Physics 309
The Rectangular Barrier

1. In class we found the general solution to the rectangular barrier problem is the following

$$\phi_1 = Ae^{ik_1x} + Be^{-ik_1x} \qquad \phi_2 = Ce^{ik_2x} + De^{-ik_2x} \qquad \phi_3 = Fe^{ik_1x} + Ge^{-ik_1x}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \qquad .$$

We expressed the boundary conditions in the form of a transfer matrix so that

$$\begin{aligned} \zeta_1 &= \mathbf{t}\zeta_3 \\ &= \mathbf{d}_{12}\mathbf{p}_2\mathbf{d}_{21}\zeta_3 \end{aligned}$$

where ζ_1 and ζ_3 are vectors representing the amplitudes of the incoming and outgoing waves, $\mathtt{t}$ is the transfer matrix, and $\mathtt{d}_{12}$, $\mathtt{p}_2$, and $\mathtt{d}_{21}$ are the propagation and discontinuity matrices that compose $\mathtt{t}$. The propagation and discontinuity matrices for the rectangular barrier are

$$\mathbf{d_{12}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_2}{k_1} & 1-\frac{k_2}{k_1} \\ 1-\frac{k_2}{k_1} & 1+\frac{k_2}{k_1} \end{pmatrix} \qquad \mathbf{d_{21}} = \frac{1}{2}\begin{pmatrix} 1+\frac{k_1}{k_2} & 1-\frac{k_1}{k_2} \\ 1-\frac{k_1}{k_2} & 1+\frac{k_1}{k_2} \end{pmatrix}$$

$$\mathbf{p_1^{-1}} = \begin{pmatrix} e^{ik_12a} & 0 \\ 0 & e^{-ik_12a} \end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix} e^{-ik_22a} & 0 \\ 0 & e^{+ik_22a} \end{pmatrix} .$$

(a) In ϕ_3 above one of the coefficients (F or G) is zero because there are no incoming waves moving to the left in Region 3. Which coefficent is zero? Explain. (Hint: use the time dependent form of the eigenfunction to construct an argument.)

(b) Starting with ϕ_1 and ϕ_2 above apply the boundary conditions (the wave functions and their derivatives must be continuous at $x = 0$ and $x = a$) and show the following.

$$A = \frac{1}{2}\left[C\left(1+\frac{k_2}{k_1}\right) + D\left(1-\frac{k_2}{k_1}\right)\right]$$

$$B = \frac{1}{2}\left[C\left(1-\frac{k_2}{k_1}\right) + D\left(1+\frac{k_2}{k_1}\right)\right]$$

(c) Show that the matrix

$$\mathbf{p_1} = \begin{pmatrix} e^{-ik_12a} & 0 \\ 0 & e^{ik_12a} \end{pmatrix}$$

is the inverse of $\mathbf{p_1^{-1}}$.

(d) What is the transfer matrix for the rectangular barrier in terms of the appropriate wave numbers?

(e) Obtain an analytical expression for the transmission coefficient in terms of the appropriate wave numbers. Is your expression consistent with the formula in your text? How would you check?

(f) Obtain an analytical expression for the reflection coefficient in terms of the appropriate wave numbers. How can you check your results for R and T?

(g) Obtain expressions for the remaining amplitudes that form the wave function. Assume the magnitude of the coefficient A is known (this is the amount of "beam" hitting the barrier) and the sign of the coefficient B relative to A is also known.

2. An electron beam is incident on a barrier of height $V_0 = 8\ eV$. At $E = 8.10\ eV$, the transmission coefficient is $T = 5.18 \times 10^{-2}$. What is the width a of the barrier? The expression for T is

$$\begin{aligned} T &= \frac{1}{1 + \frac{1}{4}\frac{V_0^2}{E(E-V_0)}\sin^2(2k_2 a)} \qquad E > V_0 \\ &= \frac{1}{1 + \frac{1}{4}\frac{V_0^2}{E(V_0-E)}\sinh^2(2\kappa a)} \qquad E < V_0 \end{aligned}$$

where

$$k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} = 0.162\ \text{Å}^{-1} \quad \text{and} \quad \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = 0.162\ \text{Å}^{-1}$$

at $E = 8.10\ eV$ and 1.0 Å = 1 angstrom = 10^{-10} m.

3. Cold emission is a process where electrons are drawn from a metal at room temperature by an external electric field. The potential of the electrons in the metal without the external field is shown in the left-hand panel below. The electrons fill all available energy states (the Fermi sea) up to a maximum value E_F. The potential with with the field $\mathcal{E}$ on is shown in the right-hand panel.

$$V(x) = \Phi + E_F - e\mathcal{E}x$$

where E_F is the Fermi energy, Φ is the work function, $\mathcal{E}$ is the applied electric field, e is the electronic charge, and x is the position. See the figure for more information. Electrons can 'tunnel' through this barrier.

(a) Use the WKB approximation to calculate the transmission coefficient

$$T = \exp\left[-2\int_{x_1}^{x_2}\sqrt{\frac{2m(V(x) - E)}{\hbar^2}}dx\right]$$

where x_1 and x_2 are values of the position x where the energy E_F equals $V(x)$ (see the figure). Get your answer in terms of the electron mass m, Φ, e, $\mathcal{E}$, and any other necessary constants.

(b) The electric current inside the metal is described by $J_{inc} = env$ where n is the electron density and v is the electron speed in the Fermi sea. Consider a current coming out of the metal. The most likely electrons to tunnel through the barrier are the ones at the Fermi energy E_F (the top of the Fermi sea). Calculate an expression for the electric field $\mathcal{E}$ needed to reach a current J_0 through the barrier from the Fermi sea in terms of m, e, n, E_F, Φ, and J_0.