The Quantum Program in One Dimension - So Far

1. Solve the Schroedinger equation to get eigenfunctions and eigenvalues.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x)}{\partial x^2} + V\phi(x) = E\phi(x)$$

2. For an initial wave packet $\psi(x)$ use the completeness of the eigenfunctions.

$$|\psi(x)\rangle = \sum_{n=1}^{\infty} b_n |\phi(x)_n\rangle$$

3. Apply the orthonormality $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$.

$$\langle\phi_m|\psi\rangle = \langle\phi_m|\left(\sum_{n=1}^{\infty} b_n|\phi_n\rangle\right) = b_m = \int_{-\infty}^{\infty} \phi_m^* \left(\sum_{n=1}^{\infty} b_n|\phi_n\rangle\right) dx$$

4. Get the probability P_n for measuring E_n from $|\psi\rangle$. of $|\psi\rangle$.

$$P_n = |b_n|^2$$

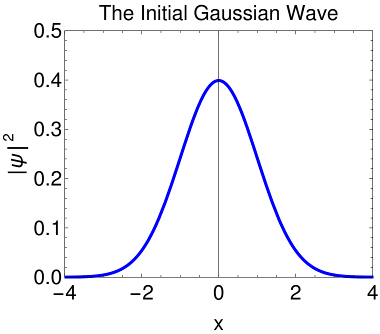
5. Do the free particle solution.
6. Put in the time evolution.

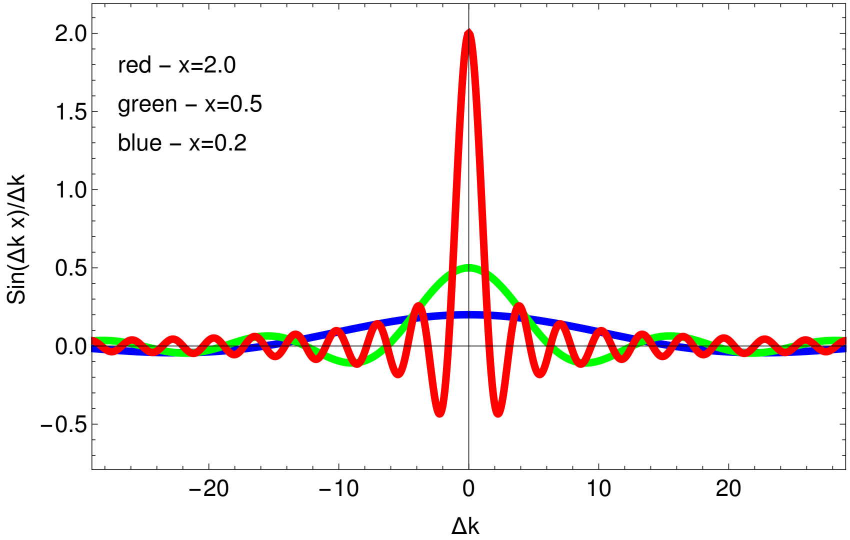
The Free Particle Problem

Consider a free particle ($V = 0$) which has an initial wave packet that is described by a gaussian function.

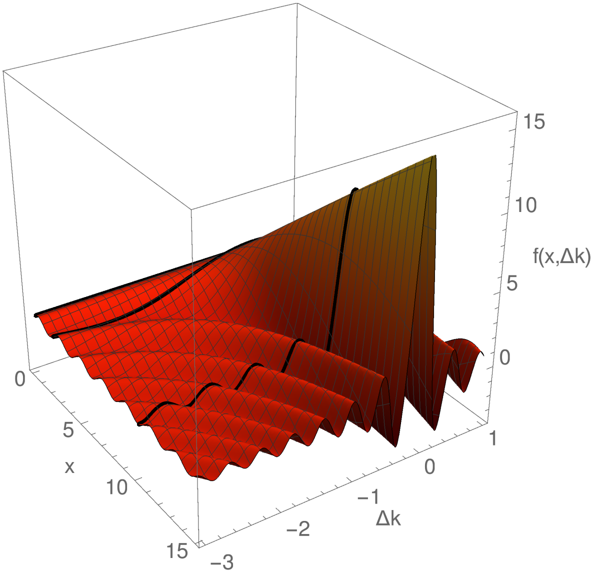
$$|\Psi(x,0)\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-x^2/4\sigma^2}$$

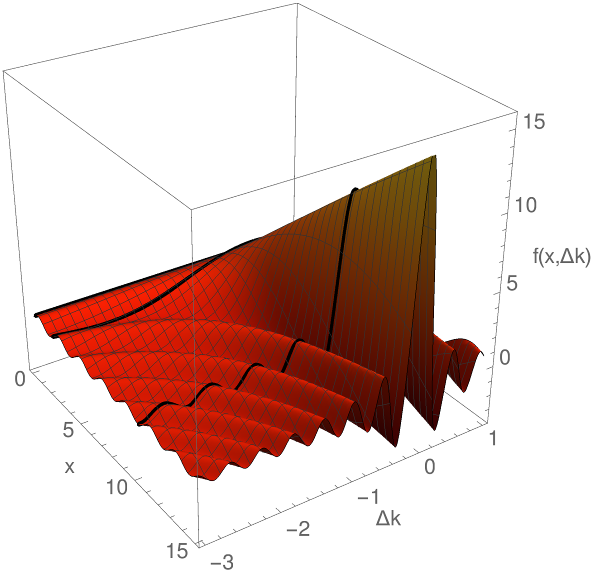
What is the spectrum of momenta that form this wave packet? How wide is that distribution?





red – x=2.0
green – x=0.5
blue – x=0.2
Sin(Δk x)/Δk
Δk
2.0
1.5
1.0
0.5
0.0
−0.5
−20
−10
0
10
20

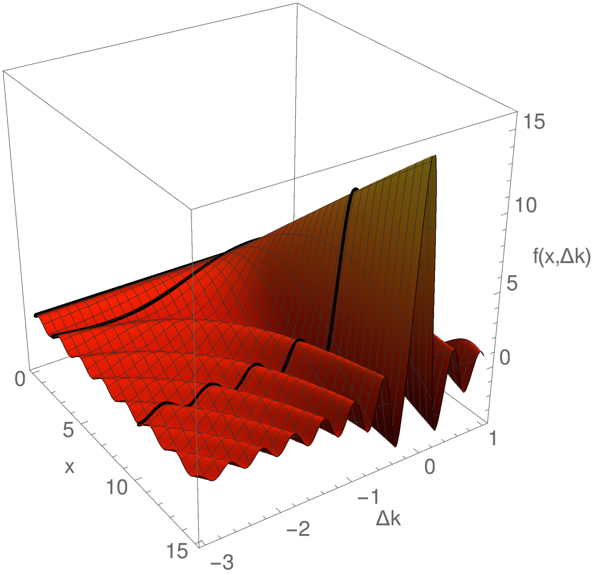

15
10
f(x,Δk)
5
0
0
5
x
10
15
1
0
−1
Δk
−2
−3



$$\int_{-\Delta k_{max}}^{\Delta k_{max}} \lim_{x\to\infty} \frac{\sin(\Delta k\, x)}{\Delta k}\, d(\Delta k)$$

x	Δk_{max}	Integral
0.01	10000	3.12445
1.0	10000	3.14178
2.0	10000	3.14151
4.0	10000	3.14158
10.0	10000	3.14161
100.0	10000	3.14159
1000.0	10000	3.14159
10000.0	10000	3.14159
100000.0	10000	3.14159

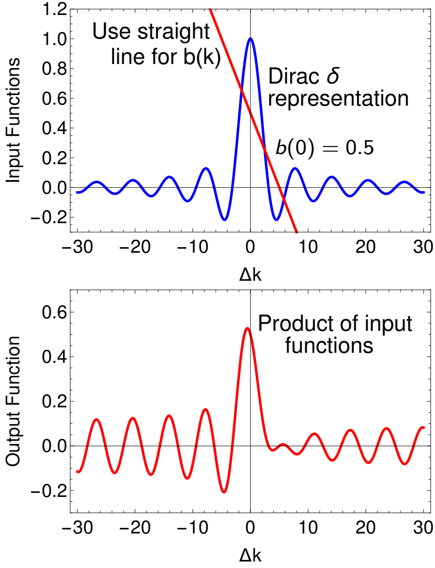
The Dirac Delta Function



$$\int_{-\Delta k_{max}}^{\Delta k_{max}} \lim_{x\to\infty} \frac{\sin(\Delta k\, x)}{\Delta k} d(\Delta k)$$

x	Δk_{max}	Integral
0.01	10000	3.12445
1.0	10000	3.14178
2.0	10000	3.14151
4.0	10000	3.14158
10.0	10000	3.14161
100.0	10000	3.14159
1000.0	10000	3.14159
10000.0	10000	3.14159
100000.0	10000	3.14159

Dirac Delta Function Demonstration



$$\int_{-\Delta k_{max}}^{\Delta k_{max}} 2b(k) \lim_{x\to\infty} \frac{\sin(\Delta k\, x)}{\Delta k} d(\Delta k) =$$

$$2b(k') \int_{-\Delta k_{max}}^{\Delta k_{max}} \lim_{x\to\infty} \frac{\sin(\Delta k\, x)}{\Delta k} d(\Delta k)$$

$$b(\Delta k = 0) = 0.5$$

x on $l.h.s.$	Δk_{max}	$l.h.s$	$r.h.s$
0.01	1000	3.31670	3.14159
1.0	1000	3.14047	3.14159
2.0	1000	3.14196	3.14159
10.0	1000	3.14178	3.14159
100.0	1000	3.14161	3.14159
1000.0	1000	3.14159	3.14159
10000.0	1000	3.14159	3.14159
100000.0	1000	3.14159	3.14159

Comparison of Bound and Free Particles

Particle in a Box

The potential

$$\begin{aligned} V =& 0 \qquad 0 < x < a \\ =& \infty \qquad \textit{otherwise} \end{aligned}$$

Eigenfunctions and eigenvalues

$$|\phi_n\rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad E_n = n^2 \frac{\hbar^2 \pi^2}{2ma^2}$$

Superposition

$$|\psi\rangle = \sum_{n=1}^{\infty} b_n |\phi_n\rangle \qquad \langle \phi_m | \phi_n \rangle = \delta_{m,n}$$

Getting the coefficients

$$b_n = \langle \phi_n | \psi \rangle \qquad P_n = |b_n|^2$$

Free Particle

The potential

$$V = 0$$

Eigenfunctions and eigenvalues

$$|\phi(k)\rangle = \frac{1}{\sqrt{2\pi}} e^{\pm ikx} \qquad E = \frac{\hbar^2 k^2}{2m}$$

Superposition

$$\begin{aligned} |\psi\rangle =& \int_{-\infty}^{\infty} b(k)\phi(k) dk \\ \langle \phi(k') | \phi(k) \rangle =& \delta(k - k') \end{aligned}$$

Getting the coefficients

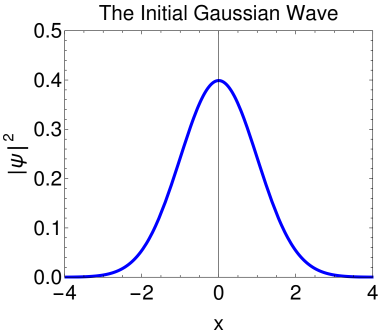
$$b(k) = \langle \phi(k) | \psi \rangle \qquad P(k)dk = |b(k)|^2 dk$$

The Free Particle Problem

Consider a free particle ($V = 0$) which has an initial wave packet that is described by a gaussian function.

$$|\Psi(x,0)\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-x^2/4\sigma^2}$$

What is the spectrum of momenta that form this wave packet? How wide is that distribution?



From The Homework (3.10)

(3.37) $$\psi(x,t) = A \exp\left[\frac{-(x-x_0)^2}{4a^2}\right] \exp\left(\frac{ip_0 x}{\hbar}\right) \exp(i\omega_0 t)$$

3.10 For the state ψ, given by (3.37), show that

$$(\Delta x)^2 = a^2$$

Argue the consistency of this conclusion with the change in shape that $|\psi|^2$ suffers with a change in the parameter a.

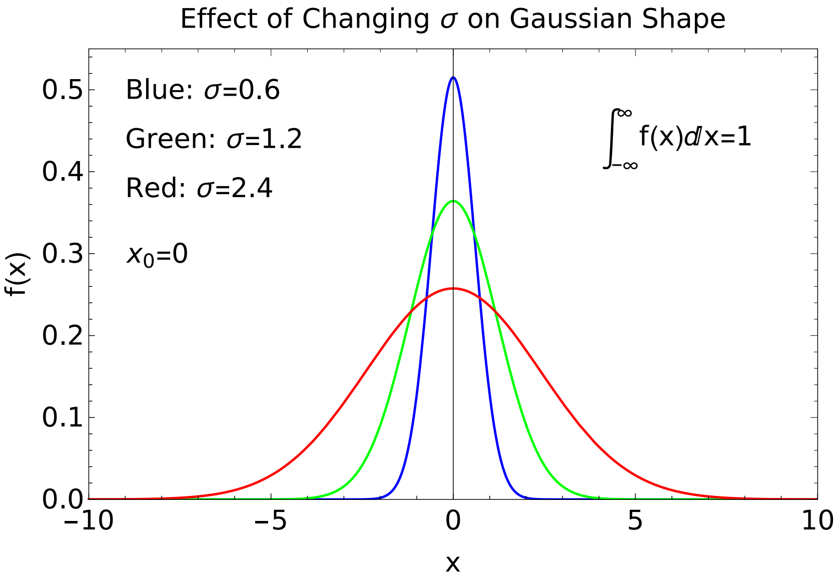
In the solution to 3.10

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 \quad \text{and} \quad \langle x^2\rangle = a^2 + x_0^2 \quad \text{and} \quad \langle x\rangle^2 = x_0^2$$

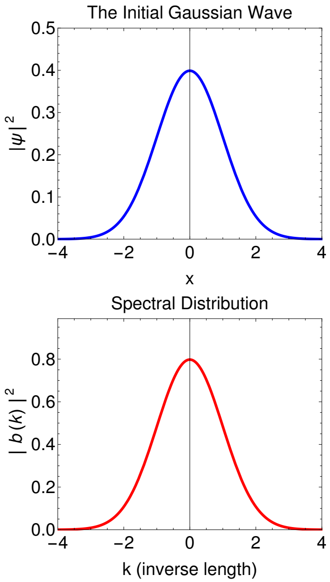
so

$$(\Delta x)^2 = a^2 + x_0^2 - x_0^2 = a^2$$

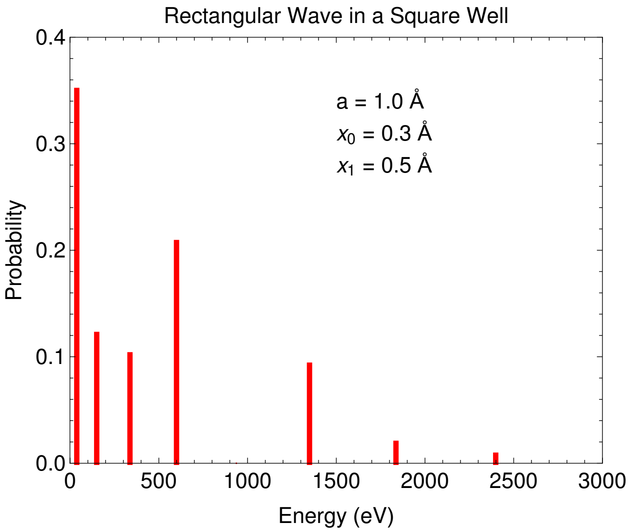
From The Homework (3.10)



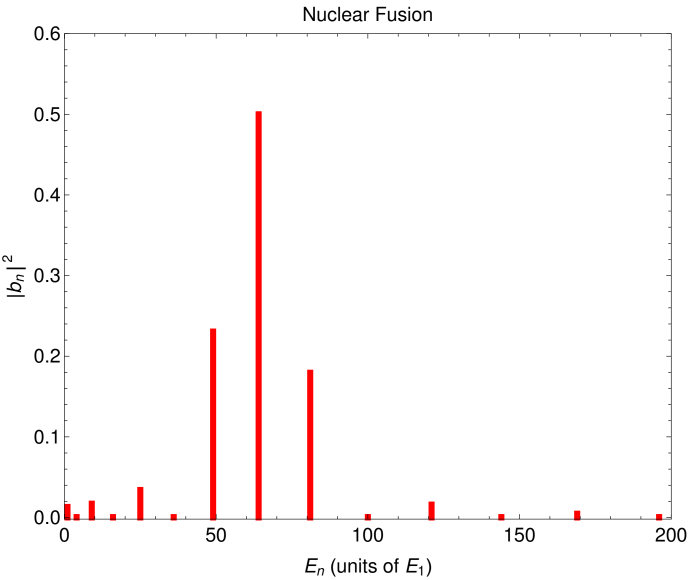
Initial Wave Packet and the Spectral Distribution



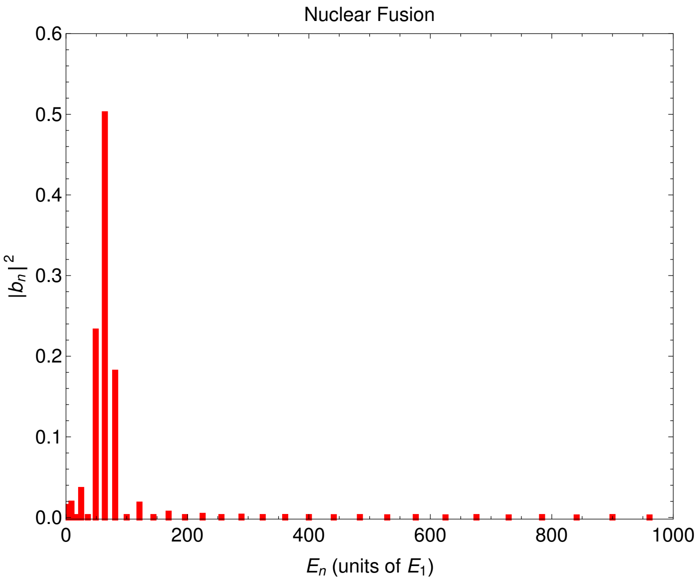
Probabilities of Different Final States



Spectral Distribution for One-Dimensional Nuclear Fusion



Spectral Distribution for One-Dimensional Nuclear Fusion



Spectral Distribution for One-Dimensional Nuclear Fusion