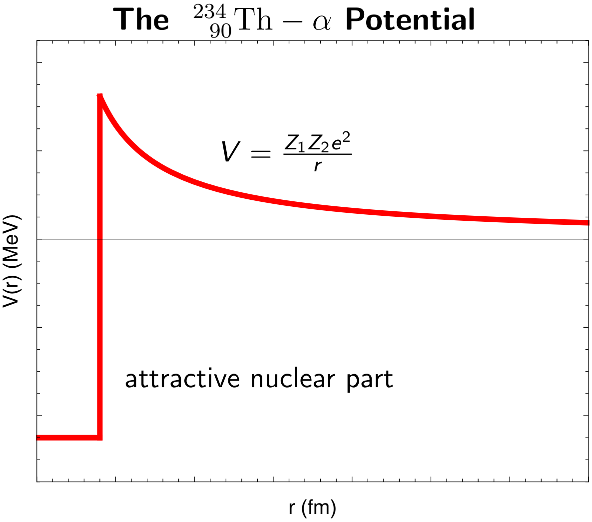
Consider the alpha decay shown below where a uranium nucleus spontaneously breaks apart into a $^4\mathrm{He}$ or alpha particle and $^{234}_{90}\mathrm{Th}$.

$$^{238}_{92}\mathrm{U} \rightarrow \ ^4\mathrm{He} \ + \ ^{234}_{90}\mathrm{Th} \qquad \mathrm{E}(^4\mathrm{He}) = 4.2 \ \mathrm{MeV}$$

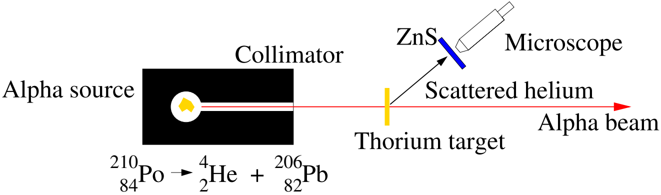
To study this reaction we first map out the $^4\mathrm{He} - ^{234}_{90}\mathrm{Th}$ potential energy. We reverse the decay above and use a beam of $^4\mathrm{He}$ nuclei striking a $^{234}_{90}\mathrm{Th}$ target. The $^4\mathrm{He}$ beam comes from the radioactive decay of another nucleus $^{210}_{84}\mathrm{Po}$ and $\mathrm{E}(^4\mathrm{He}) = 5.407 \ \mathrm{MeV}$.

1. What is the distance of closest approach of the $^4\mathrm{He}$ to the $^{234}_{90}\mathrm{Th}$ target if the Coulomb force is the only one that matters?
2. Is the Coulomb force the only one that matters?
3. What is the lifetime of the $^{238}_{92}\mathrm{U}$?


The $^{234}_{90}\mathrm{Th} - \alpha$ Potential
$V = \frac{Z_1 Z_2 e^2}{r}$
attractive nuclear part
V(r) (MeV)
r (fm)

Rutherford Scattering

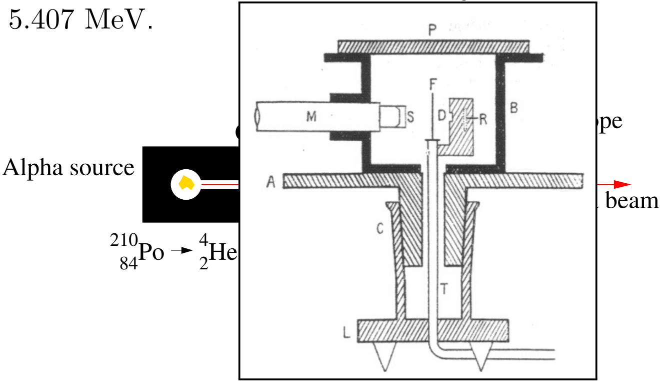
What is the distance of closest approach of the $^{4}\mathrm{He}$ to the $^{234}_{90}\mathrm{Th}$ target if only the Coulomb force is active? Is the Coulomb force the only one active? The energy of the $^{4}\mathrm{He}$ emitted by the $^{210}_{84}\mathrm{Po}$ to make the beam is $\mathrm{E}(^{4}\mathrm{He}) = 5.407\ \mathrm{MeV}$.



Demo go

Rutherford Scattering

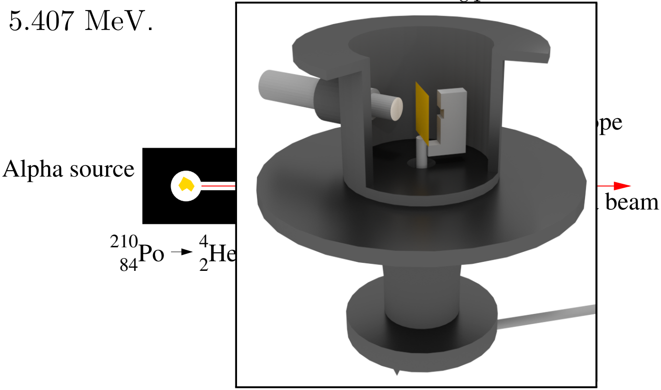
What is the distance of closest approach of the $^4\mathrm{He}$ to the $^{234}_{90}\mathrm{Th}$ target if only the Coulomb force is active? Is the Coulomb force the only one active? The energy of the $^4\mathrm{He}$ emitted by the $^{210}_{84}\mathrm{Po}$ to make the beam is $\mathrm{E}(^4\mathrm{He}) = 5.407\ \mathrm{MeV}$.



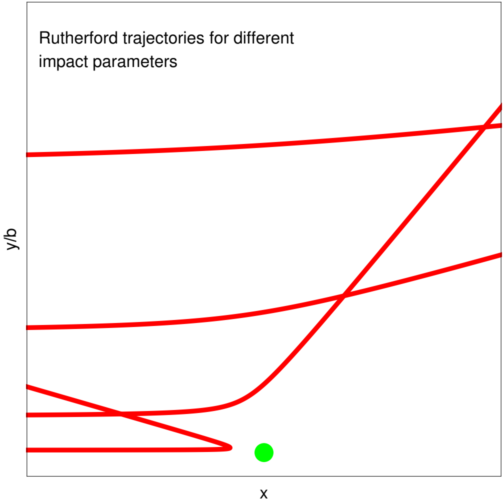
Demo go

Rutherford Scattering

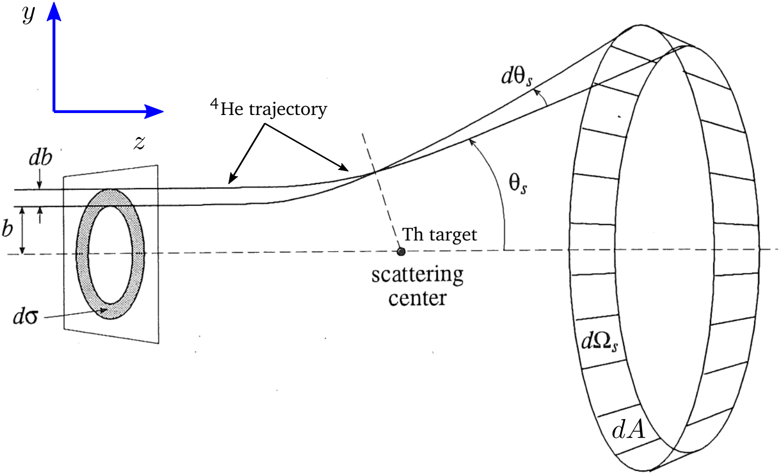
What is the distance of closest approach of the ^{4}He to the $^{234}_{90}$Th target if only the Coulomb force is active? Is the Coulomb force the only one active? The energy of the ^{4}He emitted by the $^{210}_{84}$Po to make the beam is $E(^{4}He) = 5.407$ MeV.



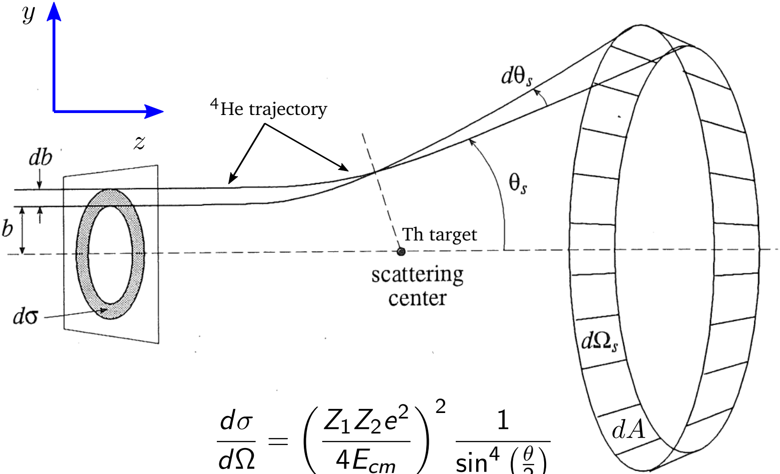
Demo


Rutherford trajectories for different
impact parameters
y/b
x

The Differential Cross Section



The Differential Cross Section



$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4E_{cm}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

particle rate scattered into dA of detector	$= \dfrac{dN_s}{dt}$	$\propto$	incident beam rate	$\times$	areal target density	$\times$	angular detector size

$$\frac{dN_s}{dt} \propto \frac{dN_{inc}}{dt} \times n_{tgt} \times d\Omega$$

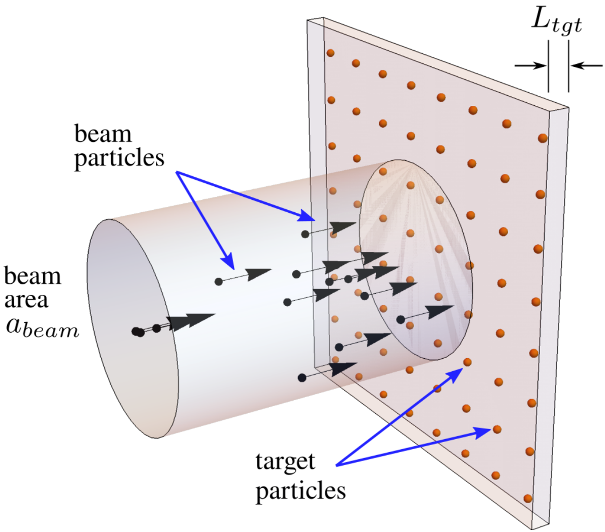
$$\frac{dN_s}{dt} = \frac{d\sigma}{d\Omega} \times \frac{dN_{inc}}{dt} \times n_{tgt} \times d\Omega$$

$$\frac{dN_{inc}}{dt} = \frac{\Delta N_{inc}}{\Delta t} = \frac{I_{beam}}{Ze}$$

I_{beam} - beam current
Z - beam charge

$$n_{tgt} = \frac{\rho_{tgt}}{A_{tgt}} N_A V_{hit} \frac{1}{a_{beam}} = \frac{\rho_{tgt}}{A_{tgt}} N_A L_{tgt}$$

ρ_{tgt} - target density
A_{tgt} - molar mass
V_{hit} - beam-target overlap
L_{tgt} - target thickness



molar mass A_{tgt}

density ρ_{tgt}

Avogadro's N_A

beam current I_{beam}

beam charge q_{beam}

target thickness L_{tgt}

beamtime on target Δt

beam area a_{beam}

particle rate scattered into dA of detector $= \frac{dN_s}{dt} \propto$ incident beam rate $\times$ areal target density $\times$ angular detector size

$$\frac{dN_s}{dt} \propto \frac{dN_{inc}}{dt} \times n_{tgt} \times d\Omega$$

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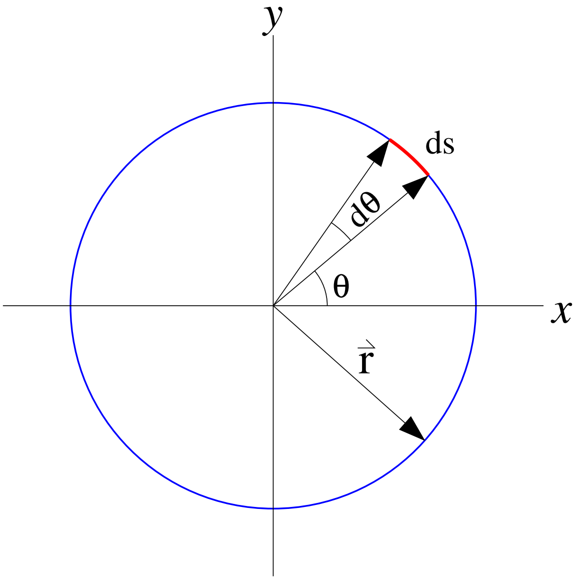
V_{hit} - beam-target overlap
L_{tgt} - target thickness

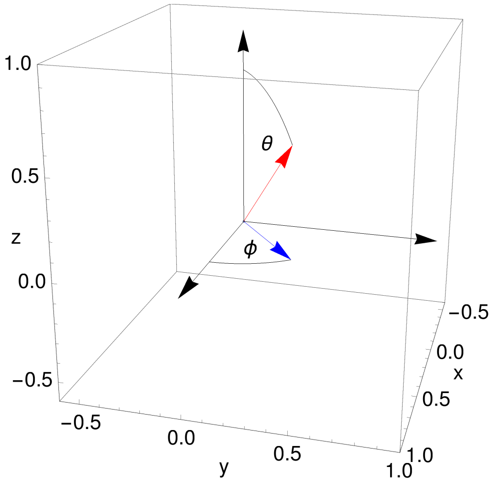
$$d\Omega = \frac{dA_{det}}{r_{det}^2} = \frac{\Delta A_{det}}{r_{det}^2} = \sin\theta d\theta d\phi$$

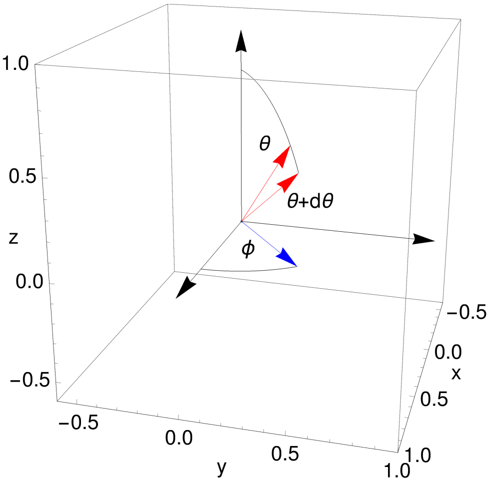
dA_{det} - detector area
r_{det} - target-detector distance

What is an Angle?

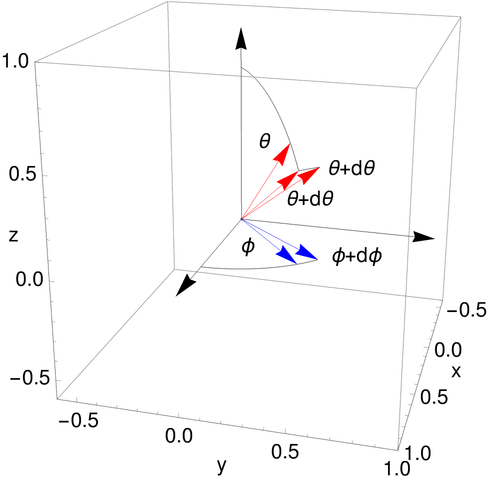


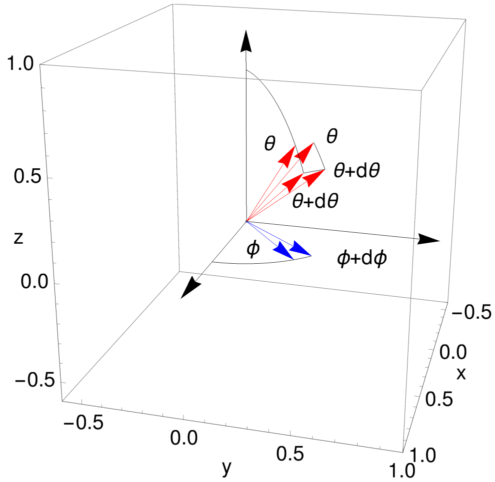
$$d\theta = \frac{ds}{|\vec{r}|}$$

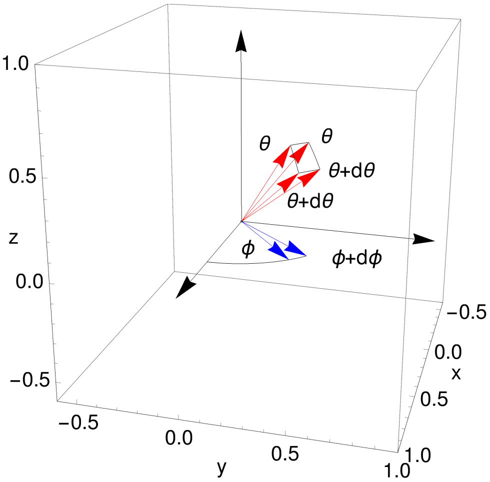

1.0
0.5
z
0.0
−0.5
θ
ϕ
−0.5
0.0
0.5
1.0
y
−0.5
0.0
x
0.5
1.0



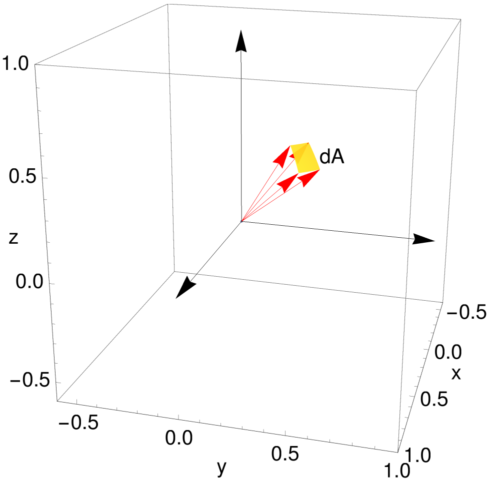
θ
θ+dθ
ϕ
1.0
0.5
z
0.0
−0.5
−0.5
0.0
0.5
1.0
y
−0.5
0.0
x
0.5
1.0


1.0
0.5
z
0.0
−0.5
θ
θ+dθ
θ+dθ
ϕ
ϕ+dϕ
−0.5
0.0
x
0.5
1.0
−0.5
0.0
0.5
y
1.0

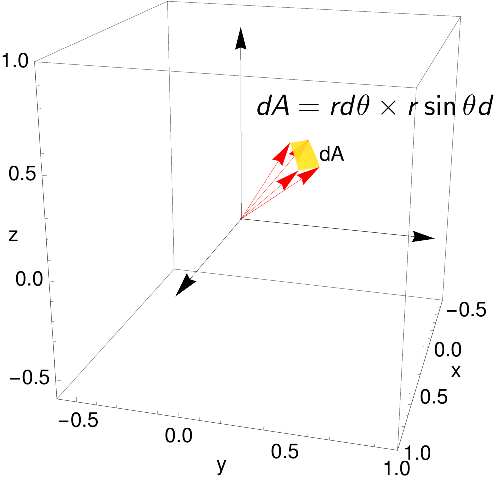

1.0
0.5
z
0.0
−0.5
−0.5
0.0
0.5
y
1.0
−0.5
0.0
x
0.5
1.0
θ
θ
θ+dθ
θ+dθ
ϕ
ϕ+dϕ

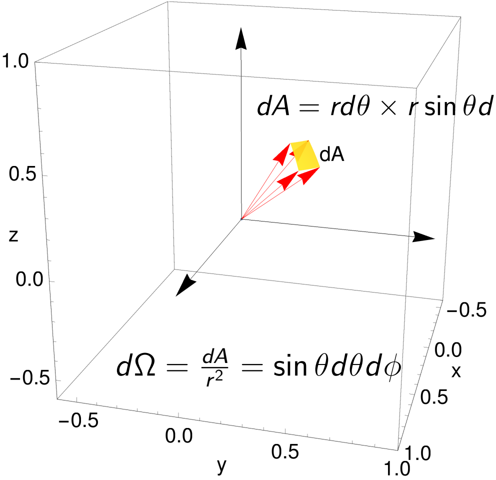

1.0
0.5
z
0.0
−0.5
−0.5
0.0
y
0.5
1.0
−0.5
0.0
x
0.5
1.0
θ
θ
θ+dθ
θ+dθ
ϕ
ϕ+dϕ

Solid Angle



$$dA = rd\theta \times r \sin\theta d\phi = r^2 \sin\theta d\theta d\phi$$





$$dA = rd\theta \times r\sin\theta d\phi = r^2 \sin\theta d\theta d\phi$$

$$d\Omega = \frac{dA}{r^2} = \sin\theta d\theta d\phi$$

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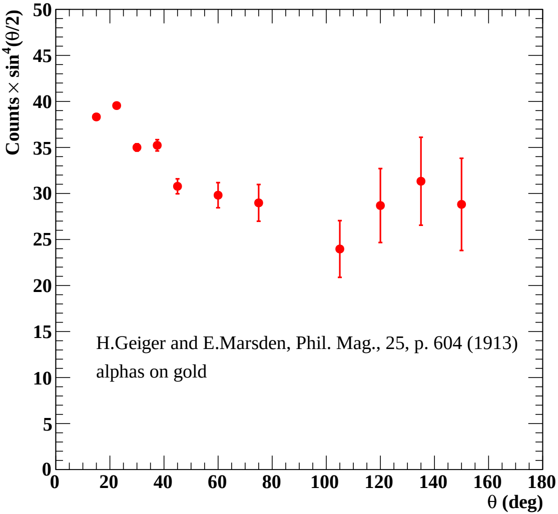
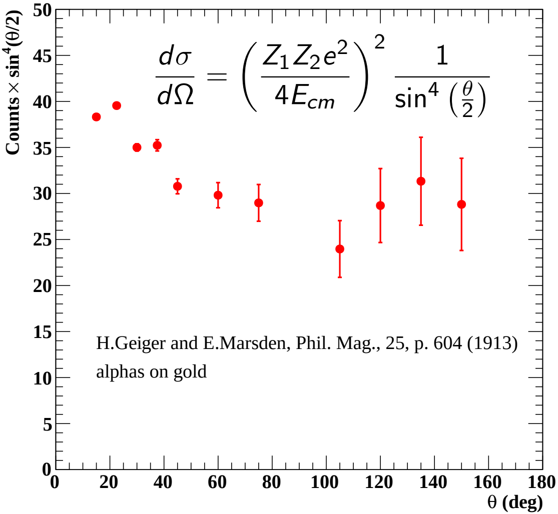
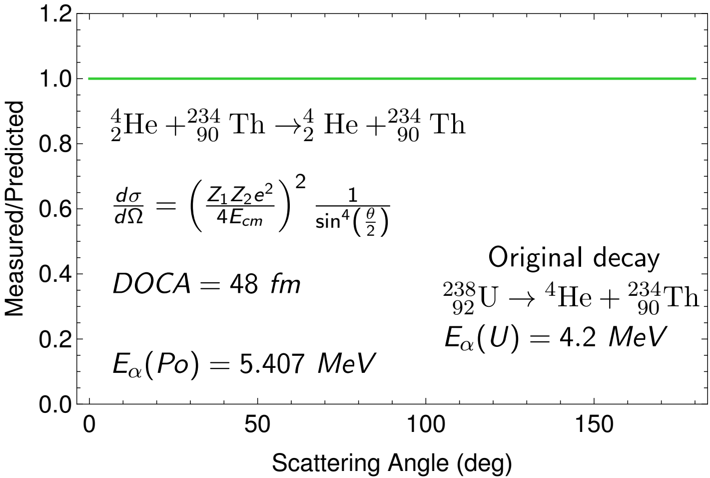
dA_{det} - detector area
r_{det} - target-detector distance

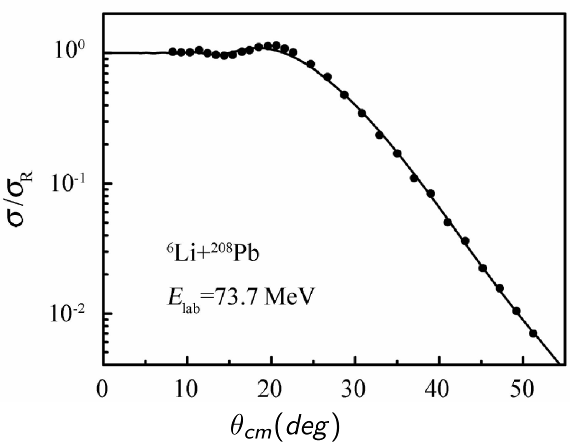

Counts × sin⁴(θ/2)
50
45
40
35
30
25
20
15
10
5
0
0
20
40
60
80
100
120
140
160
180
θ (deg)
H.Geiger and E.Marsden, Phil. Mag., 25, p. 604 (1913)
alphas on gold

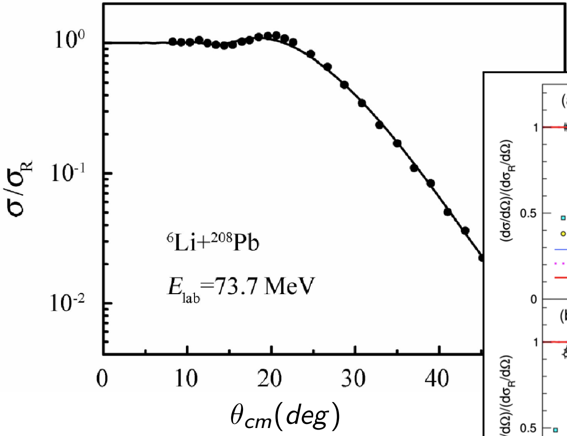
Actual Rutherford Scattering Results



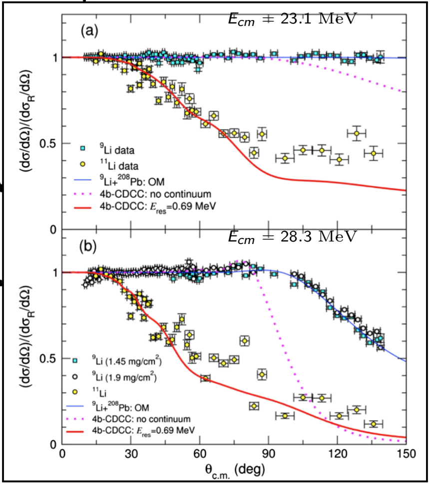


What does this say about the $^4_2\mathrm{He} - ^{234}_{90}\mathrm{Th}$ potential energy?

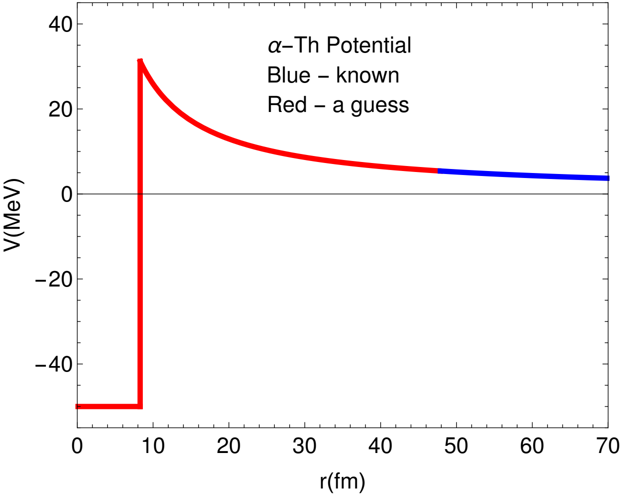

10^0
10^{-1}
10^{-2}
σ/σ_R
$^6Li+^{208}Pb$
E_{lab}=73.7 MeV
0
10
20
30
40
50
$\theta_{cm}(deg)$



PRL **109**, 262701 (2012)



The $^{4}\text{He} - {}^{234}_{90}\text{Th}$ Potential



1. We have probed the ${}^{234}_{90}\mathrm{Th}$ potential into an internuclear distance of $r_{DOCA} = 48\ fm$ using a ${}^{4}\mathrm{He}$ beam of $\mathrm{E}({}^{4}\mathrm{He}) = 5.407\ \mathrm{MeV}$.
2. The data are consistent with the Coulomb force and no others.
3. The radioactive decay ${}^{238}_{92}\mathrm{U} \to {}^{234}_{90}\mathrm{Th} + {}^{4}\mathrm{He}$ emits an α (or ${}^{4}\mathrm{He}$) with energy $E_{\alpha} = 4.2\ \mathrm{MeV}$.
4. For a classical 'decay' the emitted α should have an energy of at least $E_{min} = 5.407\ \mathrm{MeV}$.
5. It appears the 'decay' α starts out at a distance $r_{emit} = 62\ fm$.
6. How do we explain this?

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Quantum Tunneling!

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Quantum Tunneling!

7. What do we measure?

1. We have probed the $^{234}_{90}\mathrm{Th}$ potential into an internuclear distance of $r_{DOCA} = 48\ fm$ using a $^{4}\mathrm{He}$ beam of $\mathrm{E}(^{4}\mathrm{He}) = 5.407\ \mathrm{MeV}$.
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5. It appears the 'decay' α starts out at a distance $r_{emit} = 62\ fm$.
6. How do we explain this?

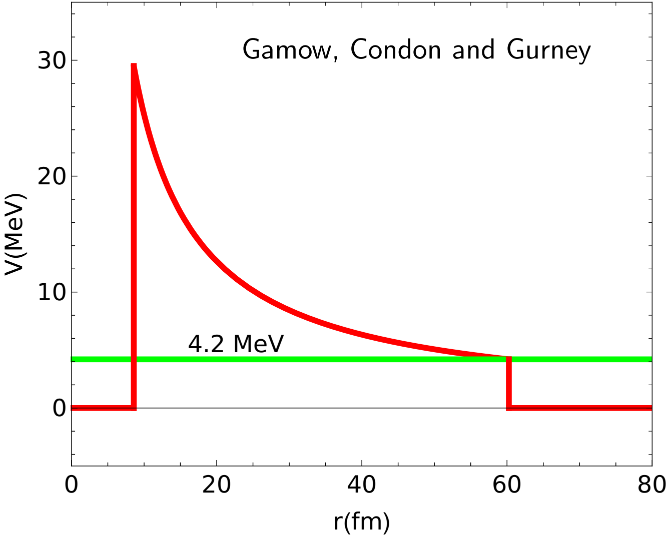
Quantum Tunneling!

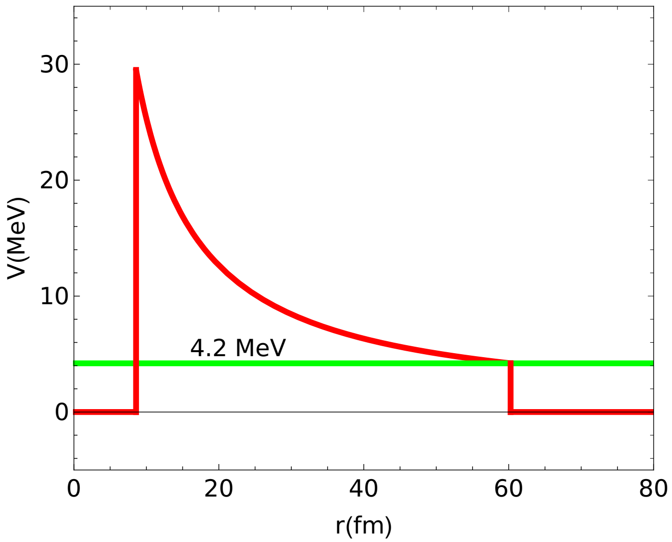
7. What do we measure?

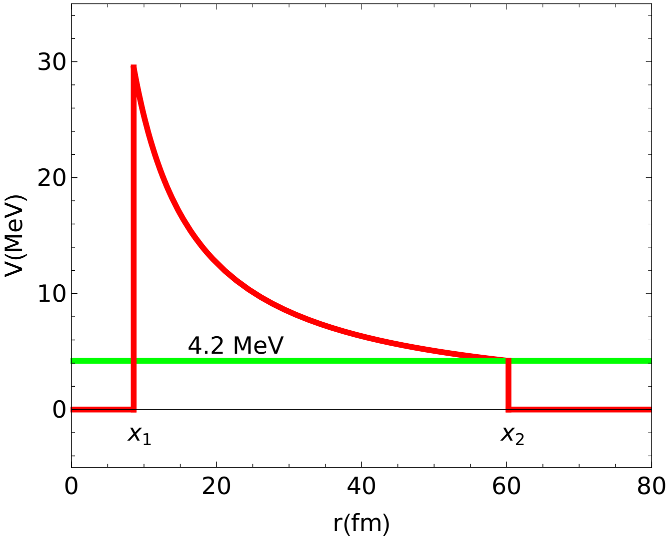
Lifetimes $\quad t_{1/2}(^{238}\mathrm{U}) = 4.5 \times 10^{9}\ \mathrm{yr}$

1. The α particle (^4He) is confined by the nuclear potential and 'bounces' back and forth between the walls of the nucleus. Assume its energy is the same as the emitted nucleon so $v = \sqrt{\frac{2E_\alpha}{m}}$.
2. Each time it 'bounces' off the nuclear wall it has a finite probability of tunneling through the barrier equal to the transmission coefficient T.
3. The decay rate will the product of the rate of collisions with a wall and the probability of transmission equal to $\frac{v}{2R} \times T$.
4. The lifetime is the inverse of the decay rate $\frac{2R}{vT} = 2R\sqrt{\frac{m}{2E}}\frac{1}{T}$.
5. The radius of a nucleus has been found to be described by $r_{nuke} = 1.2A^{1/3}$ where A is the mass number of the nucleus.
6. We are liberally copying the work of Gamow, Condon, and Gurney. Like them we will assume $V = 0$ inside the nucleus and $V = 0$ from the classical turning point to infinity.

The $^{4}\text{He} - {}^{234}_{90}\text{Th}$ Potential




30
20
10
0
V(MeV)
4.2 MeV
0
20
40
60
80
r(fm)


30
20
10
0
V(MeV)
4.2 MeV
x_1
x_2
0
20
40
60
80
r(fm)

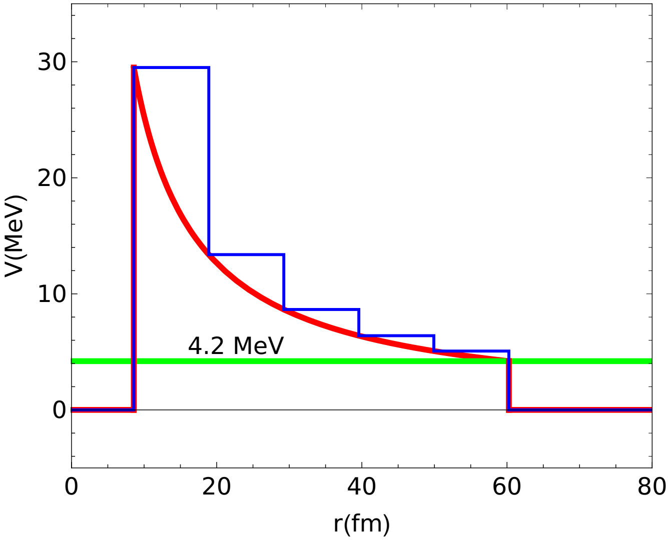
$$\zeta_1 = \mathbf{t}\zeta_3 = \mathbf{d}_{12}\mathbf{p}_2\mathbf{d}_{21}\mathbf{p}_2^{-1}\zeta_3 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \zeta_3 \qquad T = \frac{1}{|t_{11}|^2}$$

$$\mathbf{d_{12}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_2}{k_1} & 1-\frac{k_2}{k_1} \\ 1-\frac{k_2}{k_1} & 1+\frac{k_2}{k_1} \end{pmatrix} \quad \mathbf{d_{21}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_1}{k_2} & 1-\frac{k_1}{k_2} \\ 1-\frac{k_1}{k_2} & 1+\frac{k_1}{k_2} \end{pmatrix}$$

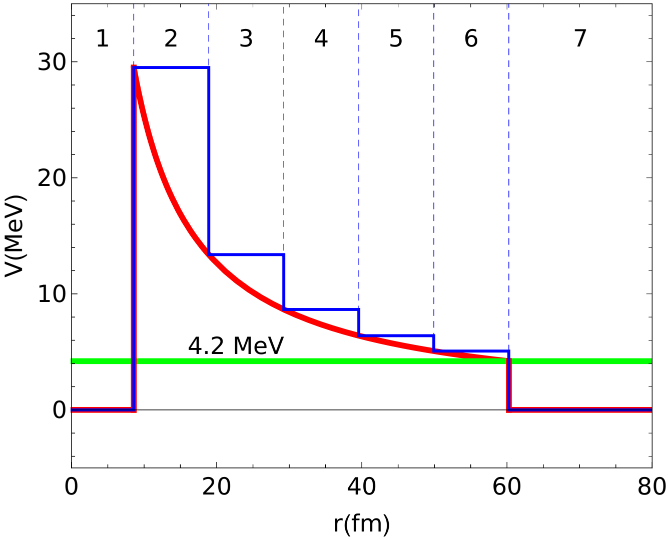
$$\mathbf{p_1^{-1}} = \begin{pmatrix} e^{ik_2 2a} & 0 \\ 0 & e^{-ik_2 2a} \end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix} e^{-ik_2 2a} & 0 \\ 0 & e^{ik_2 2a} \end{pmatrix}$$

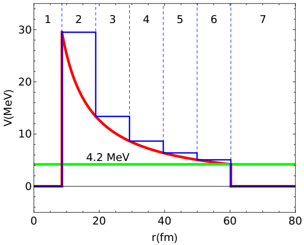
$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

$$t_{11} = \frac{1}{4}\left[\left(1+\frac{k_2}{k_1}\right)e^{-ik_2 2a}\left(1+\frac{k_1}{k_2}\right) + \left(1-\frac{k_2}{k_1}\right)e^{ik_2 2a}\left(1-\frac{k_1}{k_2}\right)\right]$$



30
20
10
0
V(MeV)
4.2 MeV
0
20
40
60
80
r(fm)

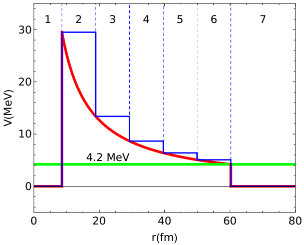

1
2
3
4
5
6
7
30
20
10
0
V(MeV)
4.2 MeV
0
20
40
60
80
r(fm)



$$\mathbf{d_{nm}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_m}{k_n} & 1-\frac{k_m}{k_n} \\ 1-\frac{k_m}{k_n} & 1+\frac{k_m}{k_n} \end{pmatrix} \quad \mathbf{p_m} = \begin{pmatrix} e^{-ik_m s} & 0 \\ 0 & e^{ik_m s} \end{pmatrix}$$

$$k_0 = \sqrt{\frac{2mE}{\hbar^2}} = k_6 \qquad k_n = \sqrt{\frac{2m(E-V_n)}{\hbar^2}}$$

n - left side of barrier
m - right side of barrier
V_n - potential of nth step.
s - step size.



$$\mathbf{d_{nm}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_m}{k_n} & 1-\frac{k_m}{k_n} \\ 1-\frac{k_m}{k_n} & 1+\frac{k_m}{k_n} \end{pmatrix} \quad \mathbf{p_m} = \begin{pmatrix} e^{-ik_m s} & 0 \\ 0 & e^{ik_m s} \end{pmatrix}$$

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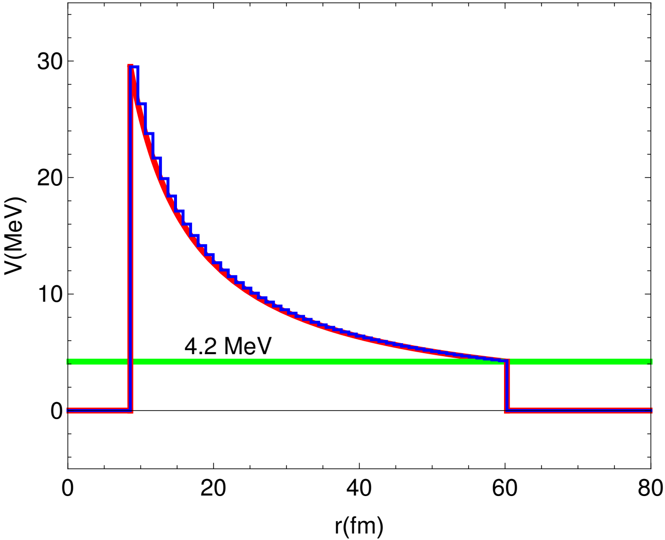
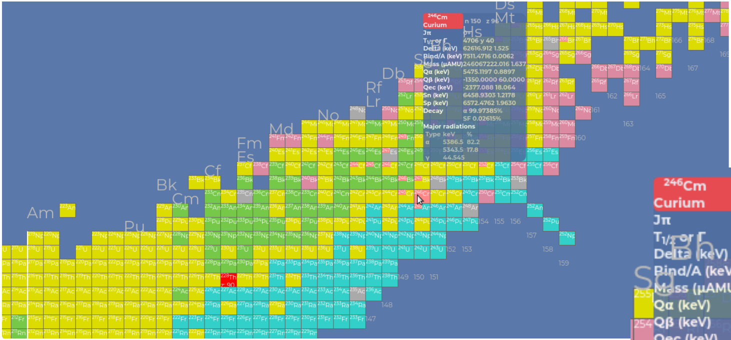
m - right side of barrier
V_n - potential of nth step.
s - step size.

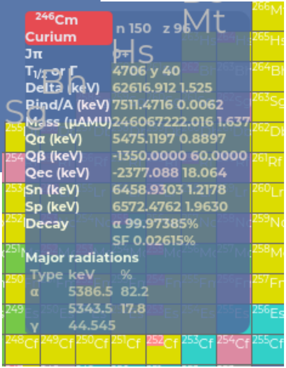
$$\underset{\sim}{\psi'_1} = \mathbf{d}_{12}\mathbf{p}_2 \cdot \mathbf{d}_{23}\mathbf{p}_3 \cdot \mathbf{d}_{34}\mathbf{p}_4 \cdot \underbrace{\mathbf{d}_{45}\mathbf{p}_5}_{\text{unit cell}} \cdot \mathbf{d}_{56}\mathbf{p}_6 \cdot \mathbf{d}_{67}\mathbf{p}_7 \; \underset{\sim}{\psi'_7}$$


30
20
10
0
V(MeV)
4.2 MeV
0
20
40
60
80
r(fm)

Getting Nuclear Data



Live Chart of the Nuclides



E_α	$t_{1/2}$ (meas/s)	Nucleus	Z	A	T (calculated)	$t_{1/2}$ (calc/s)
8.78	$3. \times 10^{-7}$	212-Po	84	212	1.11467×10^{-15}	7.27271×10^{-7}
6.78	0.15	216-Po	84	216	7.14393×10^{-22}	1.2994
8.	0.0001	215-At	85	215	3.75948×10^{-18}	0.00022696
6.26	1500.	212-Rn	86	212	4.86429×10^{-25}	1973.71
7.55	0.9	223-Th	90	223	1.81907×10^{-21}	0.488752
7.17	1500.	244-Cf	98	244	6.34009×10^{-26}	14 828.1
7.9	150.	248-Fm	100	248	7.34237×10^{-24}	122.643
4.19	1.4×10^{17}	238-U	92	238	5.10184×10^{-41}	2.39057×10^{19}
6.58	2200.	232-Pu	94	232	6.19107×10^{-27}	155 869.
6.01	4700.	239-Am	95	239	4.28498×10^{-30}	2.37989×10^{8}

E_α	$t_{1/2}$ (meas/s)	Nucleus	Z	A	T(calculated)	$t_{1/2}$ (calc/s)
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6.58	2200.	232-Pu	94	232	6.19107×10^{-27}	155869.
4.01	4.42×10^{17}	232-Th	90	232	1.62046×10^{-41}	7.62832×10^{19}
5.36	2.66×10^{11}	245-Cm	96	245	2.62605×10^{-34}	4.14617×10^{12}
5.53	4.35×10^{10}	247-Bk	97	247	1.00287×10^{-33}	1.07177×10^{12}
7.039	914.	252-Fm	100	252	3.24026×10^{-27}	295989.
5.275	2.32×10^{11}	243-Am	95	243	2.33955×10^{-34}	4.67844×10^{12}
5.49	1.36×10^{10}	241-Am	95	241	4.93363×10^{-33}	2.16868×10^{11}
4.01	332000.	222-Rn	86	222	4.01718×10^{-39}	3.03226×10^{17}

There are some differences between the formula for Rutherford scattering in the reading (go here) that are discussed below. The lecture formula is

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4E_{cm}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \tag{1}$$

while the expression in the reading is the following.

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi}{2} z^2 Z^2 \alpha^2 \left[\frac{\hbar c}{KE}\right]^2 \frac{1}{(1-\cos\theta)^2} \tag{2}$$

To go from Eq 1 to Eq 2 you need to make the following changes.

1. Change some variable names so $Z_1 = z$, $Z_2 = Z$, $E_{cm} = KE$.
2. Use $d\Omega = \sin\theta d\theta d\phi = d\cos\theta d\phi$ and integrate over all ϕ or $\phi = 0 \to 2\pi$. This gives you a factor of 2π in front of Eq 1.

$$\frac{d\sigma}{d\cos\theta} = \int_0^{2\pi} \frac{d\sigma}{d\Omega} d\phi = 2\pi \frac{d\sigma}{d\Omega} \tag{3}$$

3. Make the following substitutions

$$e^2 = \alpha\hbar c \qquad and \qquad \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta) \tag{4}$$

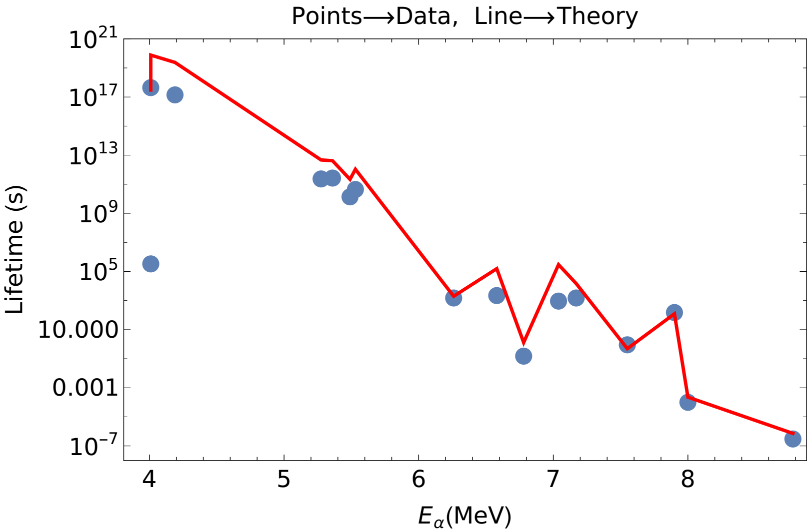
and you get Eq 2.

1. Define ALL variables with descriptive names.
2. Add comments for each 'section' of code.
3. Put inputs for individual calculation at the top of your code with comments describing each item.
4. Put constants used for all calculations in one section.
5. Indent 'new' sections.

```
Do[
  Vtest = Ve[[i,2]];
  mat1  = {{ Vtest,     0},
           {     0, Vtest} };
  mat2  = {{-Vtest,     0},
           {     0, Vtest} };
  test  = test.mat1.mat2;,
  {i,1,Ndiv+1}
 ];
```

6. Suppress printing until the end.
7. Print output at the end.

```
(* extract the transmission coefficient from the
   transition matrix here. *)
tr = Abs[1 / (Conjugate[ trans[[1, 1]] ] * trans[[1, 1]])];
Print["Transmission Coefficient: ", tr];
```


Points⟶Data, Line⟶Theory
10^{21}
10^{17}
10^{13}
10^{9}
10^{5}
10.000
0.001
10^{-7}
Lifetime (s)
4
5
6
7
8
E_α(MeV)