slit is closed? Your answers should consider the number of dots on the screen and the spacing, width, and positions of the fringes.



Problems. Clearly show all work for full credit on a separate piece of paper.

1. (10 pts.) The general solution to the classical harmonic oscillator is $x(t) = A\sin(\omega_0 t + \delta)$. Get an expression for the period of the motion (the time to make one complete oscillation) in terms of the parameters of the general solution. How is this result related to the frequency?

2. (10 pts.) Consider the following functions defined over the interval $(-a/2, a/2)$.

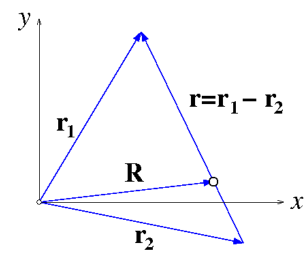
$$\phi_k = \frac{1}{\sqrt{a}} e^{ikx}$$

Show these functions form an orthogonal set in the limit $a \to \infty$.

3. (10 pts.) In studying rotational motion, we take advantage of the center-of-mass system to make life easier. Consider the two-particle system shown in the figure including the center-of-mass vector $\mathbf{R_{cm}}$. For convenience we will place our origin at the center-of-mass of the system ($\mathbf{R_{cm}} = \mathbf{0}$). Show the classical mechanical energy of the two-particle system in the center-of-mass frame can be written as

$$E_{cm} = \frac{1}{2}\mu v^2 + V(r) \qquad \text{where} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \text{and} \qquad v = \frac{dr}{dt}$$

and r is the relative coordinate between the two particles as shown in the figure. Notice that $V(r)$ depends only on the relative coordinate.



4. (10 pts.) Two hundred anyons are in a one-dimensional box with walls at $x_0 = 0$ and $x_1 = a$. At $t = 0$, the state of each particle is the following.

$$\psi(x, 0) = Ax^2(x - a) \quad \text{where} \quad A = \sqrt{\frac{105}{a^7}}$$

The eigenfunctions and eigenvalues are

$$E_n = \frac{n^2\hbar^2\pi^2}{2ma^2} \qquad \begin{aligned} \phi_n &= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad && 0 \le x \le a \\ &= 0 && x < 0 \text{ and } x > a \end{aligned} \quad .$$

How many particles have energy E_2 at $t = 0$?

5. (15 pts.) Consider Legendre's associated differential equation shown below.

$$(1-z^2)\frac{d^2\Theta}{dz^2} - 2z\frac{d\Theta}{dz} + \left(A - \frac{m^2}{1-z^2}\right)\Theta = 0$$

For the case $m = 0$ what is the recursion relationship for the series solution to this differential equation? In other words, let $\Theta = \sum a_k z^k$ and set $m = 0$. Clearly justify all your reasoning.

6. (15 pts.) A particle beam has a continuous wave function that can be described by

$$\psi(x,t) = e^{i(k_0 x - \omega t)} \qquad .$$

This equation describes a wave train moving in the positive x direction. A beam 'pulse' of length L is produced by sending the beam through a 'chopper' that opens long enough to let part of the original beam through and then closes again, cutting off the remainder. The wave function of the pulse at time $t = 0$ is the following.

$$\begin{aligned} \psi(x,0) &= \frac{1}{\sqrt{L}} e^{ik_0 x} & \qquad |x| \le L/2 \\ &= 0 & \qquad |x| > L/2 \end{aligned}$$

The free particle eigenfunction is the following.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

(a) What is the spectral distribution necessary to produce such a wave packet?

(b) Generate an uncertainty principle appropriate for this wave packet.

Physics 309 Equations and Constants

$$E = h\nu = \hbar\omega \qquad v_{wave} = \lambda\nu \qquad I \propto |\vec{E}|^2 \qquad \lambda = \frac{h}{p} \qquad p = \hbar k \qquad K = \frac{p^2}{2m}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \quad \hat{p}_{\ x} = -i\hbar\frac{\partial}{\partial x} \quad \hat{A}\ |\phi\rangle = a|\phi\rangle \quad \langle \hat{A}\ \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A}\ \psi dx$$

$$\langle \phi_{n'}|\phi_n\rangle = \int_{-\infty}^{\infty} \phi^*_{n'}\phi_n dx = \delta_{n',n} \quad \langle \phi(k')|\phi(k)\rangle = \int_{-\infty}^{\infty} \phi^*_{k'}\phi_k\ dx = \delta(k-k') \quad e^{i\phi} = \cos\phi + i\sin\phi$$

$$|\psi\rangle = \sum b_n|\phi_n\rangle \rightarrow b_n = \langle\phi_n|\psi\rangle \qquad |\phi\rangle = e^{\pm ikx} \qquad |\psi\rangle = \int b(k)|\phi(k)\rangle dk \rightarrow b(k) = \langle\phi(k)|\psi\rangle$$

$$|\psi(t)\rangle = \sum b_n|\phi_n\rangle e^{-i\omega_n t} \quad |\psi(t)\rangle = \int b(k)|\phi(k)\rangle e^{-i\omega(k)t}dk \quad \Delta p\Delta x \geq \frac{\hbar}{2} \qquad (\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$$

The wave function, $\Psi(\vec{r},t)$, contains all we know of a system and its square is the probability of finding the system in the region $\vec{r}$ to $\vec{r}+d\vec{r}$. The wave function and its derivative are (1) *finite*, (2) *continuous*, and (3) *single-valued* ($\psi_1(a) = \psi_2(a)$ and $\psi_1'(a) = \psi_2'(a)$) .

$$V_{HO} = \frac{\kappa x^2}{2} \quad \omega = 2\pi\nu = \sqrt{\frac{\kappa}{m}} \quad E_n = (n+\frac{1}{2})\hbar\omega_0 = \hbar\omega \quad |\phi_n\rangle = e^{-\xi^2/2}H_n(\xi) \quad \xi = \beta x \quad \beta^2 = \frac{m\omega_0}{\hbar}$$

$$\psi_1 = \mathbf{t}\psi_3 = \mathbf{d_{12}p_2d_{21}p_1^{-1}}\psi_3 \qquad T = \frac{1}{|t_{11}|^2} \qquad R+T = 1 \quad T_{WKB} = \exp\left[-2\int_{x_0}^{x_1}\sqrt{\frac{2m(V(x)-E)}{\hbar^2}}\ dx\right]$$

$$E = \frac{\hbar^2k^2}{2m} \quad k = \sqrt{\frac{2m(E-V)}{\hbar^2}} \quad T = \frac{\text{transmitted flux}}{\text{incident flux}} \quad R = \frac{\text{reflected flux}}{\text{incident flux}} \quad \text{flux} = |\psi|^2 v$$

$$V(r) = \frac{Z_1Z_2e^2}{r} \quad \frac{d\sigma}{d\Omega} = \left(\frac{Z_1Z_2e^2}{4E}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \quad E = \frac{1}{2}\mu v^2 + V(r) \quad \vec{R}_{cm} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \quad \mu = \frac{m_1m_2}{m_1+m_2}$$

$$\psi(x) = \sum_{n=1}^{\infty} a_n x^n \quad \langle K\rangle = \frac{3}{2}kT \qquad n(v) = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-mv^2/2k_BT} \quad \vec{L} = \vec{r}\times\vec{p} = I\vec{\omega}$$

$$I = \sum_i m_i r_1^2 = \int r^2 dm \quad KE_{rot} = \frac{L^2}{2I} \quad E_\ell = \frac{\ell(\ell+1)\hbar^2}{2I} \quad V_{coul} = \frac{Z_1Z_2e^2}{r} \quad ME = \frac{p_r^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r)$$

$$L_z|nlm\rangle = m\hbar|nlm\rangle \quad L^2|nlm\rangle = \ell(\ell+1)\hbar^2|nlm\rangle$$

$$\mathbf{d_{ij}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_j}{k_i} & 1-\frac{k_j}{k_i} \\ 1-\frac{k_j}{k_i} & 1+\frac{k_j}{k_i}\end{pmatrix} \quad \mathbf{p_i} = \begin{pmatrix} e^{-ik_i2a} & 0 \\ 0 & e^{ik_i2a}\end{pmatrix} \quad \mathbf{p_i^{-1}} = \begin{pmatrix} e^{ik_i2a} & 0 \\ 0 & e^{-ik_i2a}\end{pmatrix}$$

Speed of light (c)	$2.9979 \times 10^8\ m/s$	fermi (fm)	$10^{-15}\ m$
Boltzmann constant (k_B)	$1.381 \times 10^{-23}\ J/K$	angstrom (Å)	$10^{-10}\ m$
	$8.62 \times 10^{-5}\ eV/k$	electron-volt (eV)	$1.6 \times 10^{-19}\ J$
Planck constant (h)	$6.621 \times 10^{-34}\ J-s$	MeV	$10^6\ eV$
	$4.1357 \times 10^{-15}\ eV-s$	GeV	$10^9\ eV$
Planck constant ($\hbar$)	$1.0546 \times 10^{-34}\ J-s$	Electron charge (e)	$1.6 \times 10^{-19}\ C$
	$6.5821 \times 10^{-16}\ eV-s$	e^2	$\hbar c/137$
Planck constant ($\hbar c$)	$197\ MeV-fm$	Electron mass (m_e)	$9.11 \times 10^{-31}\ kg$
	$1970\ eV-$Å		$0.511\ MeV/c^2$
Proton mass (m_p)	$1.67 \times 10^{-27} kg$	atomic mass unit (u)	$1.66 \times 10^{-27}\ kg$
	$938\ MeV/c^2$		$931.5\ MeV/c^2$
Neutron mass (m_n)	$1.68 \times 10^{-27}\ kg$		
	$939\ MeV/c^2$		

$$\frac{df}{du} = \frac{df}{dx}\frac{du}{dx} \quad \frac{d}{dx}(x^n) = nx^{n-1} \quad \frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x \quad \frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\ln ax) = \frac{1}{x} \quad \int x^n dx = \frac{x^{n+1}}{n+1} \quad \int e^{ax} dx = \frac{e^{ax}}{a} \quad \int \frac{1}{x} = \ln x \quad \int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left[x + \sqrt{x^2+a^2}\right]$$

$$\int \frac{x}{\sqrt{x^2+a^2}} dx = \sqrt{x^2+a^2} \quad \int \frac{x^2}{\sqrt{x^2+a^2}} dx = \frac{1}{2} x\sqrt{x^2+a^2} - \frac{1}{2} a^2 \ln\left[x + \sqrt{x^2+a^2}\right]$$

$$\int \frac{x^3}{\sqrt{x^2+a^2}} dx = \frac{1}{3}(-2a^2 + x^2)\sqrt{x^2+a^2}$$

Hermite polynomials ($H_n(\xi)$)

$$H_0(\xi) = \frac{1}{\sqrt{\sqrt{\pi}}} \qquad H_3(\xi) = \frac{1}{\sqrt{48\sqrt{\pi}}}(8\xi^3 - 12\xi)$$

$$H_1(\xi) = \frac{1}{\sqrt{2\sqrt{\pi}}} 2\xi \qquad H_4(\xi) = \frac{1}{\sqrt{384\sqrt{\pi}}}(16\xi^4 - 48\xi^2 + 12)$$

$$H_2(\xi) = \frac{1}{\sqrt{8\sqrt{\pi}}}(4\xi^2 - 2) \qquad H_5(\xi) = \frac{1}{\sqrt{3840\sqrt{\pi}}}(32\xi^5 - 160\xi^3 + 120\xi)$$