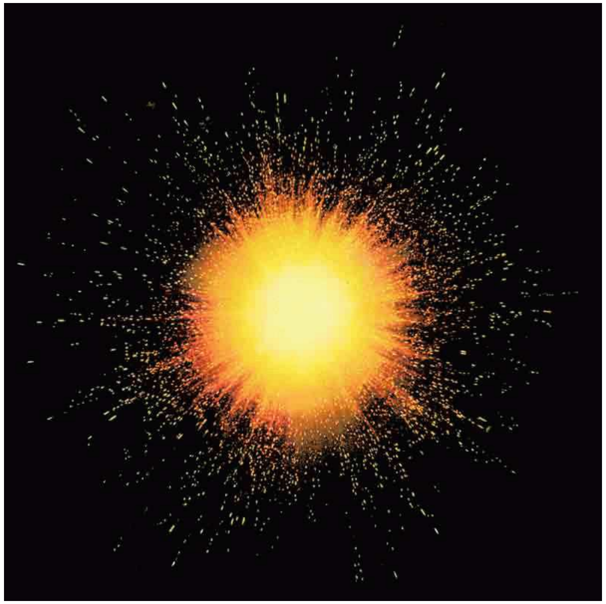
Let There Be Light!

We now finish our introduction to electromagnetism by completing the list of Maxwell's equations and deriving the properties of light.

1. Show that Maxwell's Equations in a charge-free vacuum give rise to separate differential equations for $\vec{E}$ and $\vec{B}$.
2. Show the solutions to those differential equations are waves.
3. What is the speed of those waves?
4. What constraints are imposed by Maxwell's equations?



$$\nabla \cdot \vec{\mathbf{D}} = \rho$$
$$\nabla \cdot \vec{\mathbf{B}} = 0$$
$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$
$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

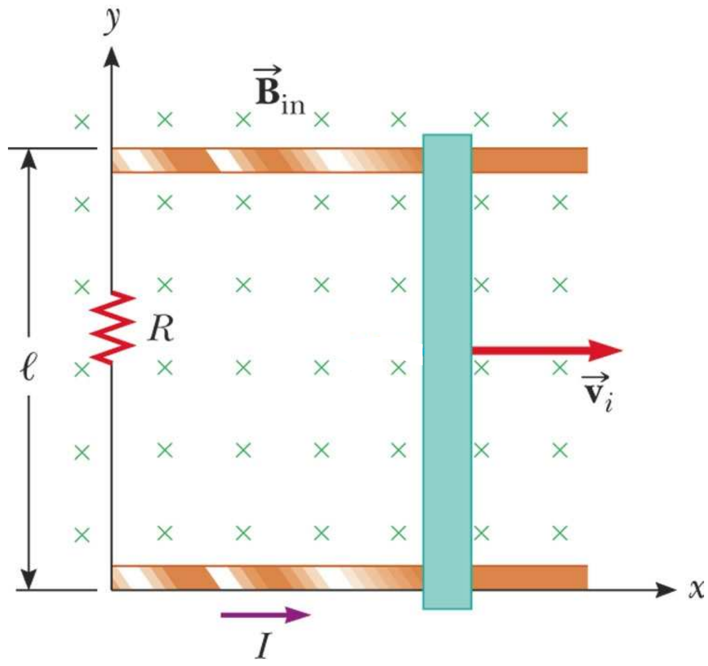
Let There Be Light!

Faraday's Law

A conducting bar moves on two frictionless, parallel rails in a uniform magnetic field directed into the plane. The bar has length l and initial velocity $\vec{v}_i$ to the right. What is the electric potential across the bar using the magnetic force law $\vec{F}_{mag} = Q\vec{v} \times \vec{B}$? Faraday's Law states that

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where $\mathcal{E}$ is the electromotive force or voltage and Φ is the magnetic flux. Does Faraday's Law agree with the result for the potential?



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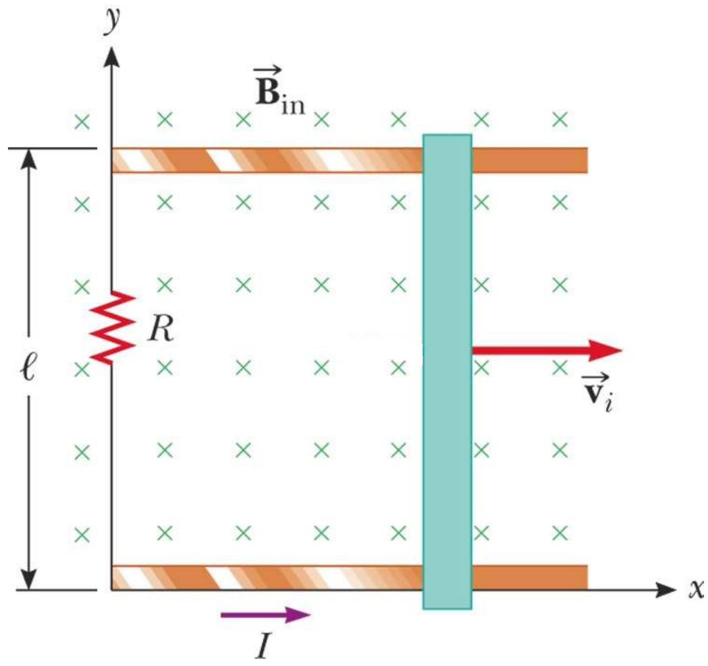

What happens if you change the $\vec{B}$ field?

Maxwell's Equations (so far)

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$ Gauss's Law

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$$ Ampere's Law

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$ Faraday's Law

$$\nabla \cdot \vec{\mathbf{B}} = 0$$ Martha's Law

Vector Identities from Griffith's Inside Cover

(1) $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

(2) $$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

(3) $$\nabla(fg) = f\nabla g + g\nabla f$$

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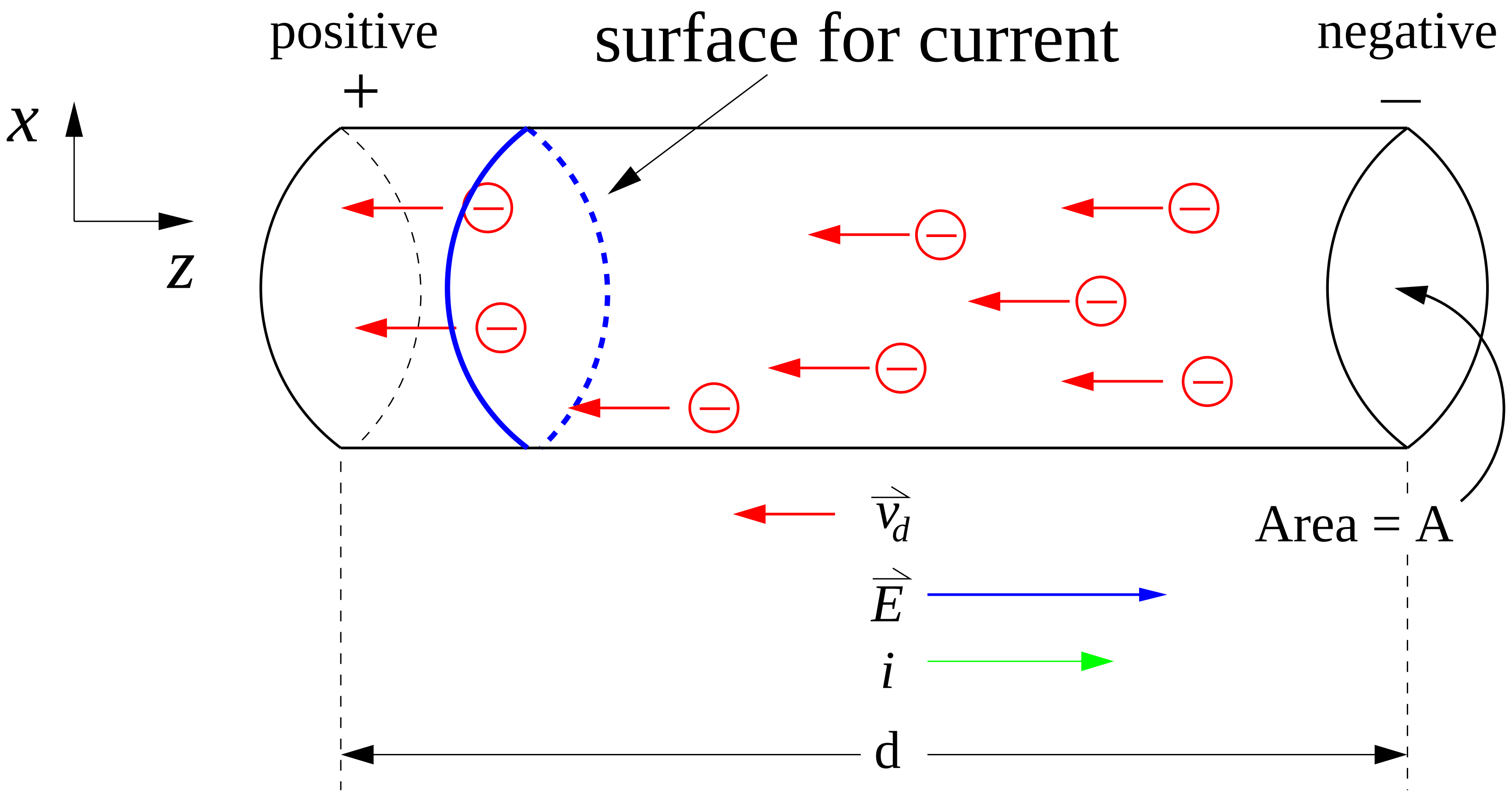
Fixing Ampere's Law



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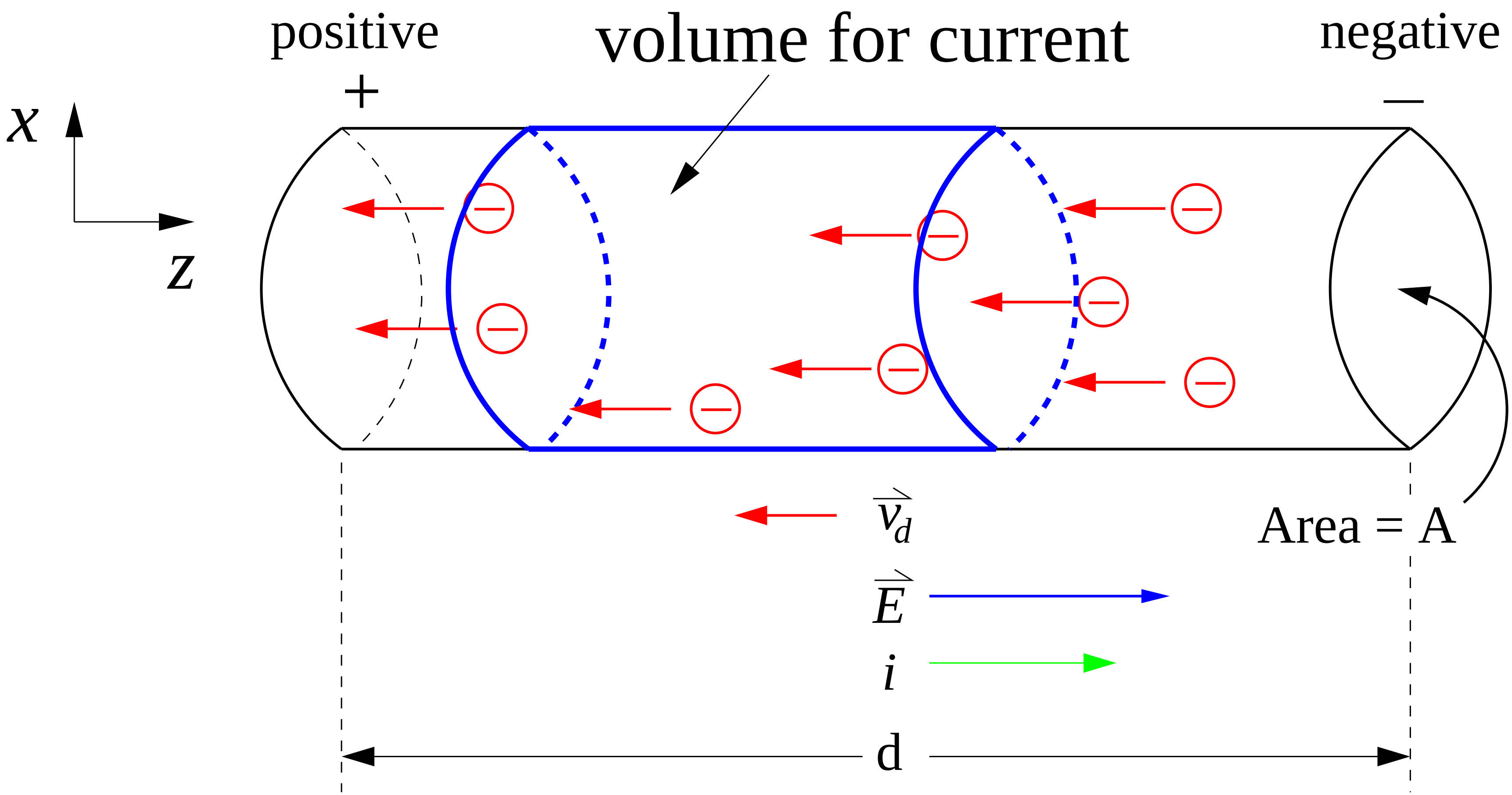

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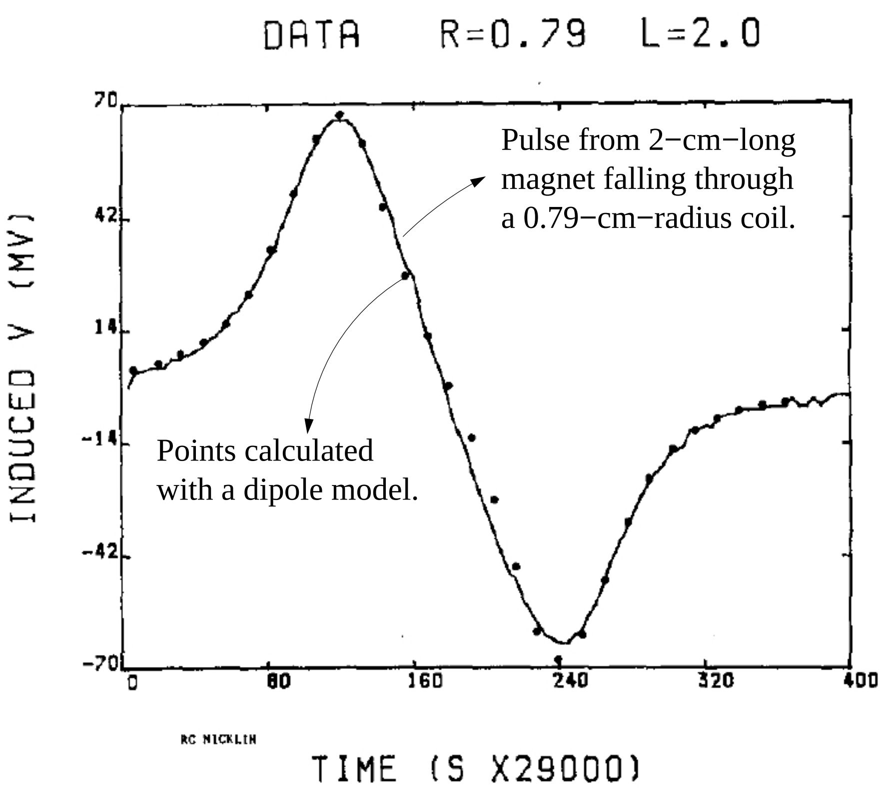
Evidence for Faraday's Law

- Lenz's Law.
- Measurement by R. C. Nicklin, Am. J. Phys. 54, 422 (1986).



Maxwell's Equations

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$

Gauss's Law

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

Ampere's Law

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Faraday's Law

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Martha's Law

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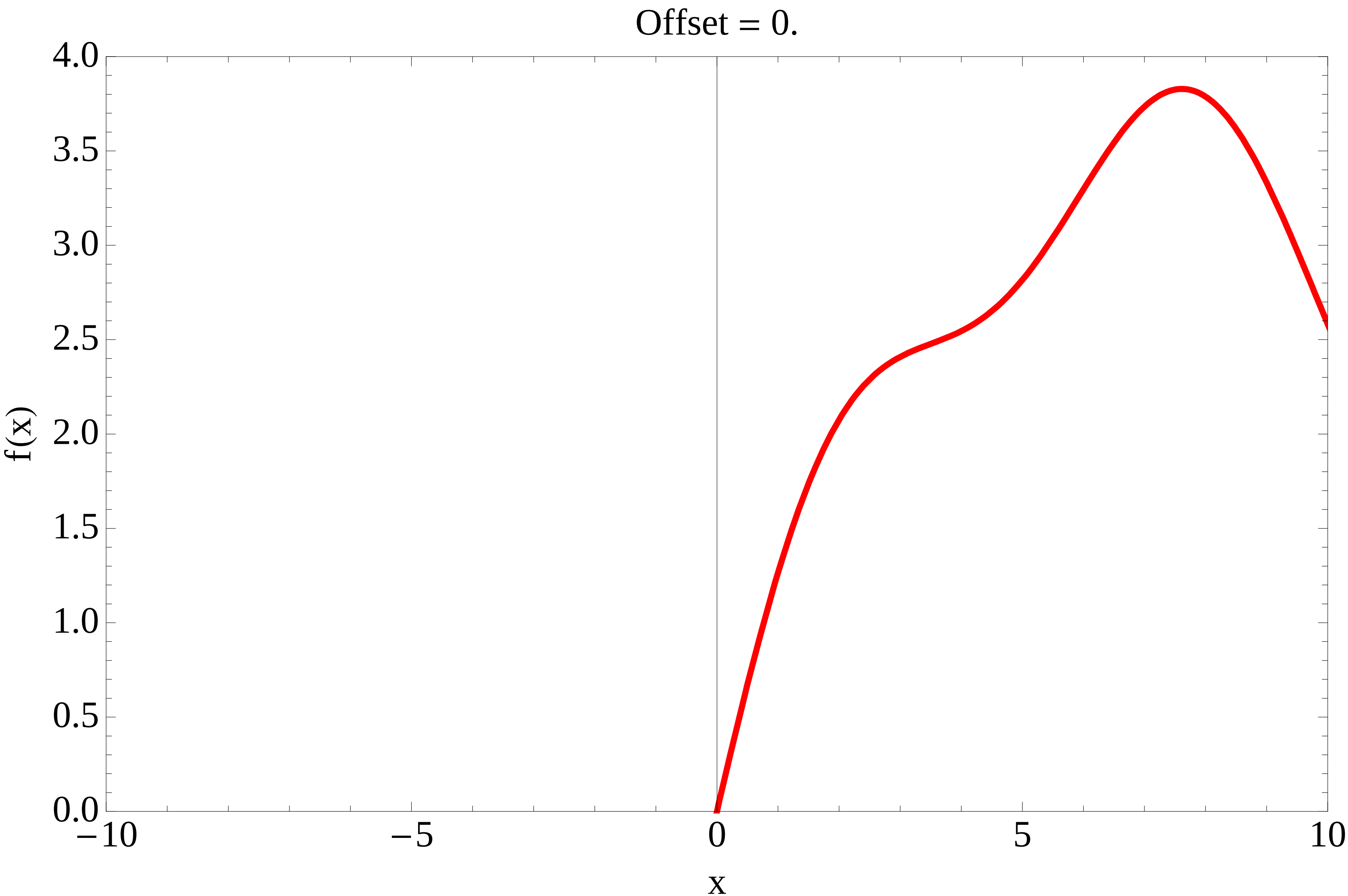

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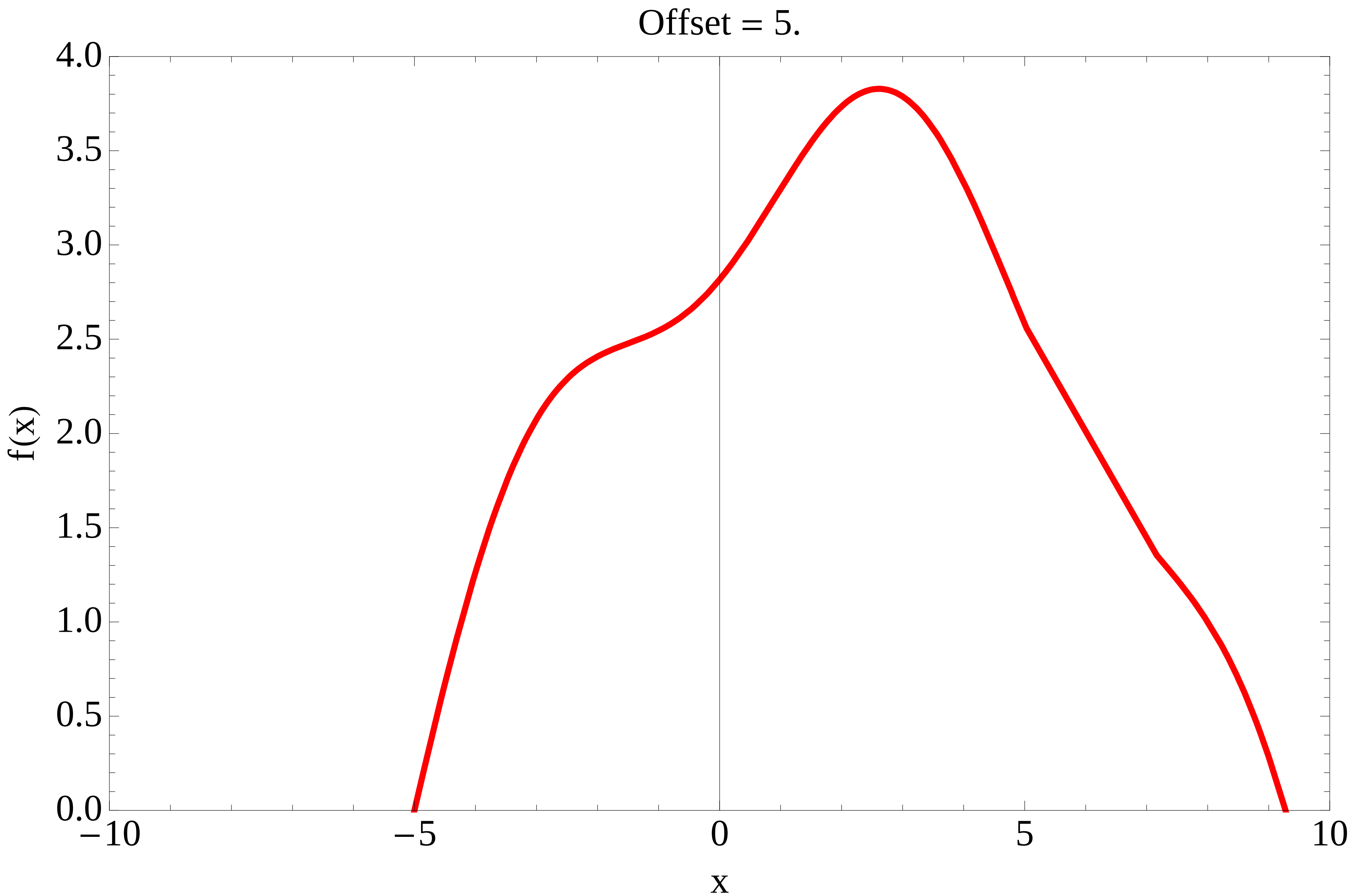

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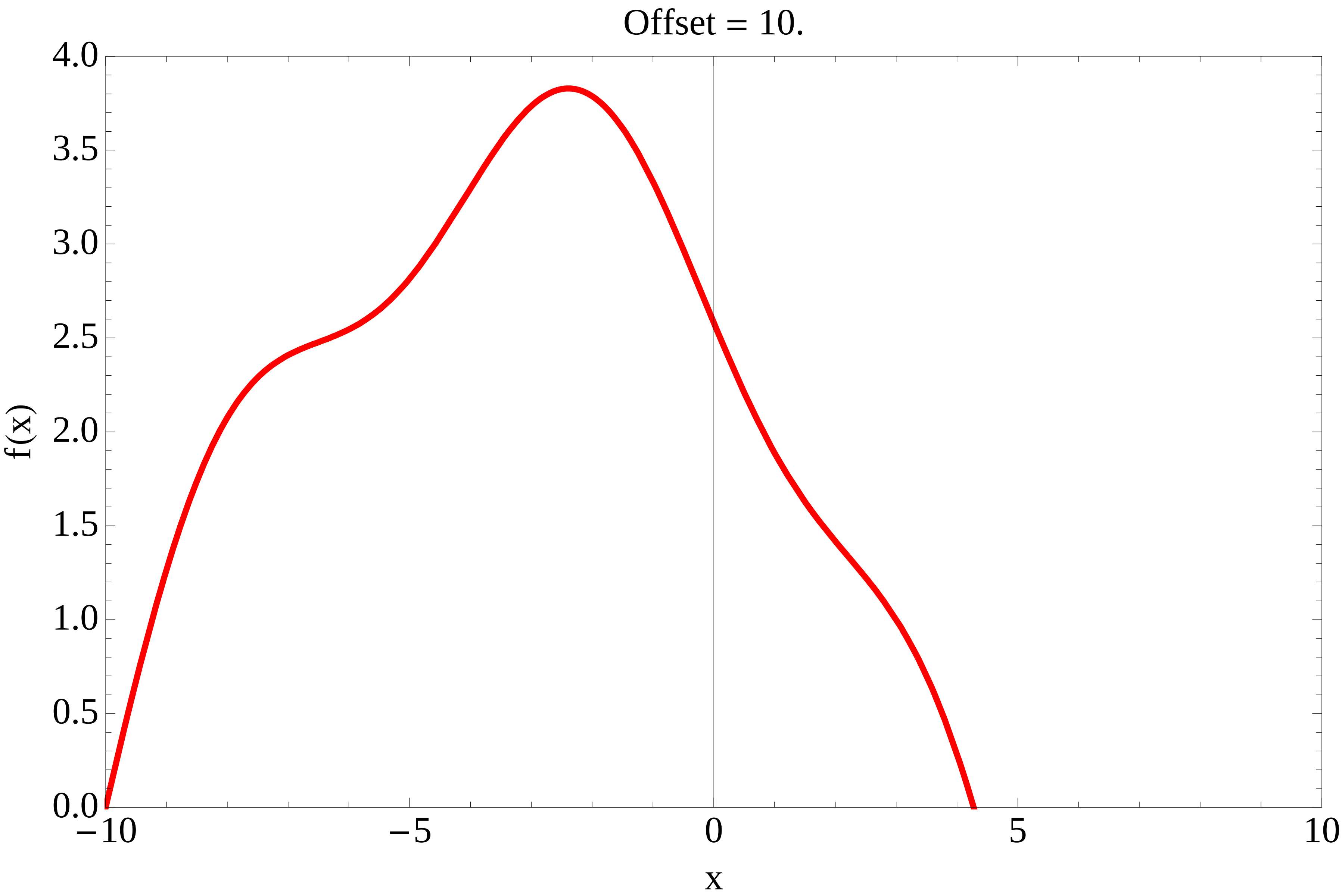

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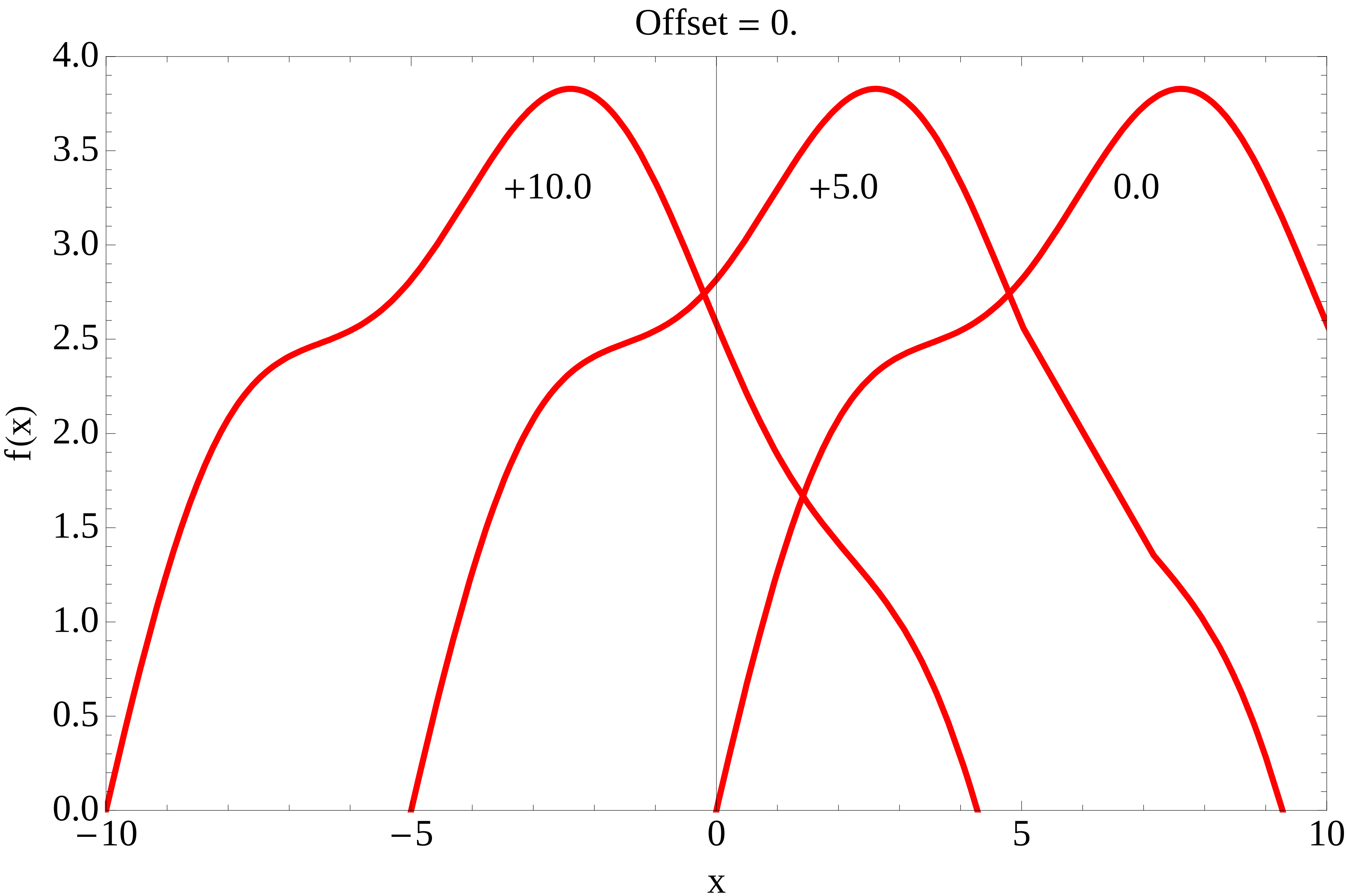
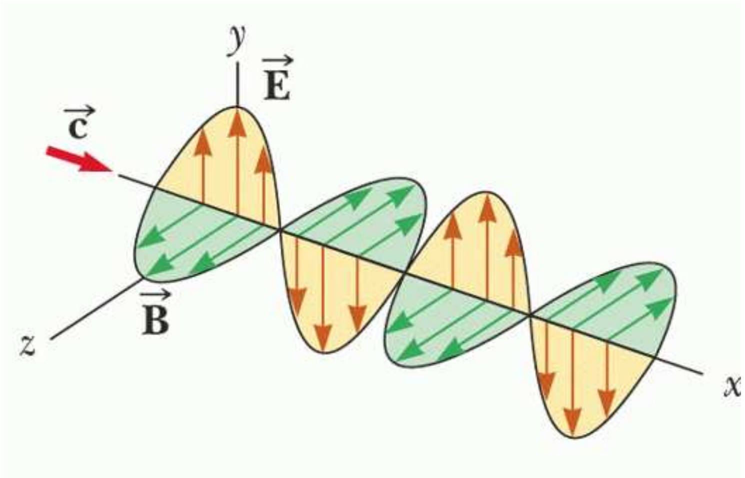


Electromagnetic Waves



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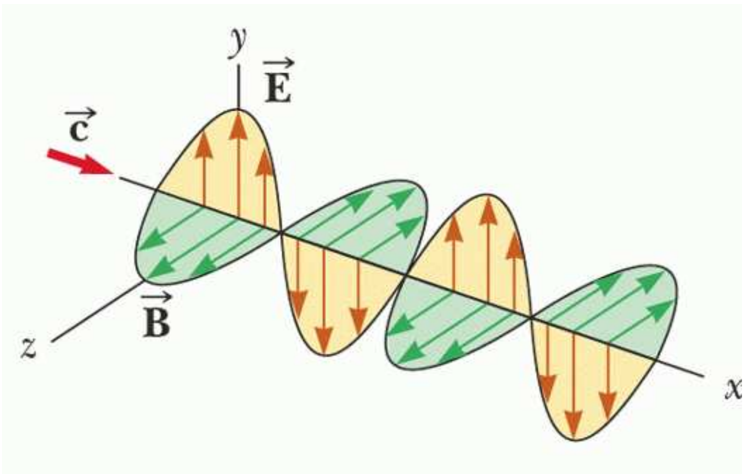

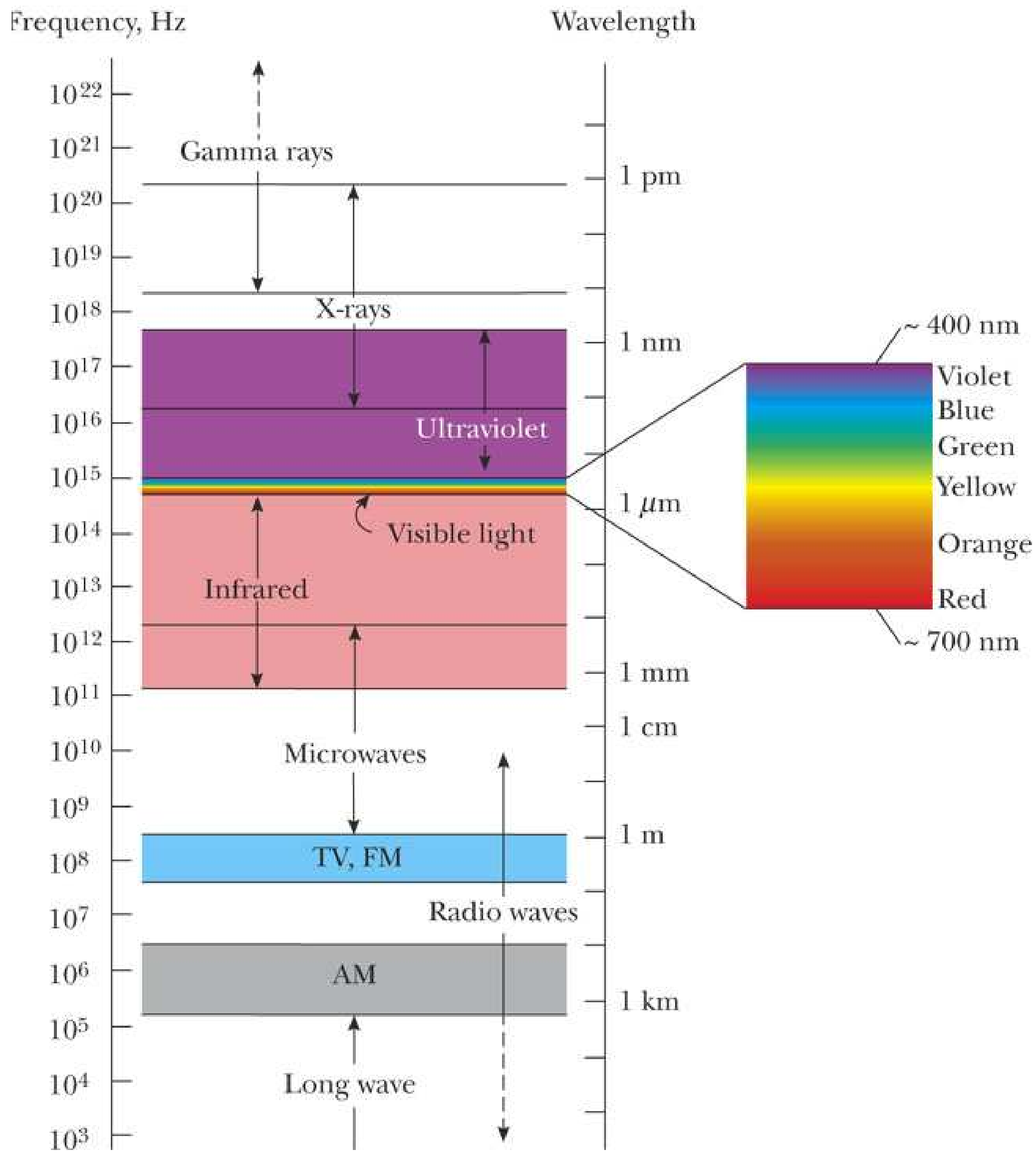

What Happens to Electromagnetic Waves in Linear Media?

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Recall Clausius-Mossotti (CM)

$$\epsilon_r = \frac{1 + \frac{2N\alpha}{3\epsilon_0}}{1 - \frac{N\alpha}{3\epsilon_0}} = 1 + \chi_e$$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

and the magnetic susceptibility

$$\chi_m = -\frac{1}{1 + \frac{16\pi m_e R_E}{2\mu_0 e^2}} = \frac{\mu}{\mu_0} - 1$$

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

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In free space with no charge:

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

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For nitrogen gas ($\mathrm{N_2}$):

$$n(CM) = \sqrt{\epsilon_r} = 1.0002939$$

$$n(measured) = 1.0002982 \rightarrow 1.0003012$$

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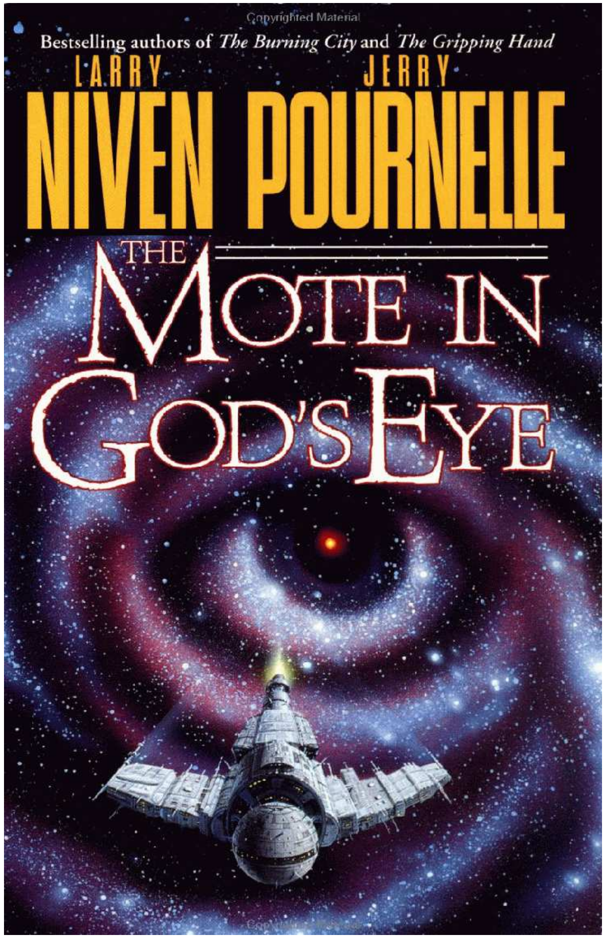
The Mote in God’s Eye

Described by Robert Heinlein as the finest Science Fiction book ever, *The Mote in God’s Eye* by Larry Niven and Jerry Pournelle is a story about our First Contact with an alien civilization. The Moties use a laser cannon to shine light on a solar sail and push it across a distance of 35 light-years.



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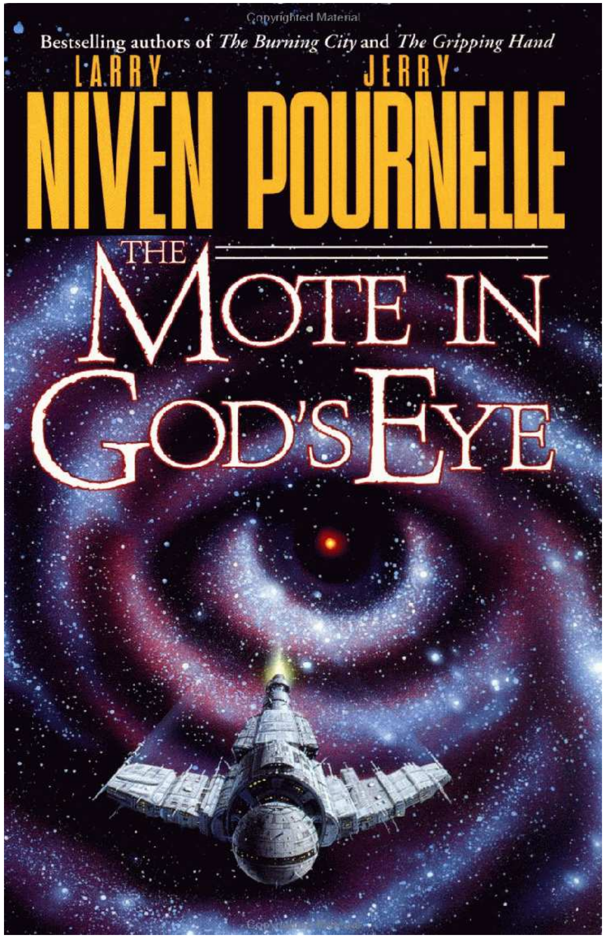
1. What is the expression for the energy per time per area transported by EM waves in vacuum?
2. What is the expression for the pressure of the electromagnetic waves?
3. Assume the sail is $3000\ km$ in diameter, the total mass is $m = 4.5 \times 10^5\ kg$, and the craft accelerates uniformly for 75 years to reach the halfway point. What is the minimum laser power required?
4. What is the minimum pressure of the light?



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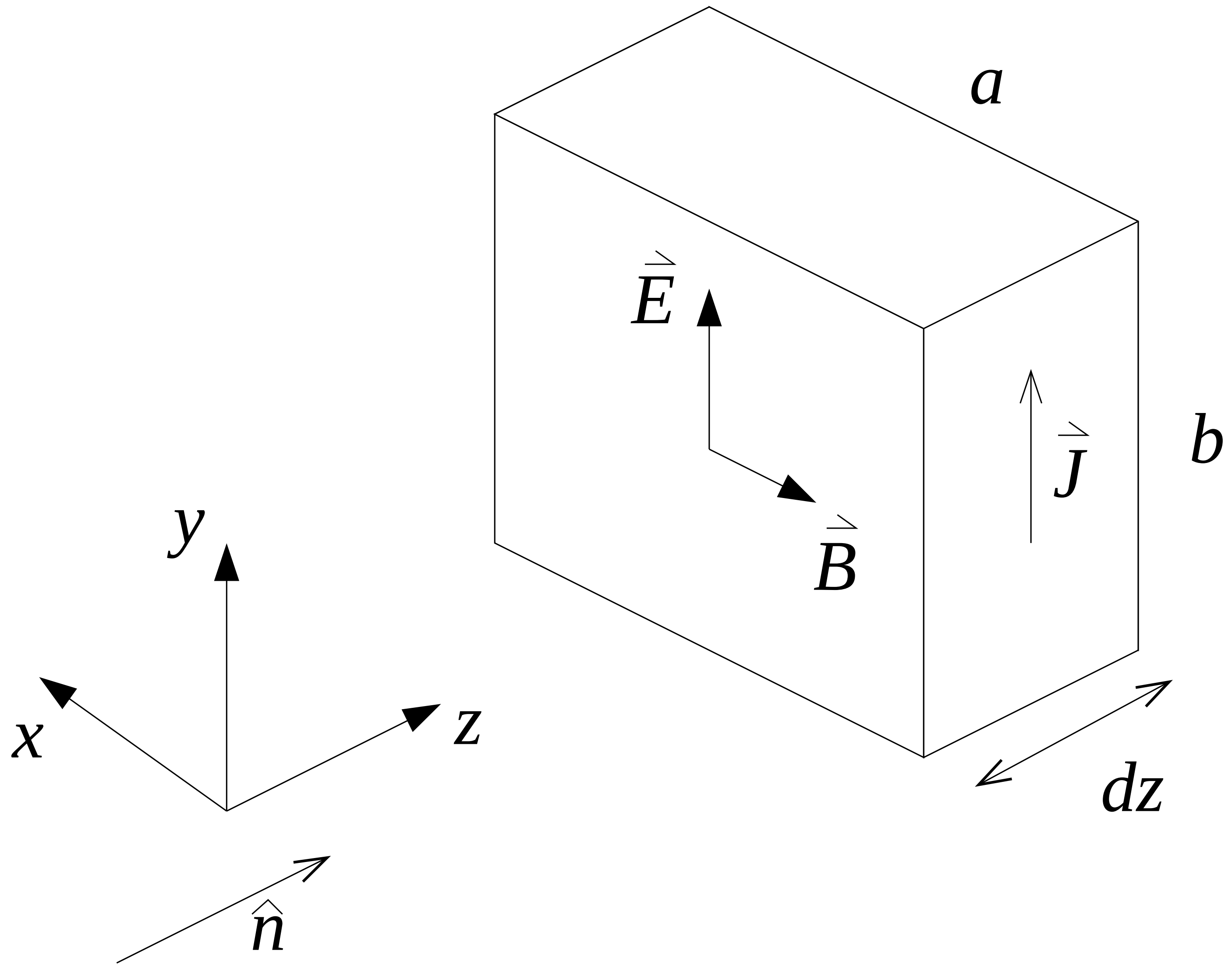
Light Pressure



Time Averaging