The Physical Pendulum and the Onset of Chaos

Consider the uniform rod rotating about an end point in the figure. Starting from the definition of the torque $\vec{\tau} = \vec{r} \times \vec{F}$,

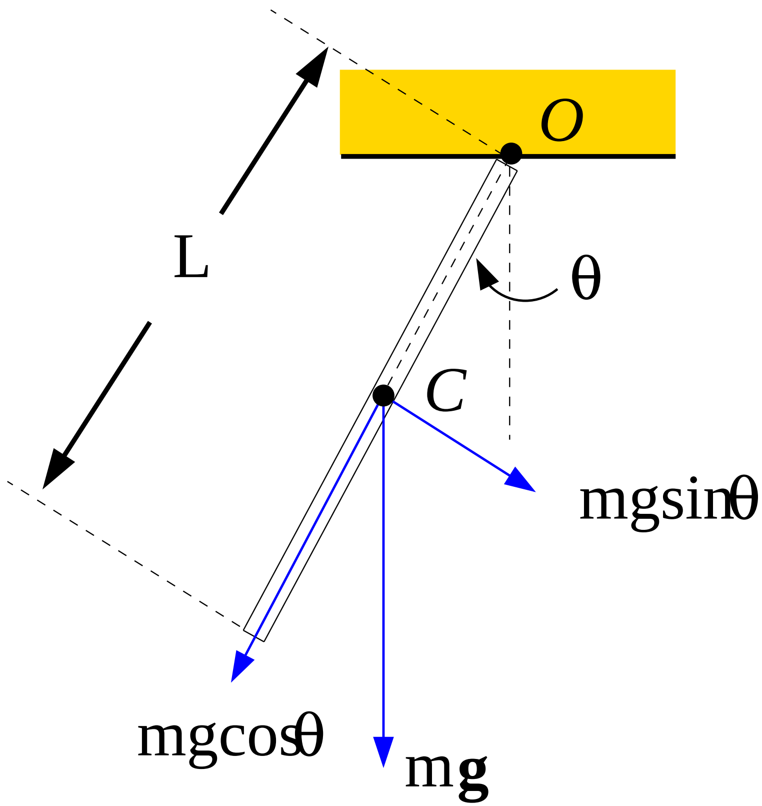
(1) derive the differential equation the angular position θ must satisfy.

(2) Derive a new differential equation if the pendulum is damped by a friction force $\vec{F}_f = -b\vec{v}$ where b is some constant describing the the pendulum.

(3) Derive a final differential equation if the pendulum is now also driven by a force $\vec{F}_{drive} = F_D \sin(\Omega t)\hat{\theta}$.

(4) Generate an algorithm for the differential equation from Part 3.

(5) What does the phase space look like for each set of conditions if the initial conditions are $\theta_0 = 25^\circ$ and $\omega_0 = 0\ rad/s$ or $\theta_0 = 24^\circ$ and $\omega_0 = 0\ rad/s$?



Getting Started - The Harmonic Oscillator

Hooke's Law states that

$$F_s = -kx$$

where F_s is the restoring force exerted by a spring and x is the displacement from equilibrium where there is no net force acting on the mass. See example here.

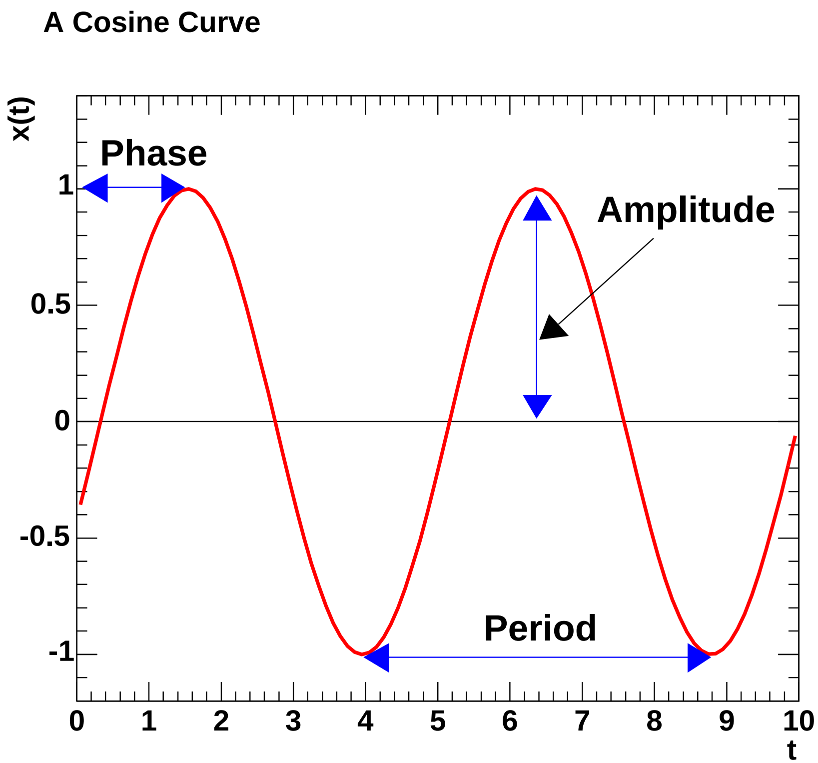
1. What differential equation does x satisfy?
2. What is the solution?
3. How would you test the solution?
4. What is the physical meaning of the constants in the solution?

The Harmonic Oscillator - The Solution

The solution for Hooke's Law is

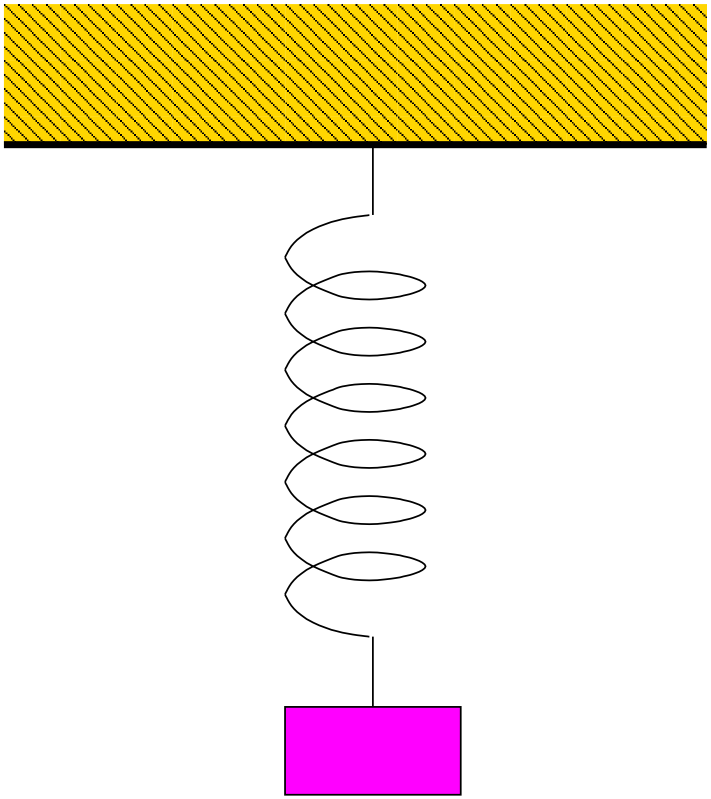
$$x(t) = A\cos(\omega t + \phi)$$

where $x(t)$ is the displacement from equilibrium.



The Simple Harmonic Oscillator - An Example

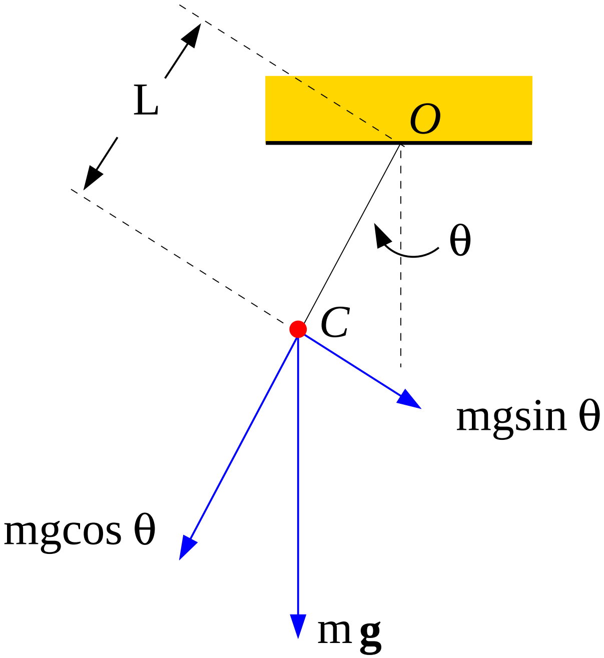
A harmonic oscillator consists of a block of mass $m = 0.33\ kg$ attached to a spring with spring constant $k = 400\ N/m$. See the figure below. At time $t = 0.0\ s$ the block's displacement from equilibrium and its velocity are $y = 0.100\ m$ and $v = -13.6\ m/s$. (1) Find the particular solution for this oscillator. (2) Use a centered derivative formula to generate an algorithm for solving the equation of motion.



The Pendulum - Stating the Problem

The simple pendulum is an example of an oscillatory system where the restoring force is provided by gravity. Consider the pendulum shown in the figure.

1. What differential equation does θ satisfy?
2. What differential equation does θ satisfy for small angles?
3. What is the solution?
4. How would you test the solution?
5. What is the physical meaning of the constants?
6. Redo Part 1 using torques.

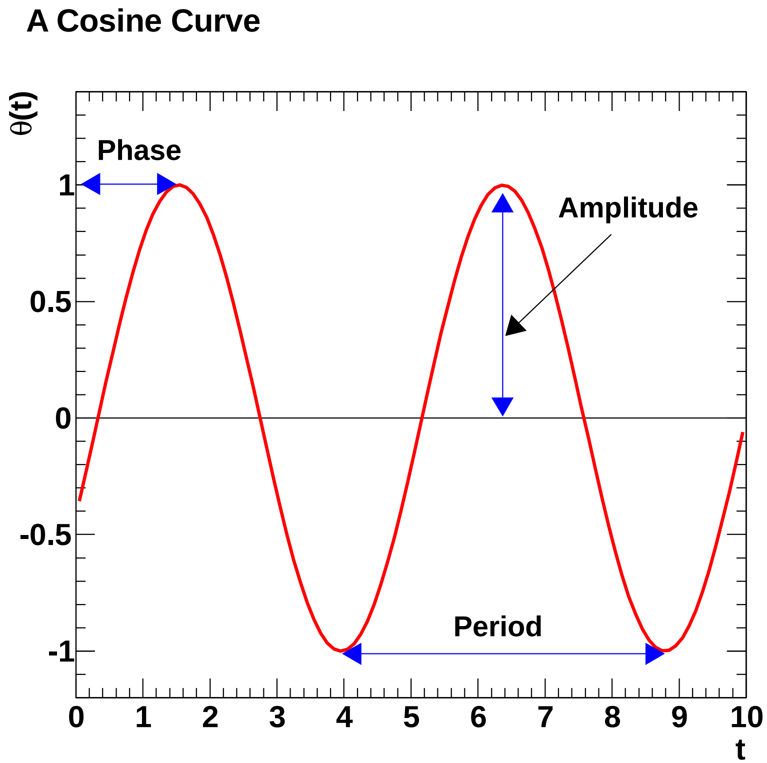


The Simple Pendulum - The Solution

The solution for simple pendulum is

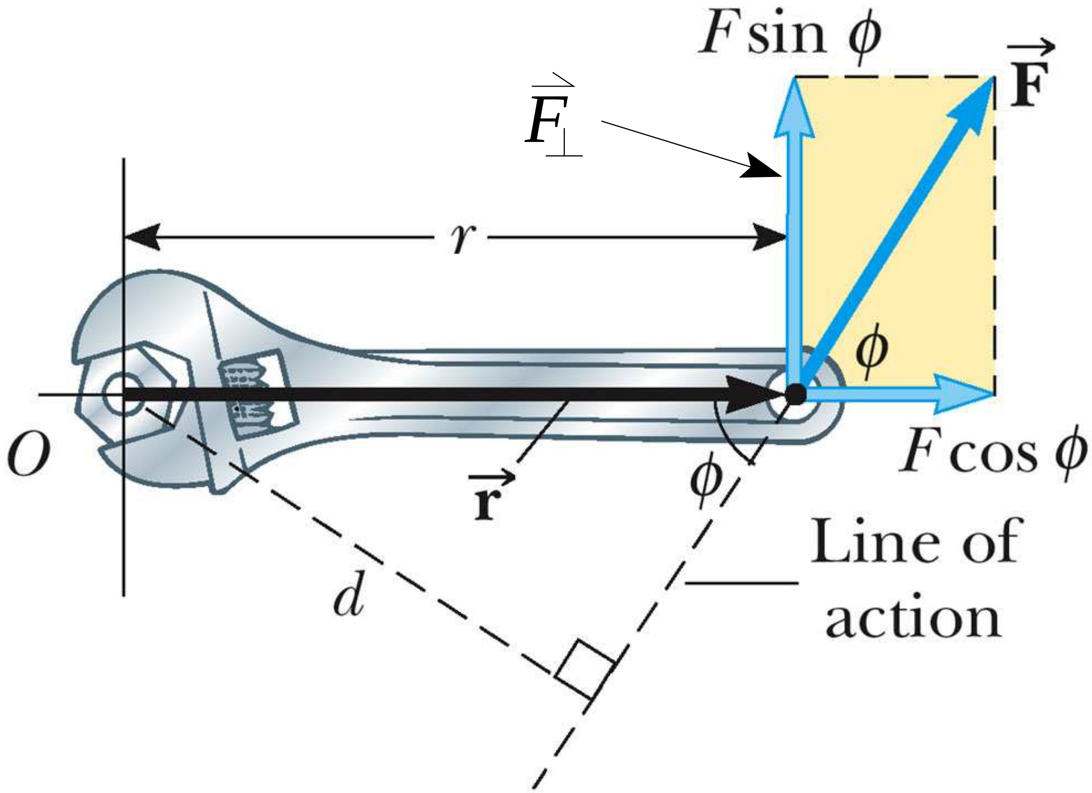
$$\theta(t) = A\cos(\omega t + \phi)$$

where $\theta(t)$ is the angular displacement from equilibrium.



Torque - Rotational Equivalent of Force

$$\vec{F} = m\vec{a} \rightarrow \vec{\tau} = r\vec{F}_{\perp}$$



Linear → Rotational Quantities

Linear Quantity	Connection	Rotational Quantity
s	$s = r\theta$	$\theta = \frac{s}{r}$
v_T	$v_T = r\omega$	$\omega = \frac{v_T}{r} = \frac{d\theta}{dt}$
a_T	$a_T = r\alpha$	$\alpha = \frac{a_T}{r} = \frac{d\omega}{dt}$
$KE = \frac{1}{2}mv^2$		$KE_R = \frac{1}{2}I\omega^2$
$\vec{F} = m\vec{a}$	$\tau = rF_\perp$	$\vec{\tau} = I\vec{\alpha}$
$\vec{p} = m\vec{v}$	$\vec{L} = \vec{r} \times \vec{p}$	$\vec{L} = I\vec{\omega}$

The Physical Pendulum and the Onset of Chaos

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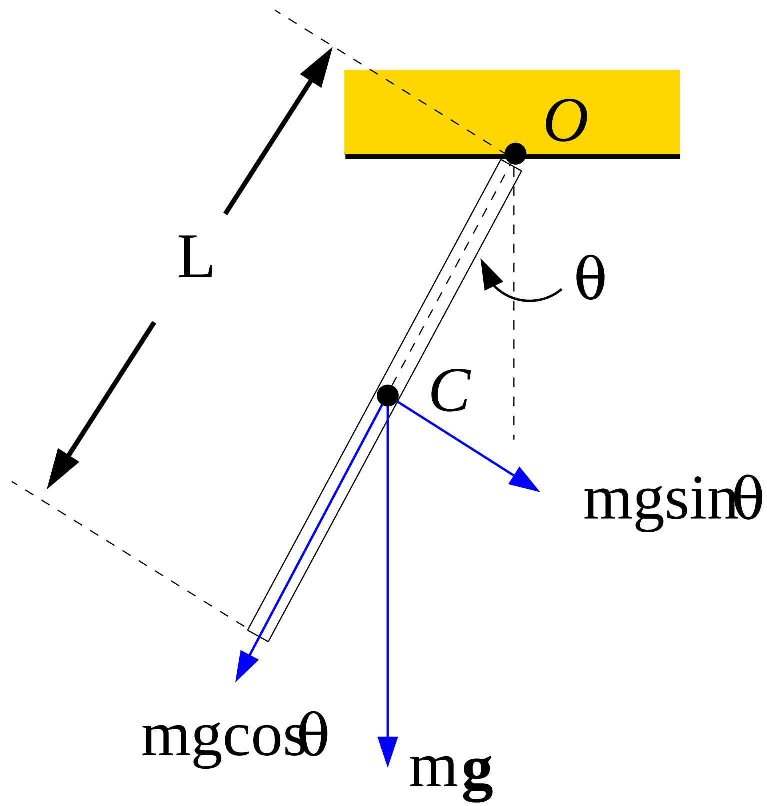
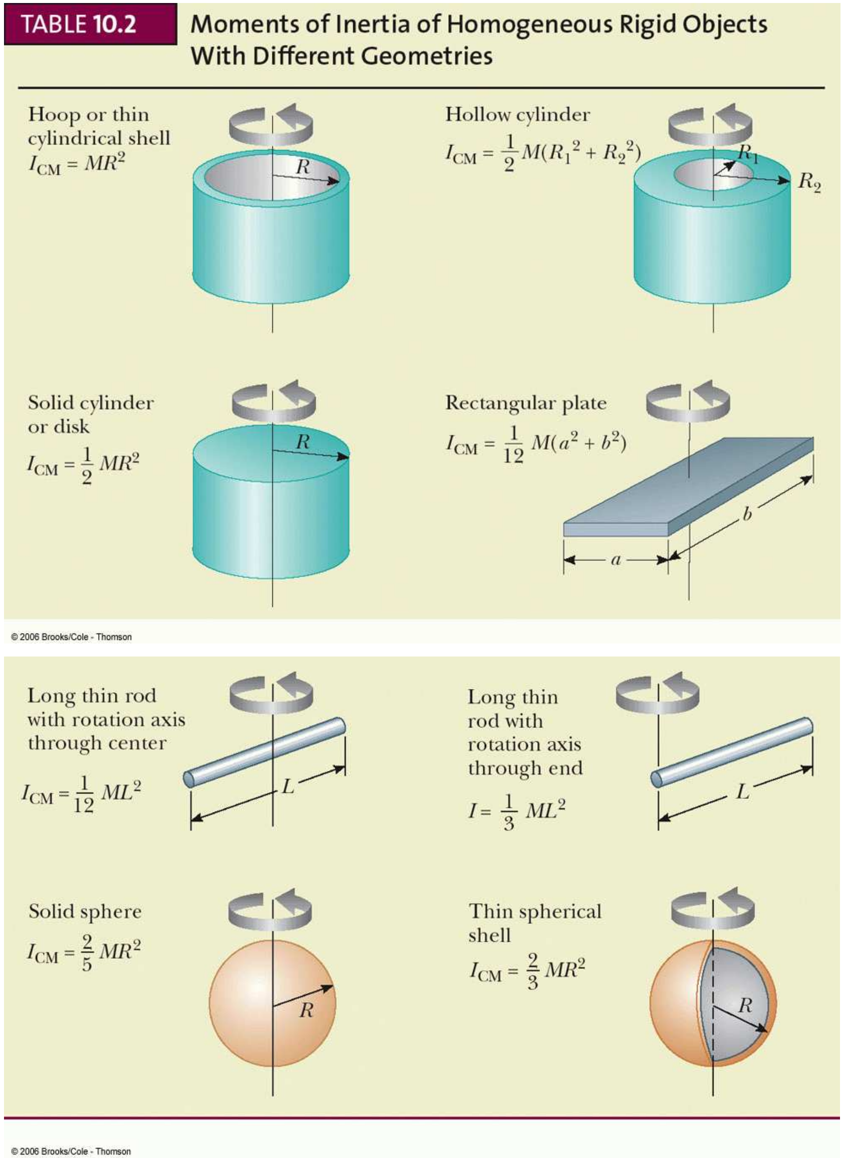
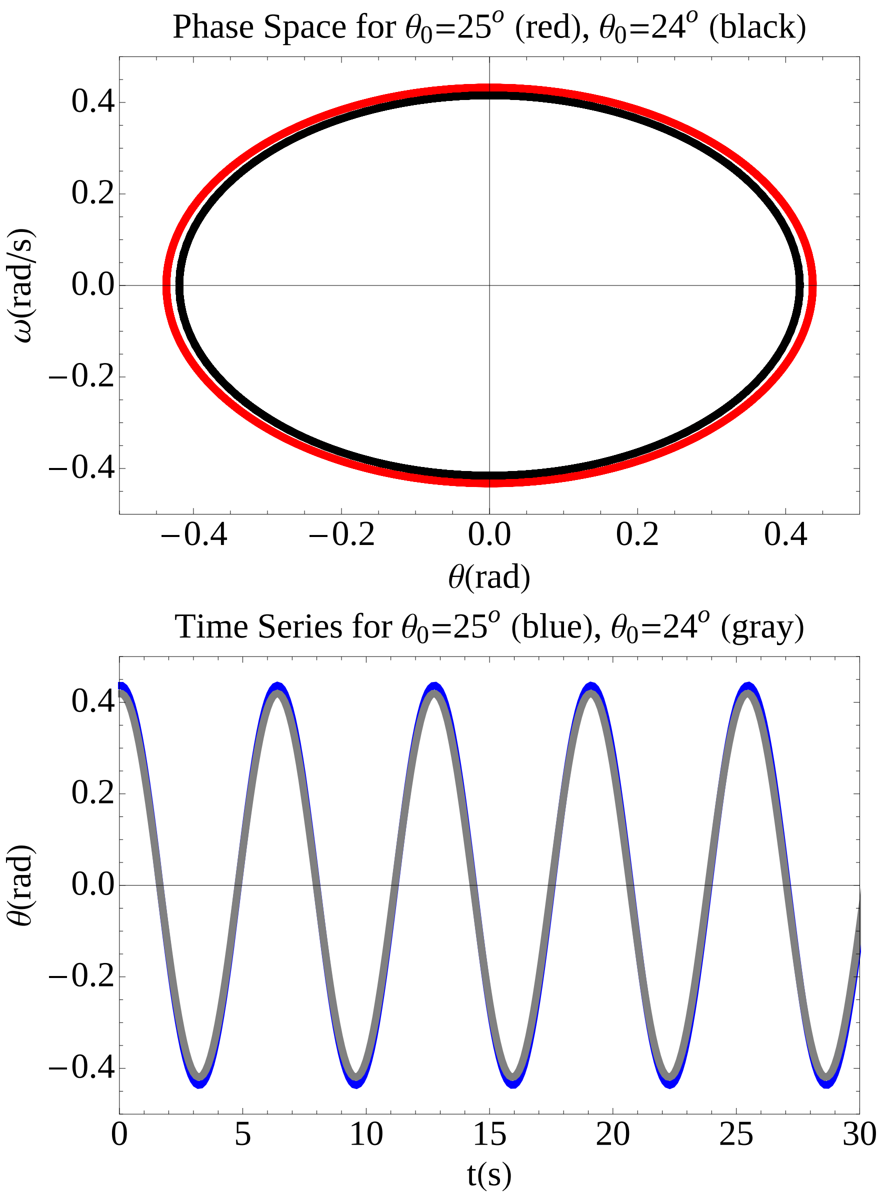


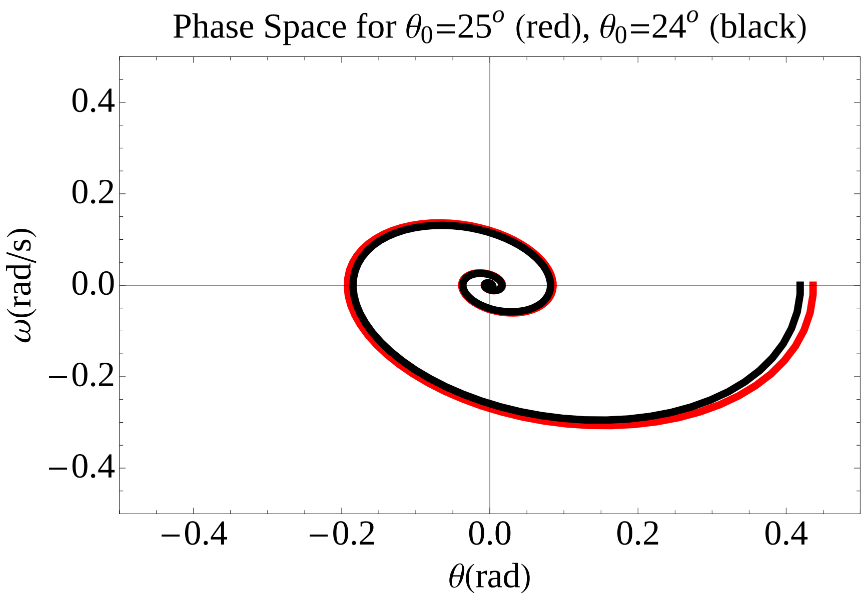
Moments of Inertia

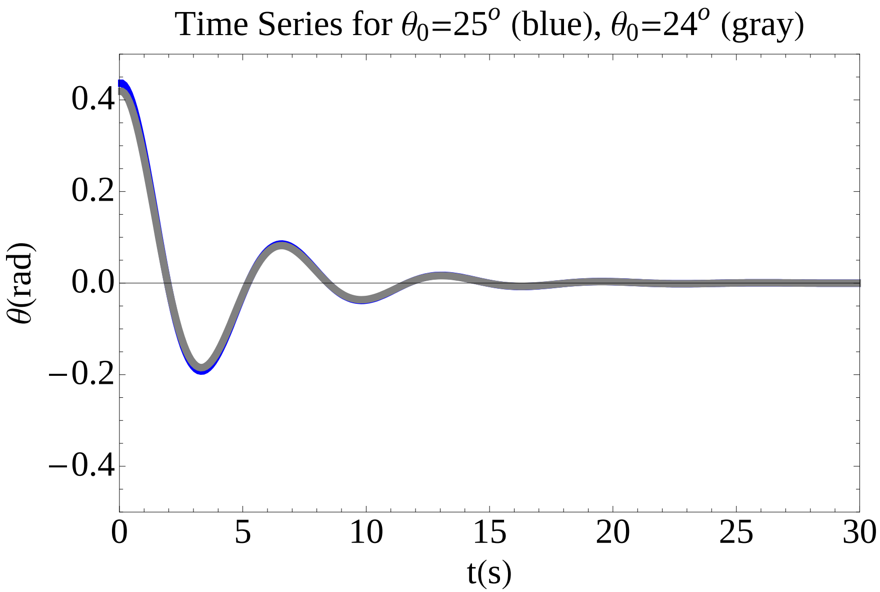


Nonlinear, Physical Pendulum Phase Space and Time Series

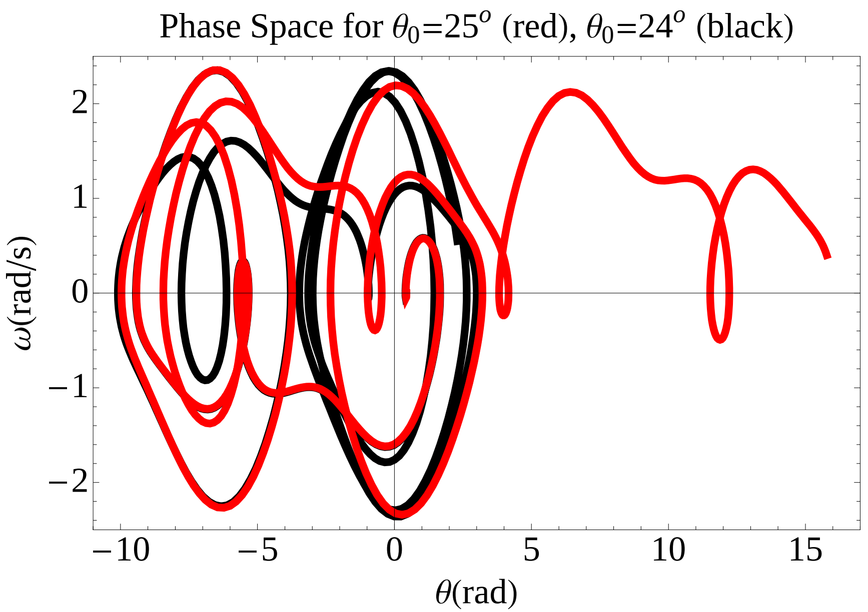


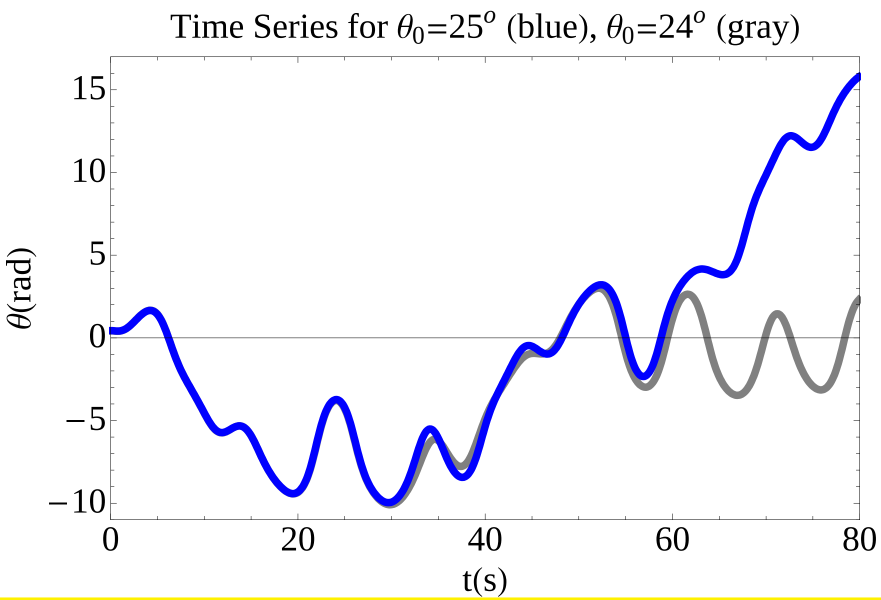
Nonlinear, Physical Pendulum Phase Space and Time Series





Nonlinear, Damped, Driven, Physical Pendulum Phase Space and Time Series





Code for Nonlinear, Damped, Driven, Physical Pendulum

```
(* Initial conditions and parameters *)
th0 =  25.0*Pi/180; (* initial position in meters *)
w0 = 0.0; (* initial velocity in m/s *)
t0 = 0.0; (* initial time in seconds *)
grav = 9.8; (* acceleration of gravity *)
length = 14.7; (* length of pendulum *)
mass = 0.245; (* mass of pendulum *)

(* driving force amplitude and friction force. See below for more *)
qDrag = 0.6; (* drag coefficient *)
DriveForce = 11.8; (* DriveForce = 11.8; cool plot value *)
DriveFreq = 0.67; (* driving force angular frequency *)
DrivePeriod = 2*Pi/DriveFreq; (* period of the driving force *)

(* step size *)
step = 0.10;
```

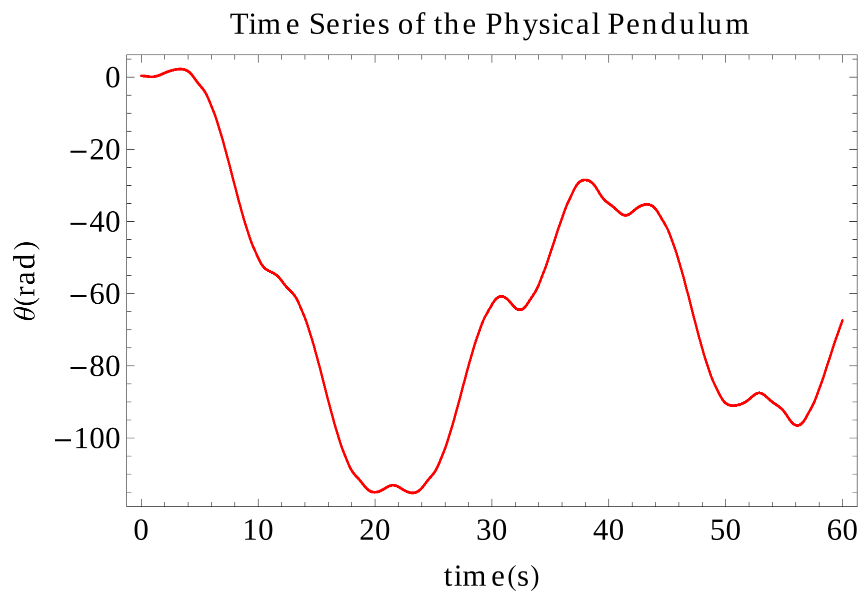
Code for Nonlinear, Damped, Driven, Physical Pendulum

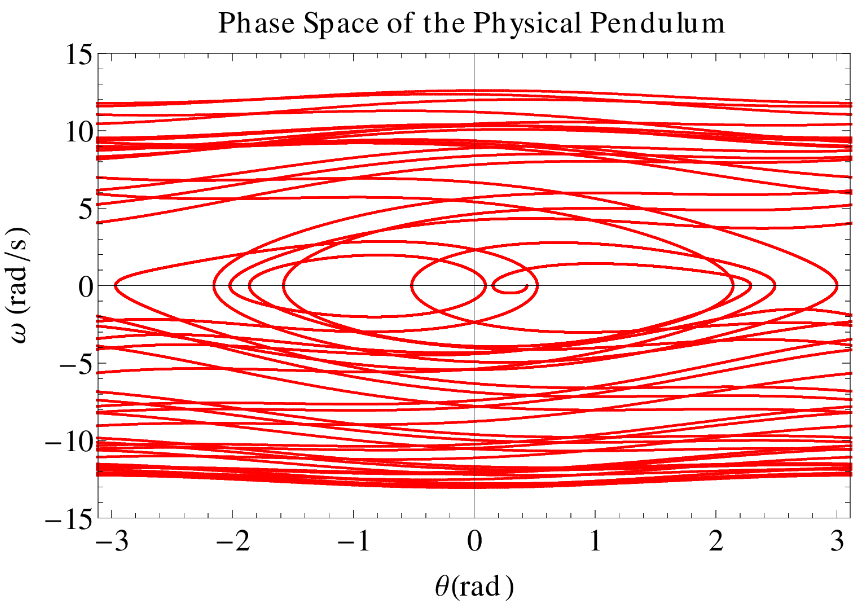
```
(* limits of the iterations. since we already have theta(t=0) and we
have calculated theta(t=step) then the first value in the table will
be for t=2*step. *)
tmin = 2*step;
tmax = 80.0;

(* condense the constants into coefficients for the appropriate terms. *)
f1 = 1 + (3*qDrag*step/(2*mass*length));
f2 = 3*DriveForce*(step^2)/(2*length);
f3 = -3*grav*(step^2)/(2*length);
f4 = -1 + (3*qDrag*step/(2*mass*length));

(* set up the first two points. *)
t1   = t0 + step;
th1 = th0 + w0*step;
(* get rid of the previous results for the table and proceed *)

Clear[pdispl]
Clear[tdispl]
```

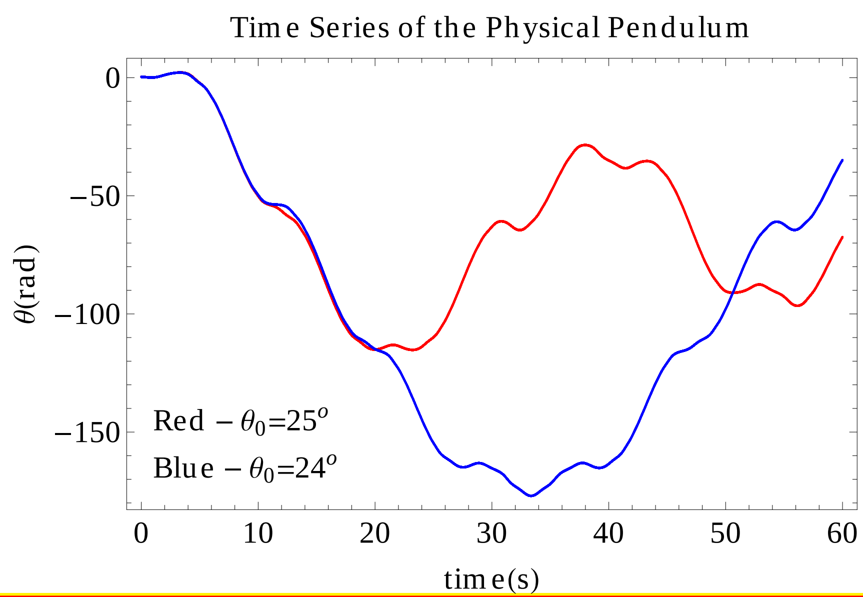
Code for Nonlinear, Damped, Driven, Physical Pendulum

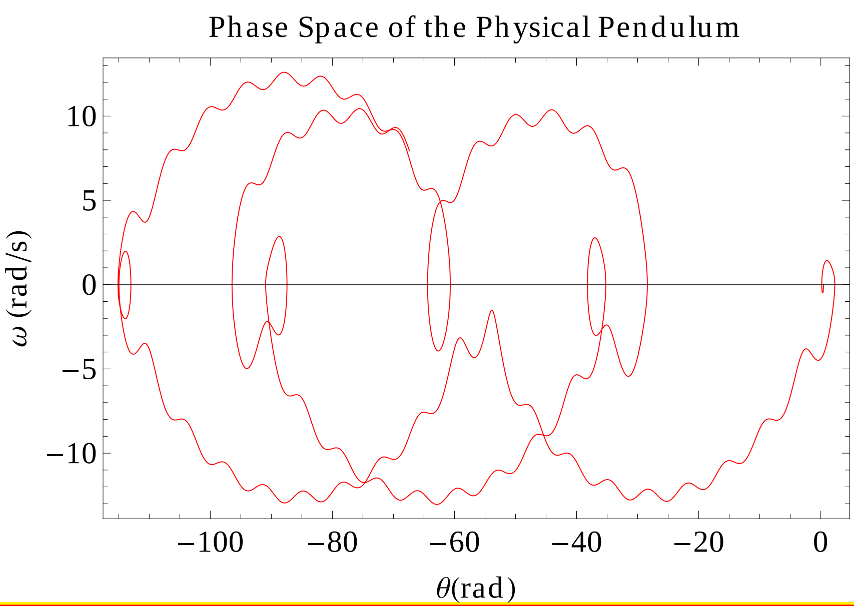
```
(* A centered second derivative formula is used to generate an iterative
solution for the mass on a spring. first load the starting poin. *)
thmid  = th0;  (*starting value of theta *)
thplus = th1;   (* second value of theta *)
tmid = t0;

(* create a table of ordered (theta,w). for each component the next value
is calculated and then the variables incremented for the next interation.
pdispl = {{th0, w0}};
tdispl = {{t0, th0}};

Do[thminus = thmid;
   thmid      = thplus;
  tmid = tmid + step;
  thplus   =  (f2*Sin[DriveFreq*t] +  2*thmid + f3*Sin[thmid] +
      f4*thminus)/f1; wmid = (thplus - thminus)/(2*step);
  pdispl = Append[pdispl, {thmid, wmid}] ;
  tdispl = Append[tdispl, {tmid, thmid}] ,
  {t, tmin, tmax, step}
  ];
```

Chaos Lab 1 Results

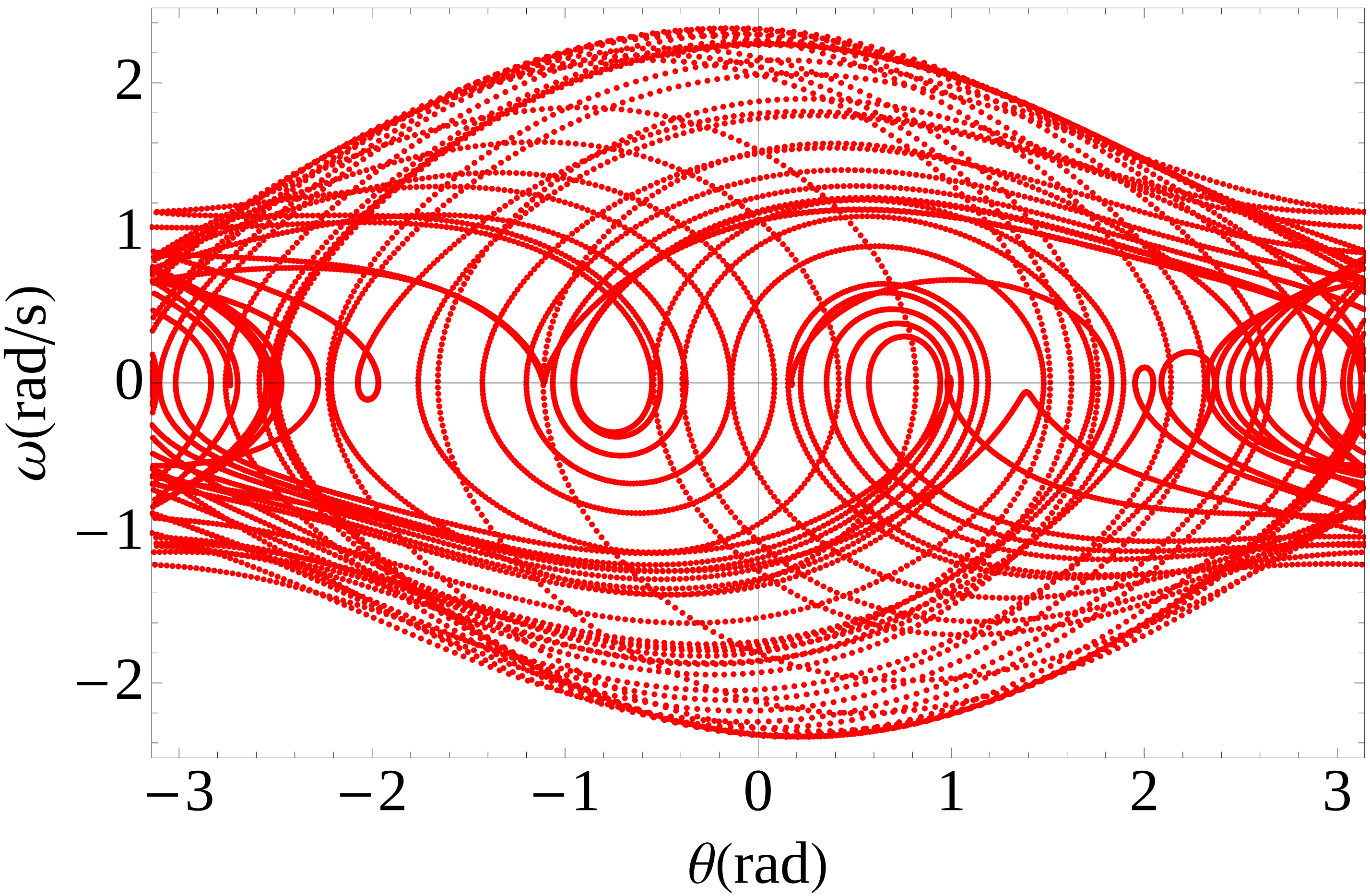






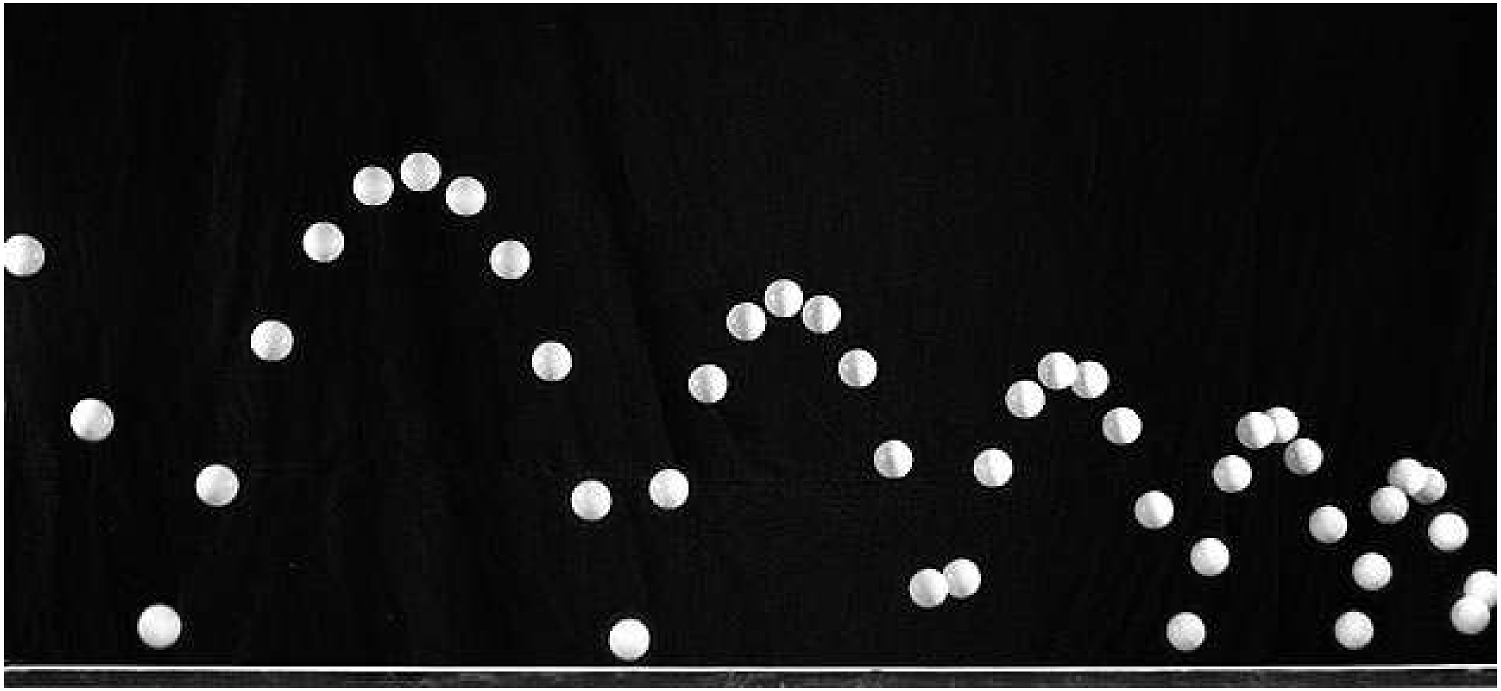


Visualizing Chaos - The Phase Space Trajectory

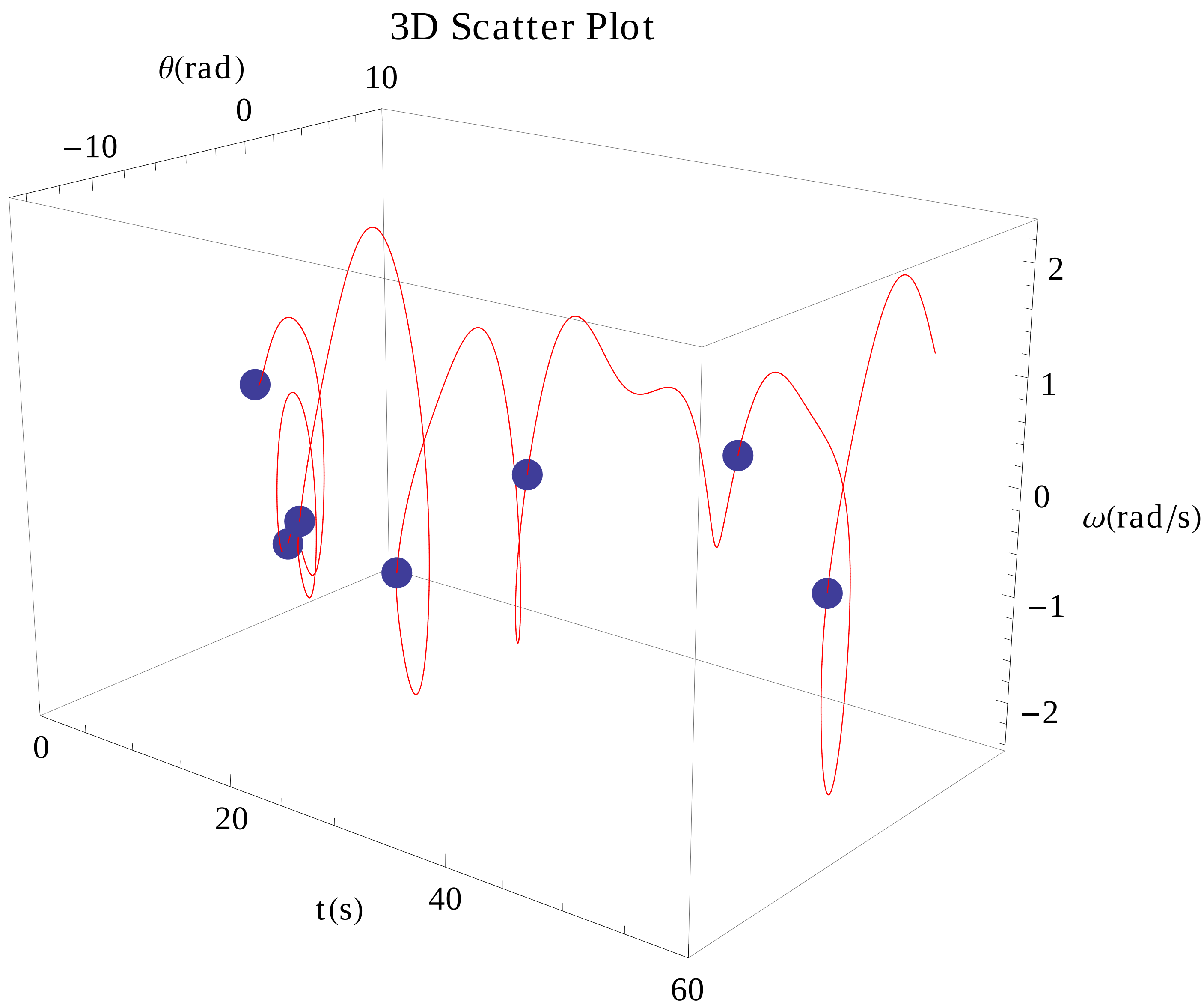


$$\theta_0 = 10^\circ$$

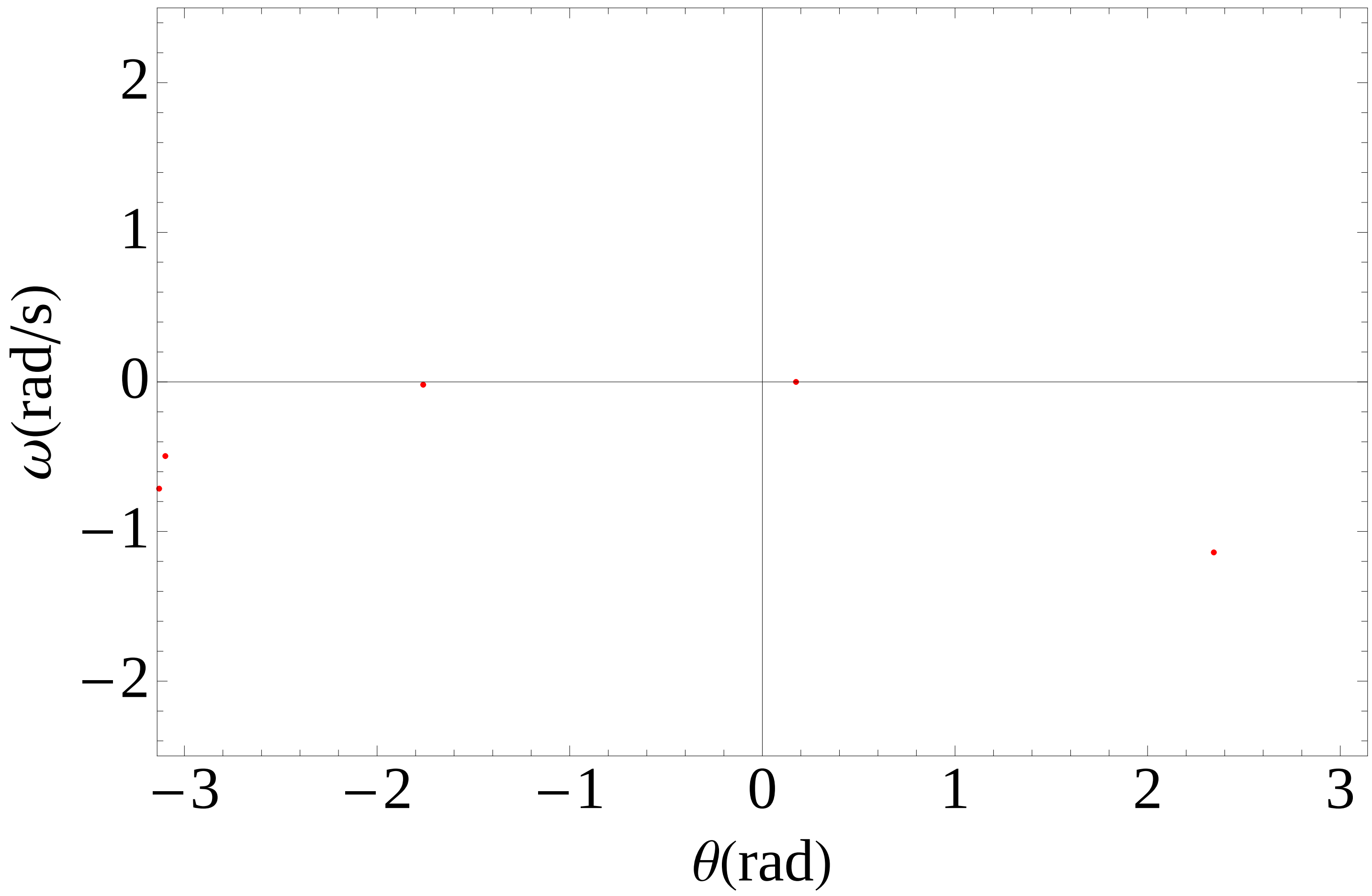
Visualizing Chaos - Stroboscopic Pictures



Visualizing Chaos - Stroboscopic Pictures

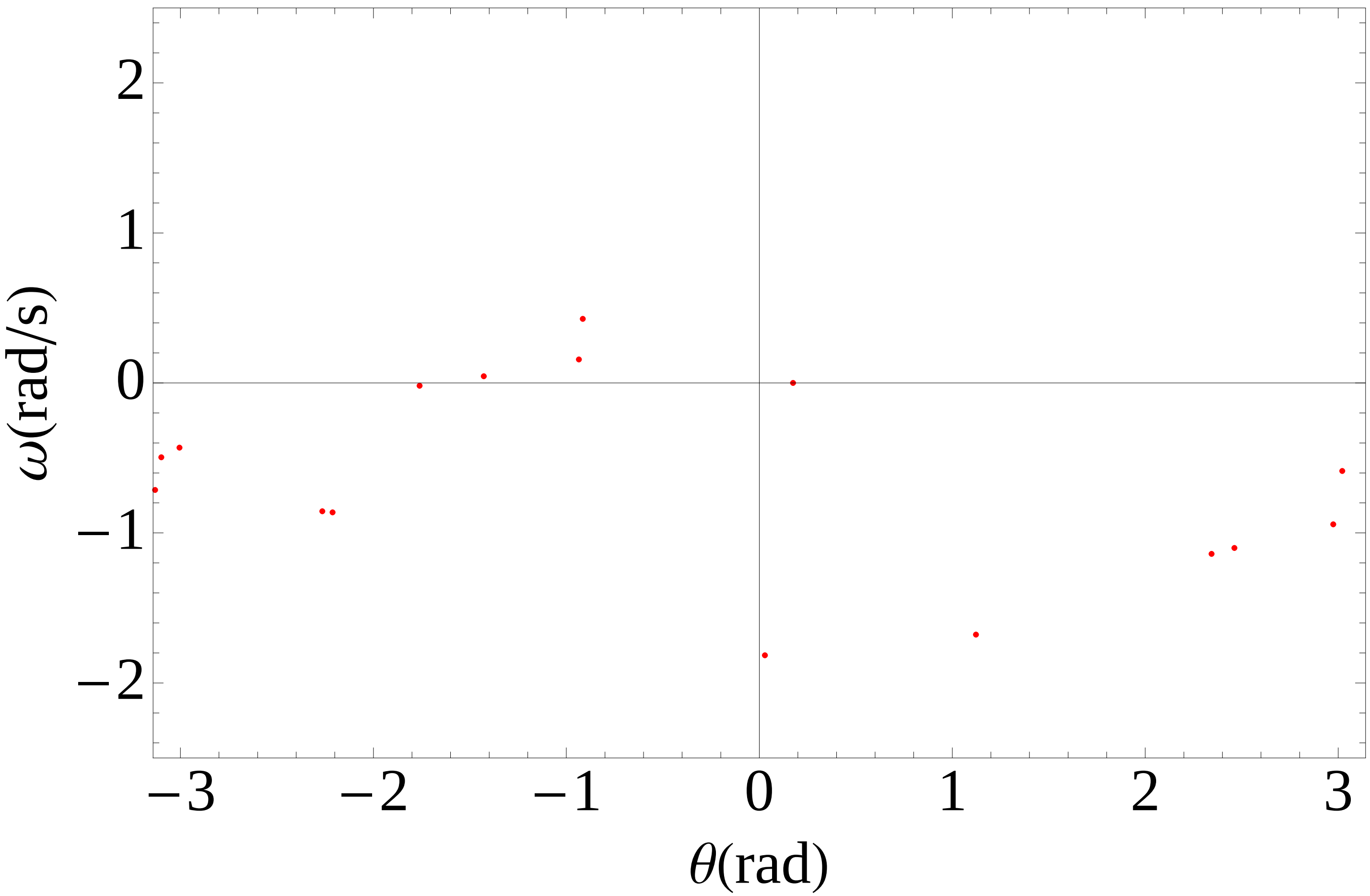


Visualizing Chaos - The Poincare Section



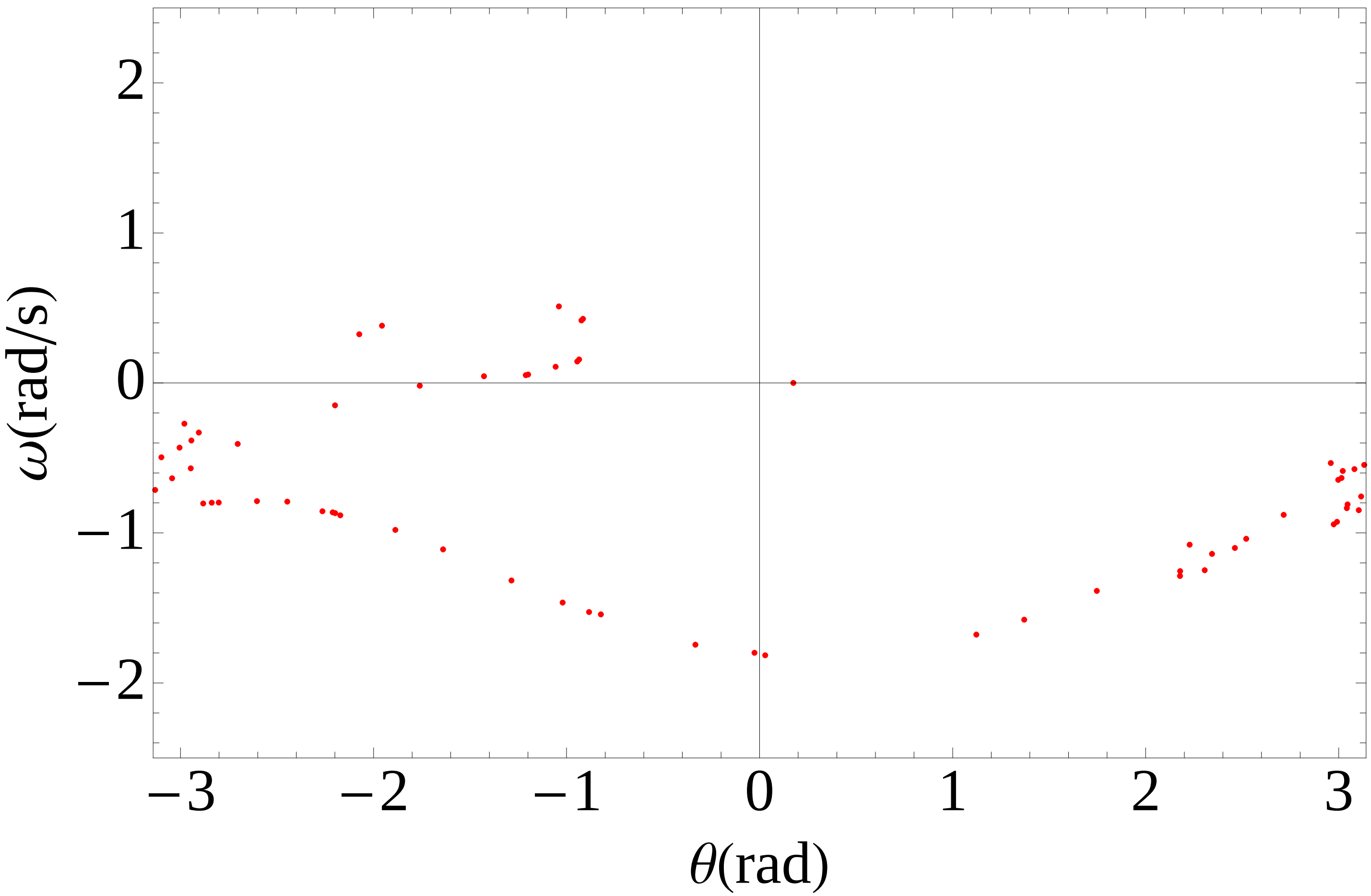
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



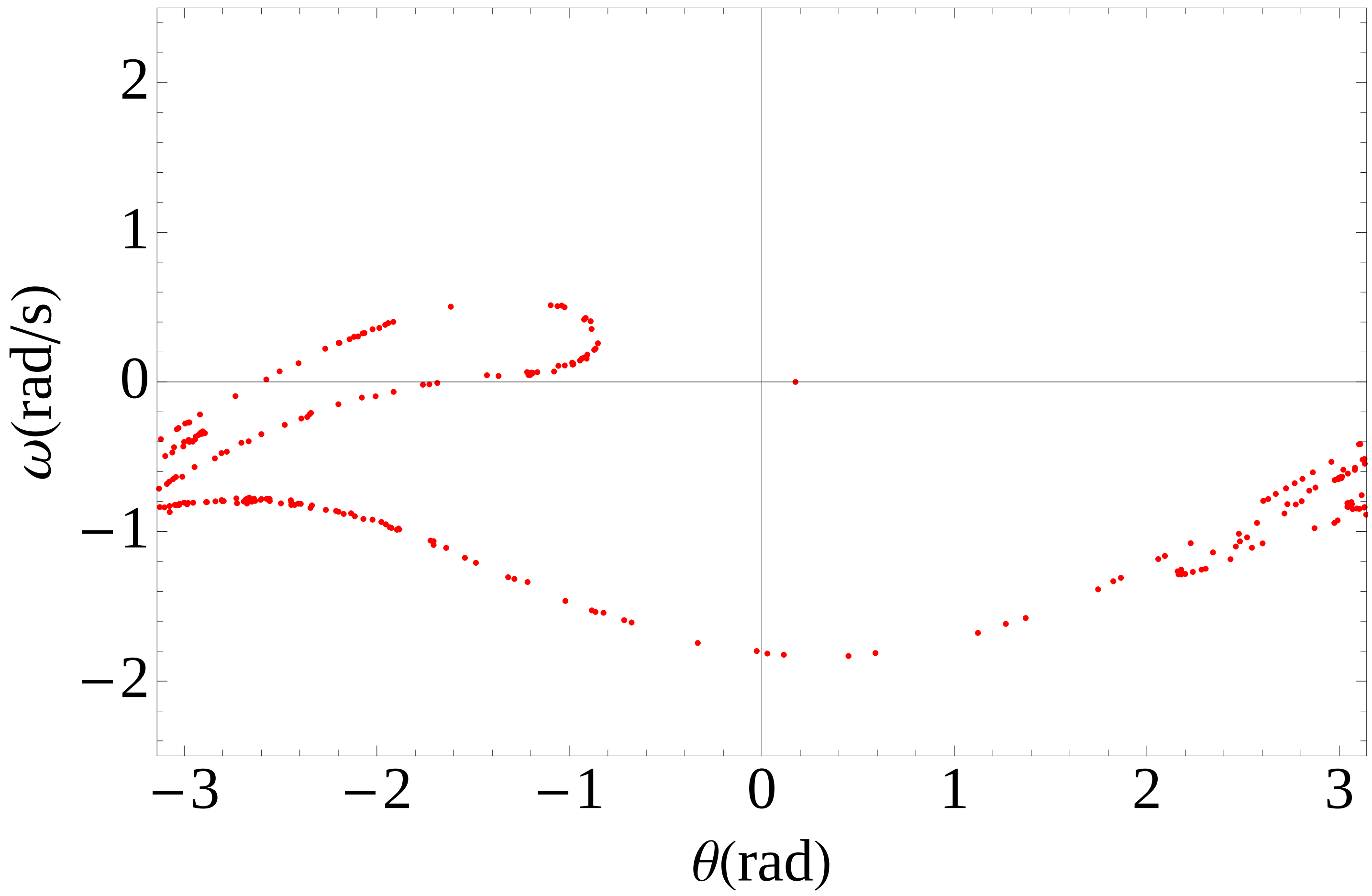
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



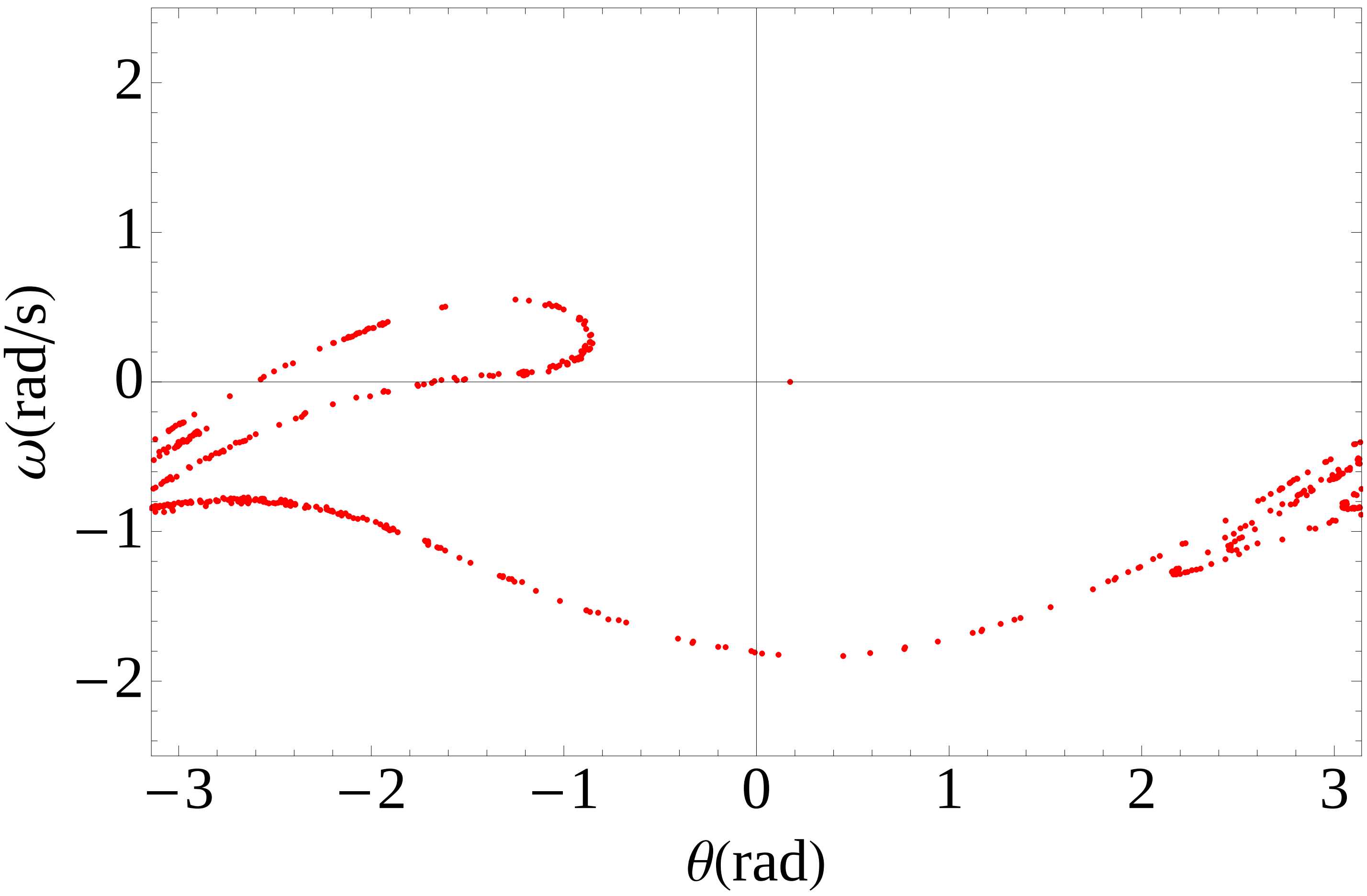
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



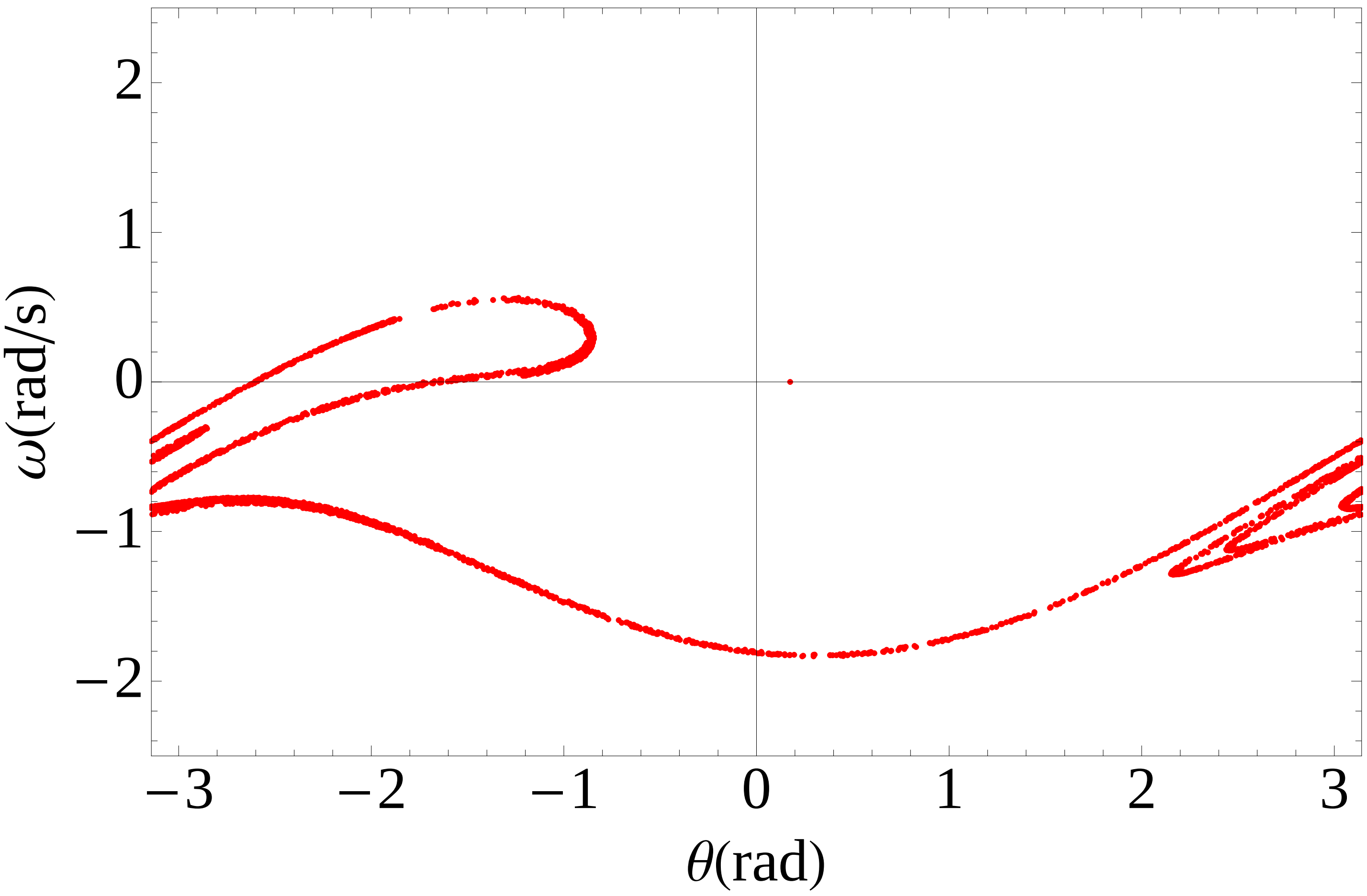
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



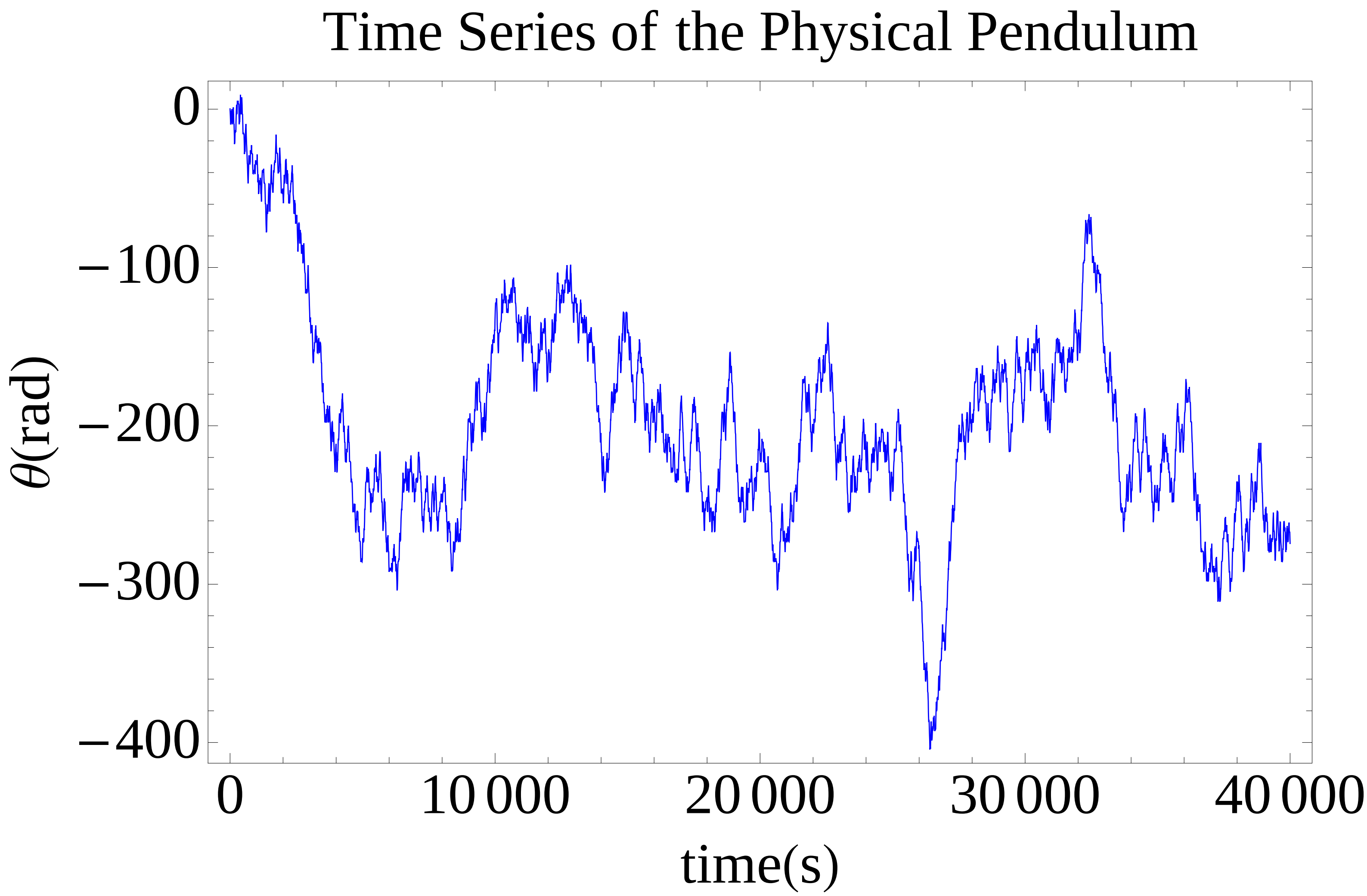
$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Poincare Section



$$\theta_0 = 10^\circ$$

Visualizing Chaos - The Time Series



Calculating Chaos - The Poincare Series - 1

```
(* initial conditions and parameters *)
t0 = 0.0; x0 = 1.0; v0 = 0.2; step = 0.01;

(* get the second and third points on the curve *)
t1 = t0 + step;
x1 = x0 + step*v0;
x2 = 2*x1 - x0 - (step*step*x1);
v1 = (x2 - x0)/(2*step);

(* put the first point in the table *)
MyTable = {{x0, v0}, {x1, v1}};

(* rename stuff for the first point of the algorithm *)
xminus = x0;
xmid = x1;
xplus = x2;
tmin = t1 + step;
tmax = 50.0;
```

Calculating Chaos - The Poincare Series - 2

```
(* Use a Do loop and store the points when t = n\[Pi]. A centered
formula is used to approximate the second derivative. Set parameters
needed to test when to store the data. *)
TimeTest = Pi;
PeriodCounter = 1;

(* main loop. *)
Do[xminus = xmid;
       xmid = xplus;
       xplus = 2*xmid - xminus - (step*step*xmid);
       vmid = (xplus - xminus)/(2*step);
    If[t > TimeTest,
            MyTable = Append[MyTable, {xmid, vmid}];
            PeriodCounter = PeriodCounter + 1;
            TimeTest = PeriodCounter*2*Pi
          ],
       {t, tmin, tmax, step}
   ]
```

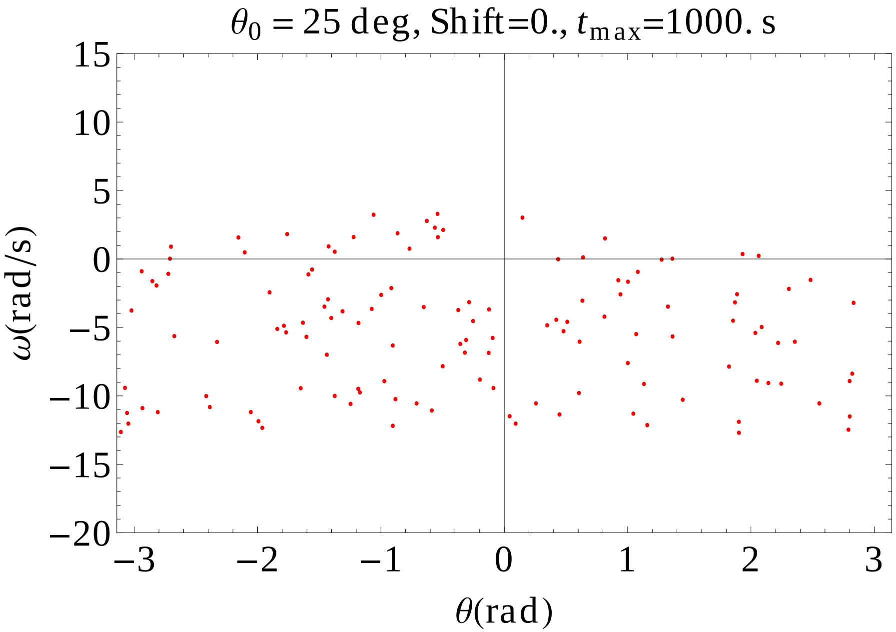
Harmonic Oscillator With Coupled Equations - 1

```
(* Solving the mass on a spring problem.
   Initial conditions and parameters *)
x0 = 0.0; (*   initial position in meters *)
v0 = 2.0; (* initial velocity in m/s *)
t0 = 0.0; (* initial time in seconds *)

(* set up the first two points.
   step size *)
step = 0.1;
t1 = t0 + step;
x1 = x0 + v0*step;
v1 = v0 - ( step*kspring*x0/mass);

xminus = x0;   (* initial value of x *)
vminus = v0;   (* initial value of v *)
xmid   = x1;
vmid   = v1;
mass = 0.33; (* the mass in kg *)
kspring = 0.5; (* spring constant in N/m *)
```

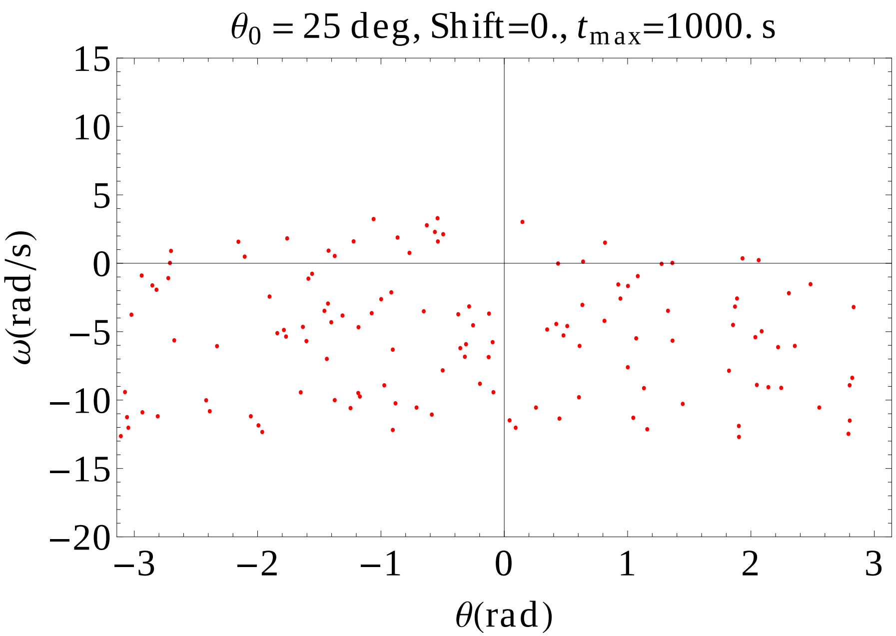
Harmonic Oscillator With Coupled Equations - 2

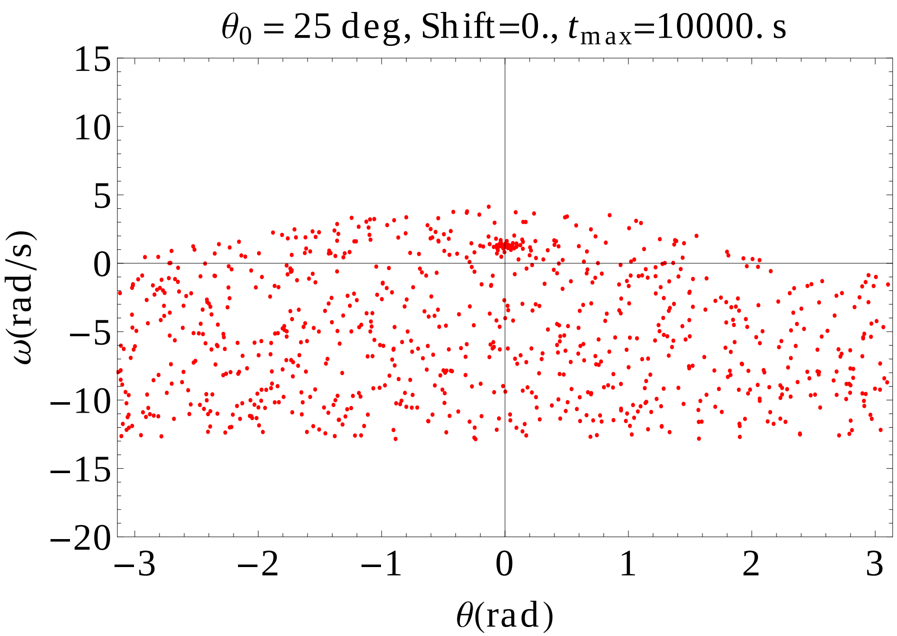
```
(* limits of the iterations. since we already have y(t=0) and we
    have calculated y(t=step), then the first value in the table
    will be for t=2*step. *)
tmin = 2*step;
tmax = 25.0;

(* create a table of ordered (t,x). for each component the next value is
   calculated and then variables are incremented for the next interation.
tpos = Table[
    {t,
      vplus = vminus - (2*step*kspring/mass)*xmid;
      xplus =  xminus + (2*step*vmid);
      vminus = vmid;
      vmid  = vplus;
      xminus = xmid;
      xmid = xplus
     },
    {t, tmin, tmax, step}
   ];
```

Chaos Lab 2 Results



Chaos Lab 2 Results





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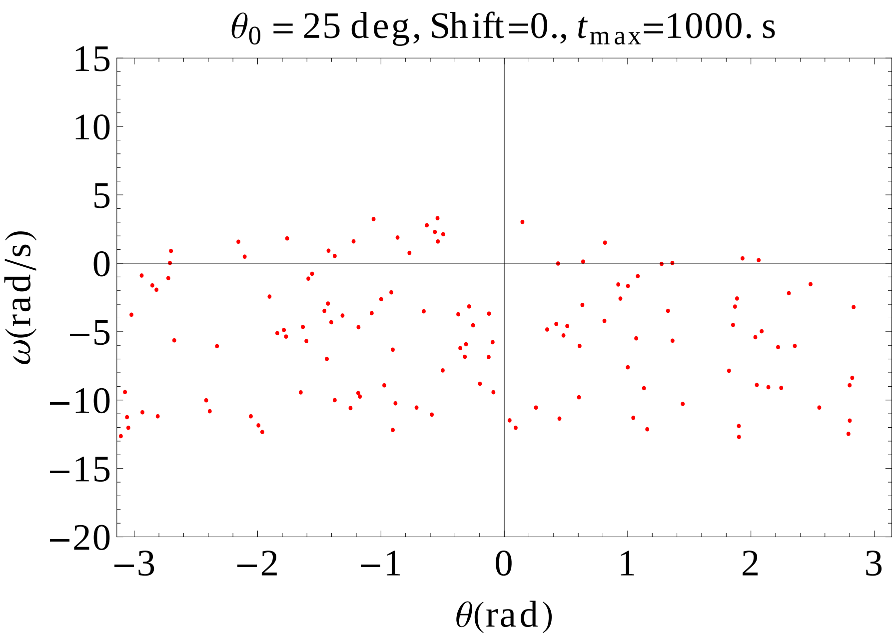

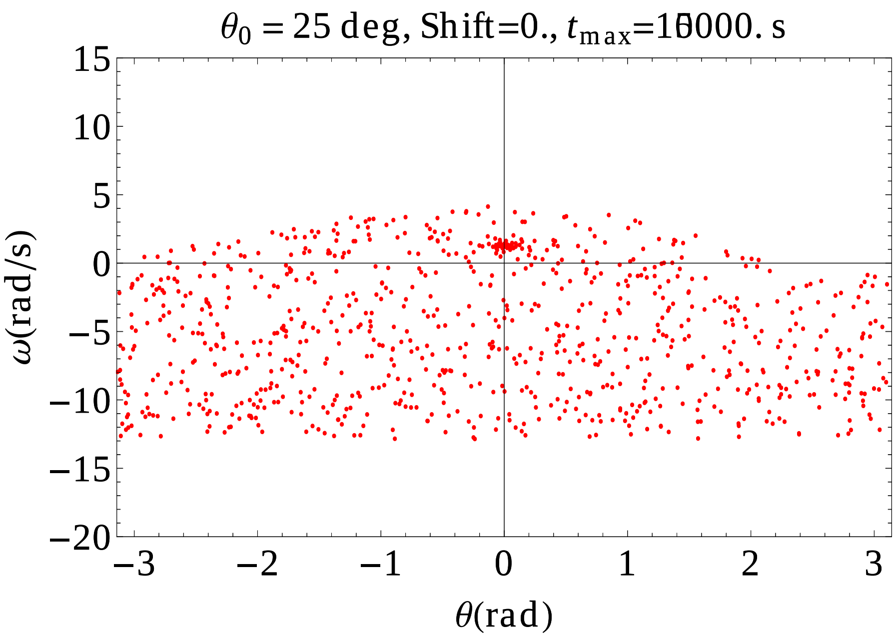

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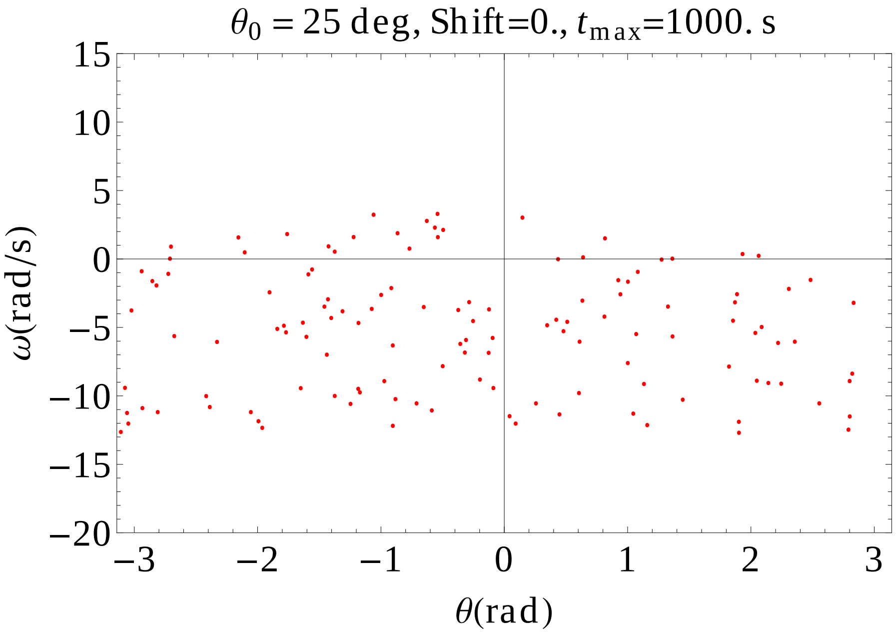

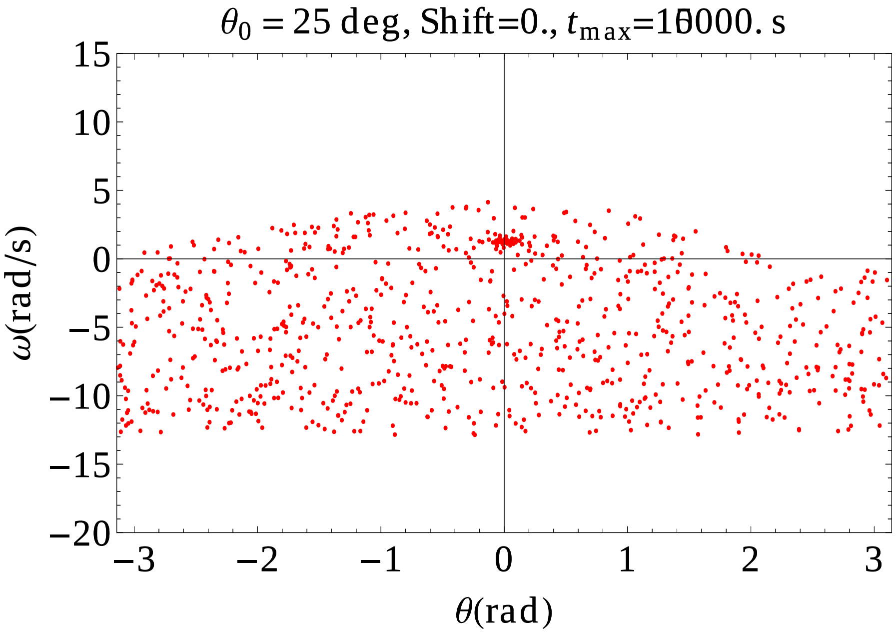

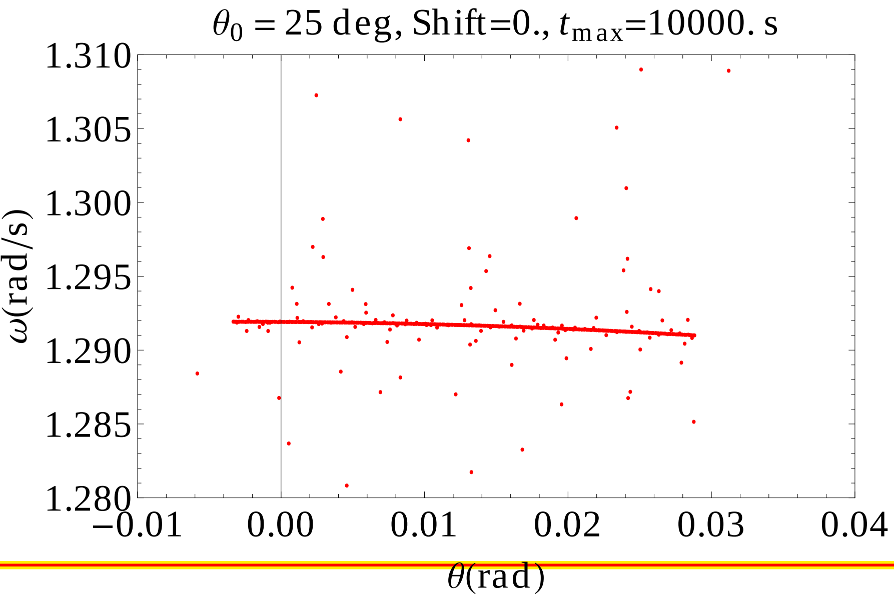

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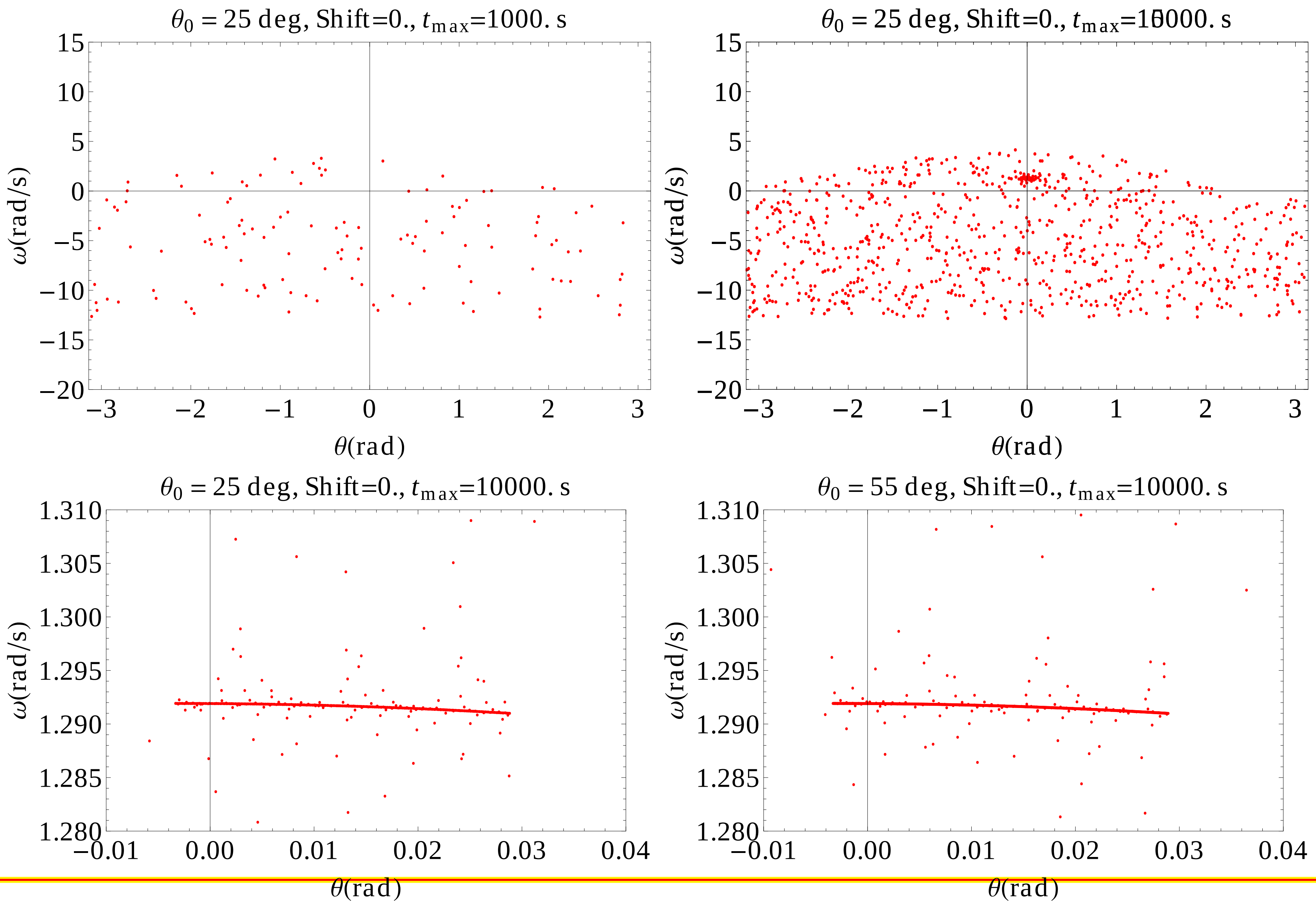
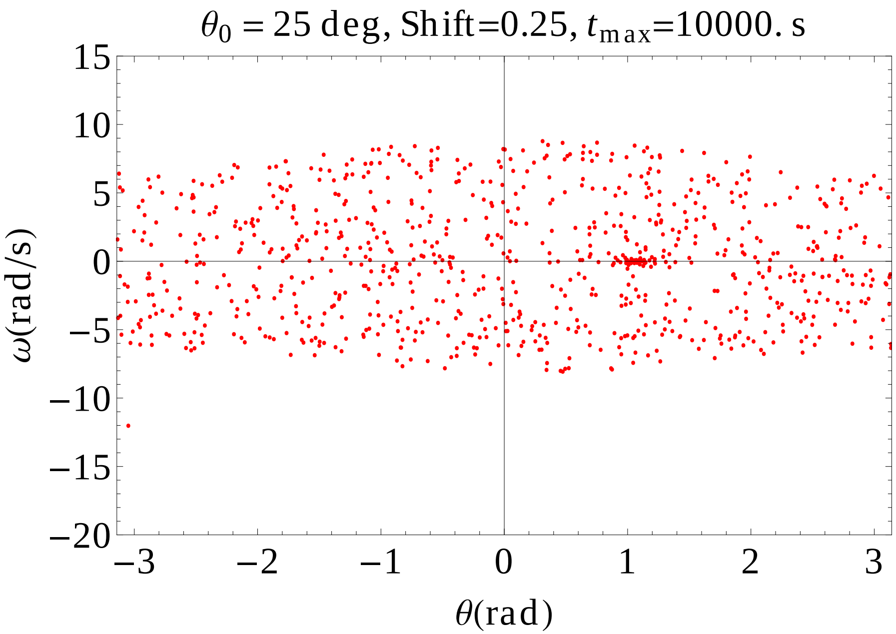


Chaos Lab 2 Results



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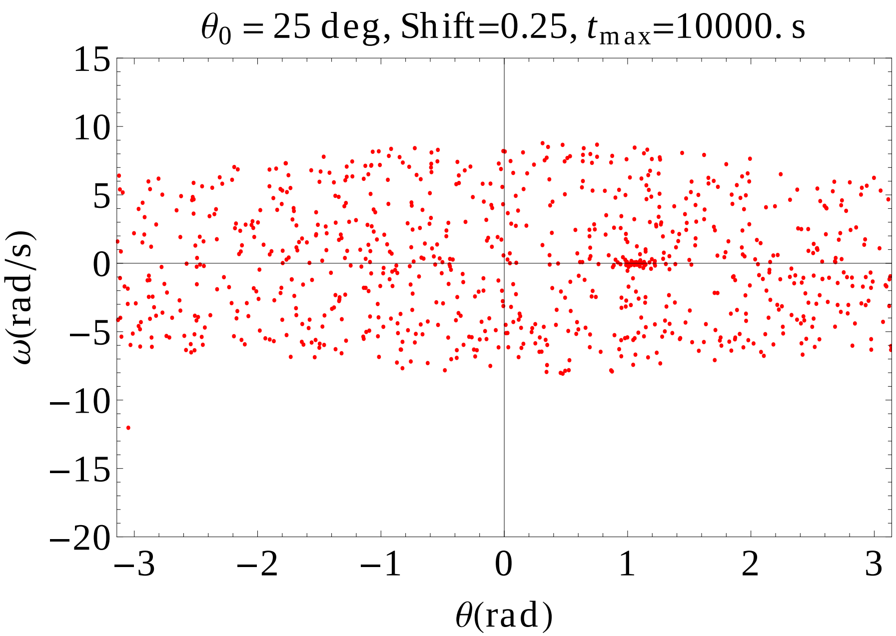
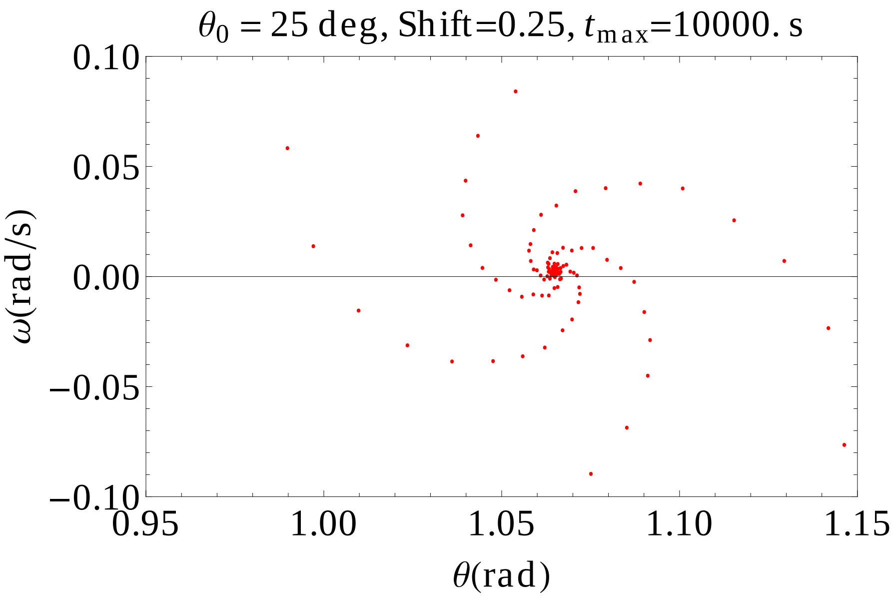
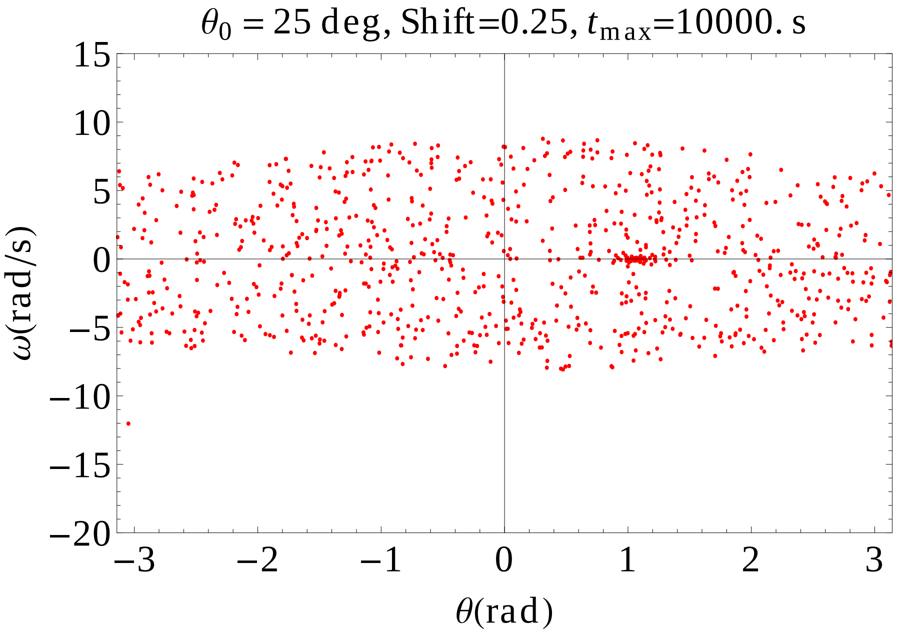
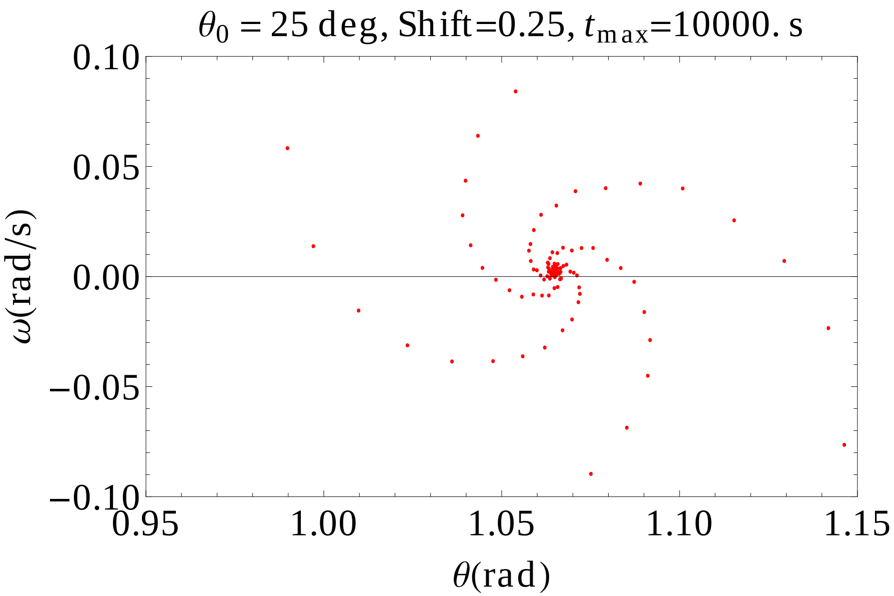
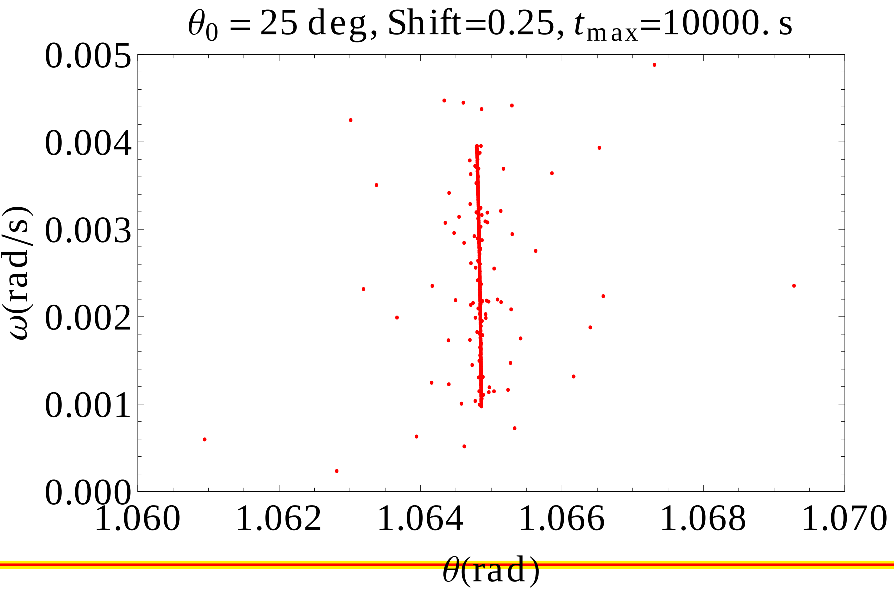




Chaos Lab 2 Results







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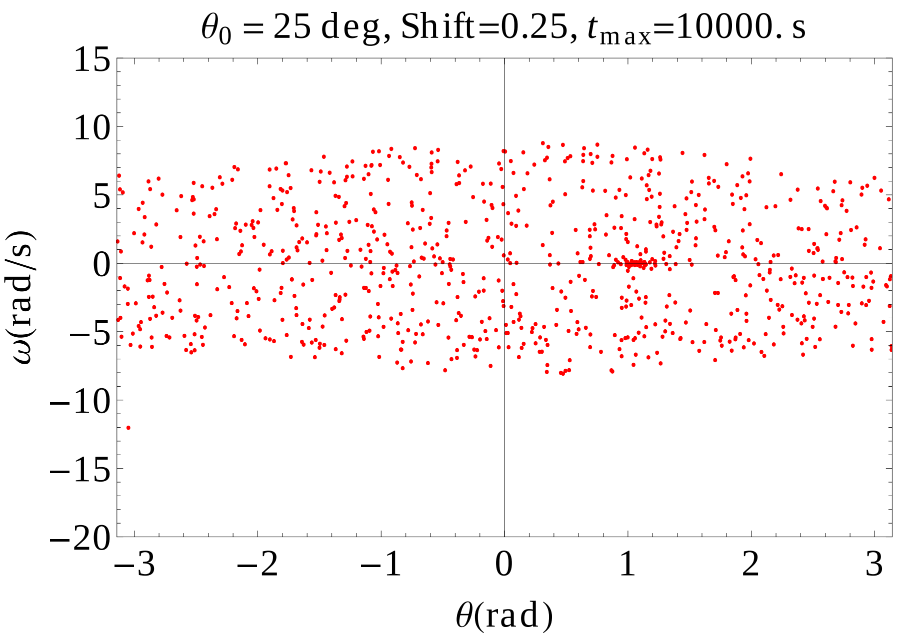
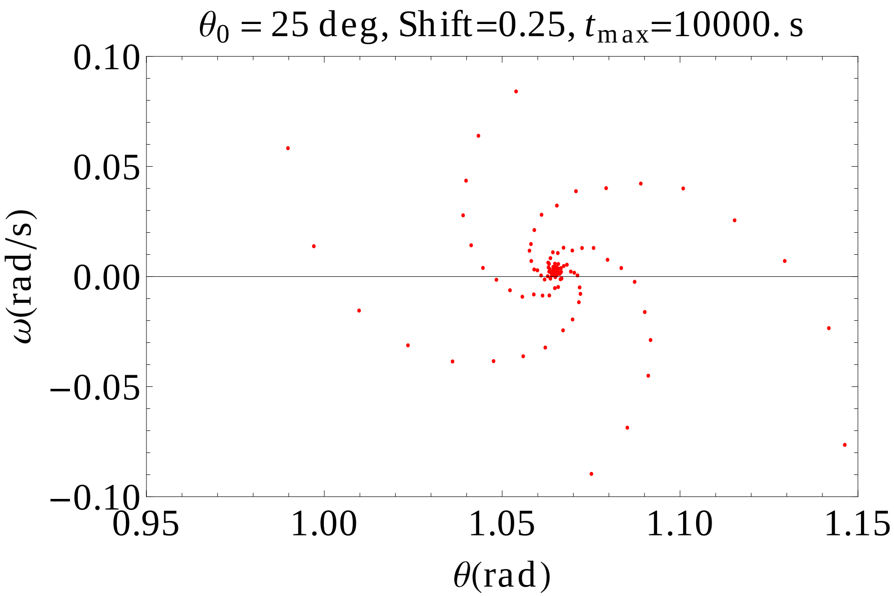
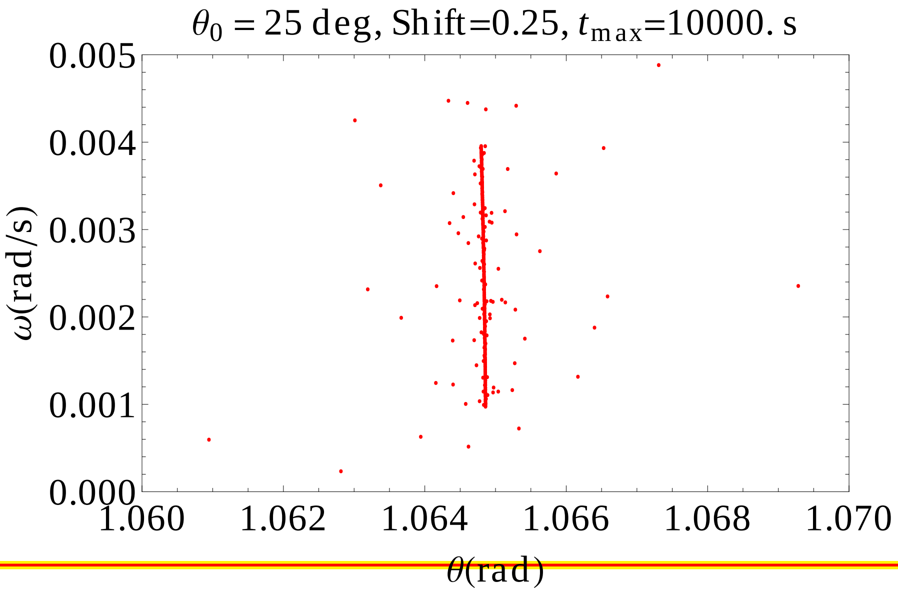
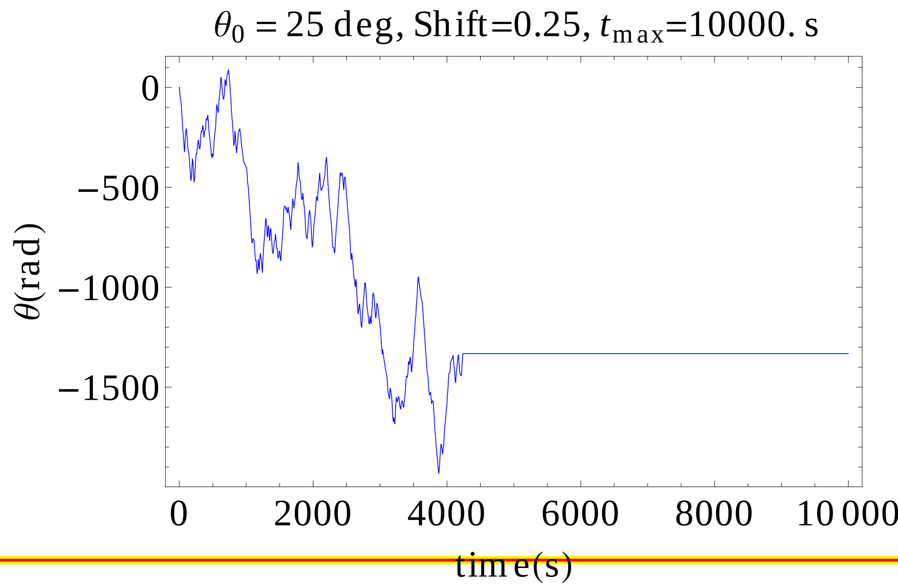
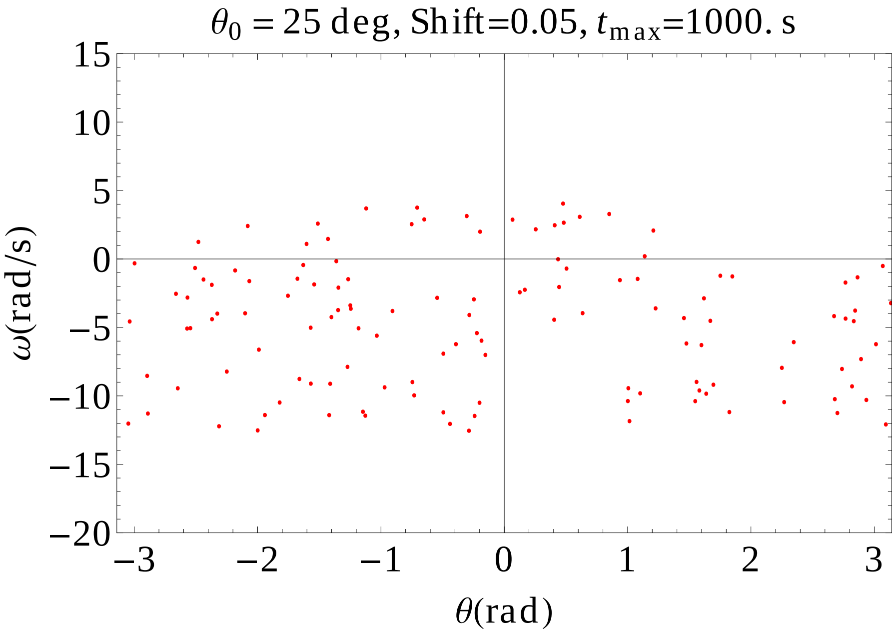




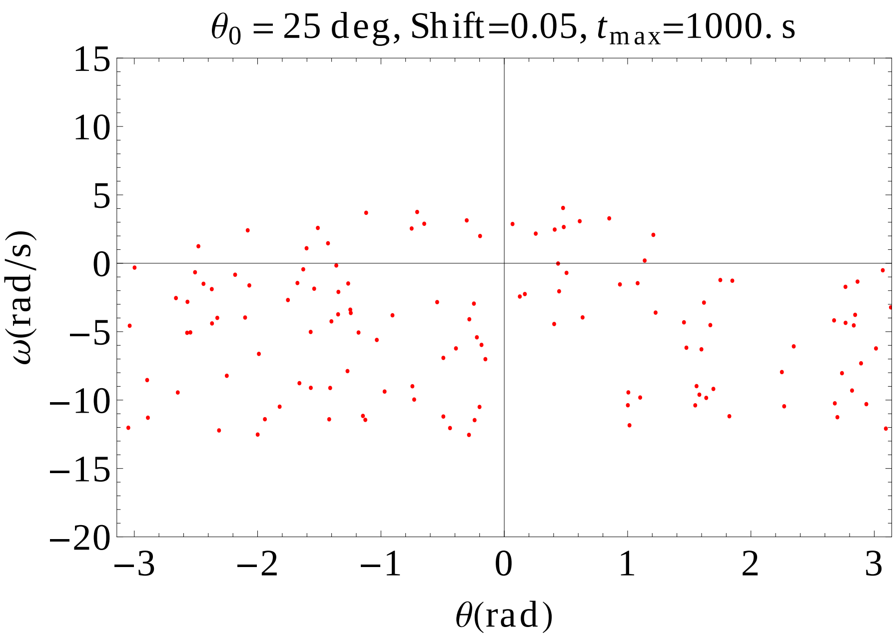


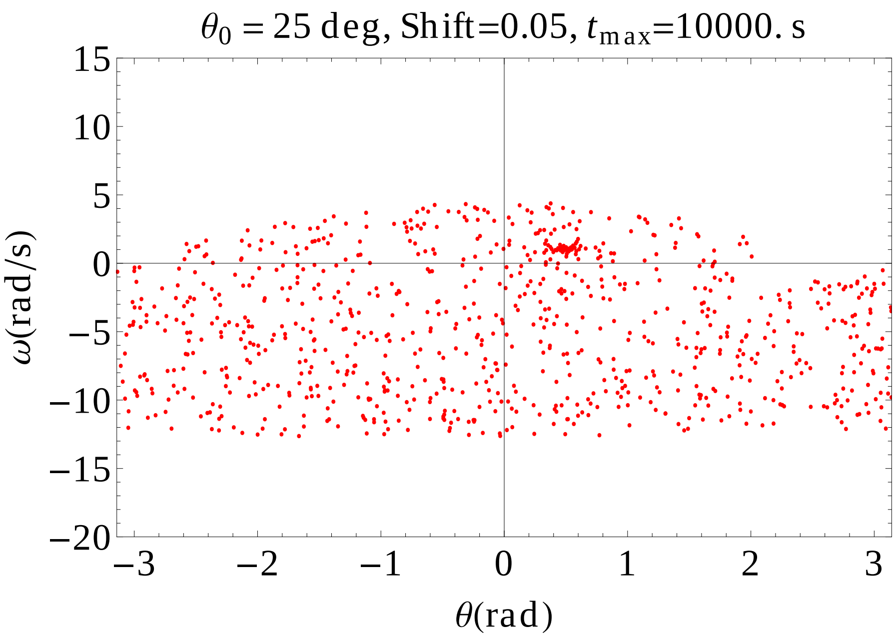


Chaos Lab 2 Results

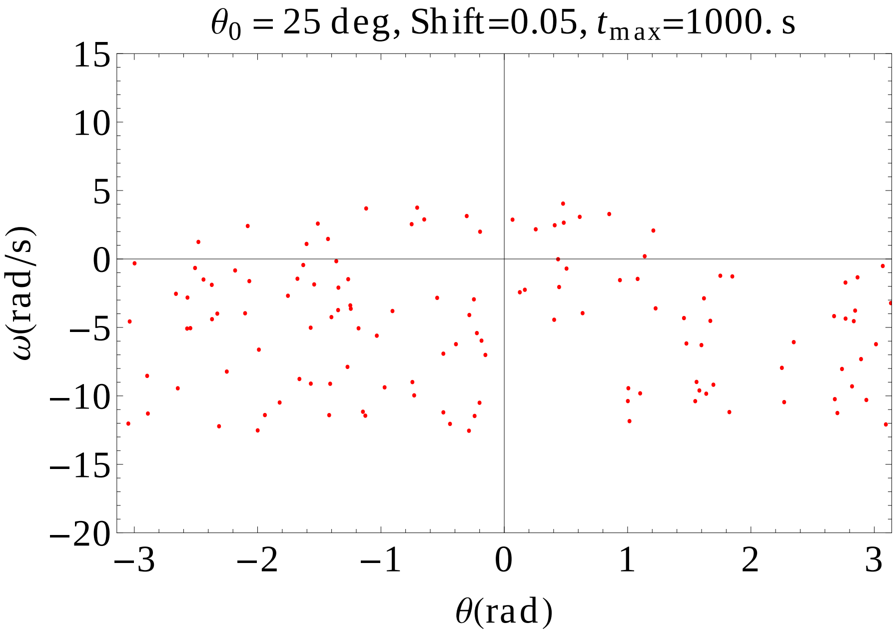


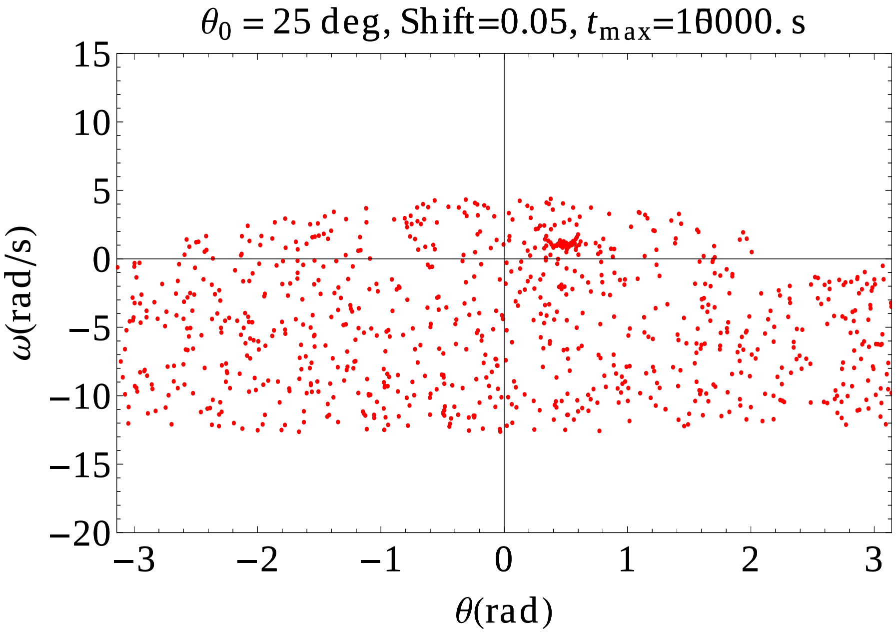
Chaos Lab 2 Results



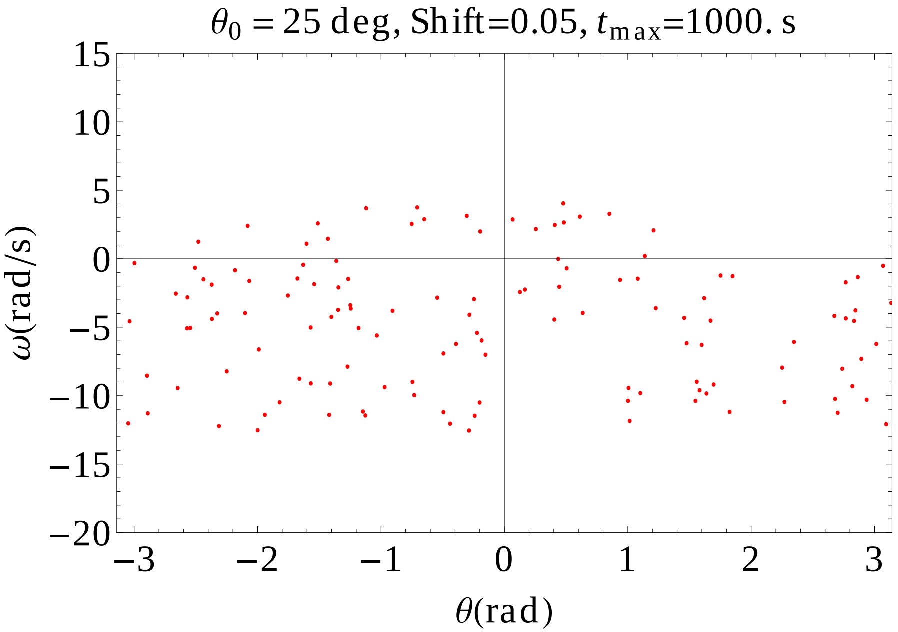


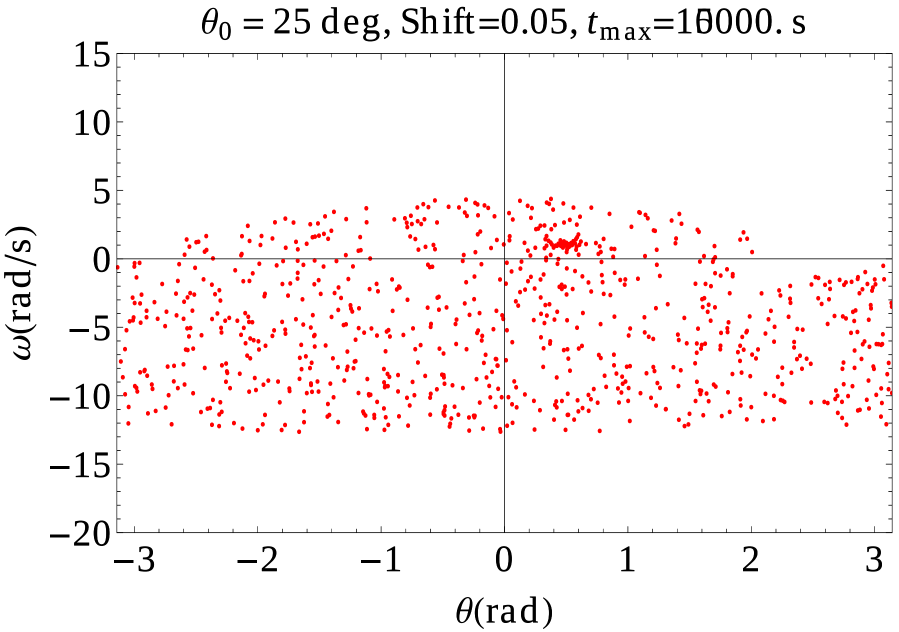
Chaos Lab 2 Results

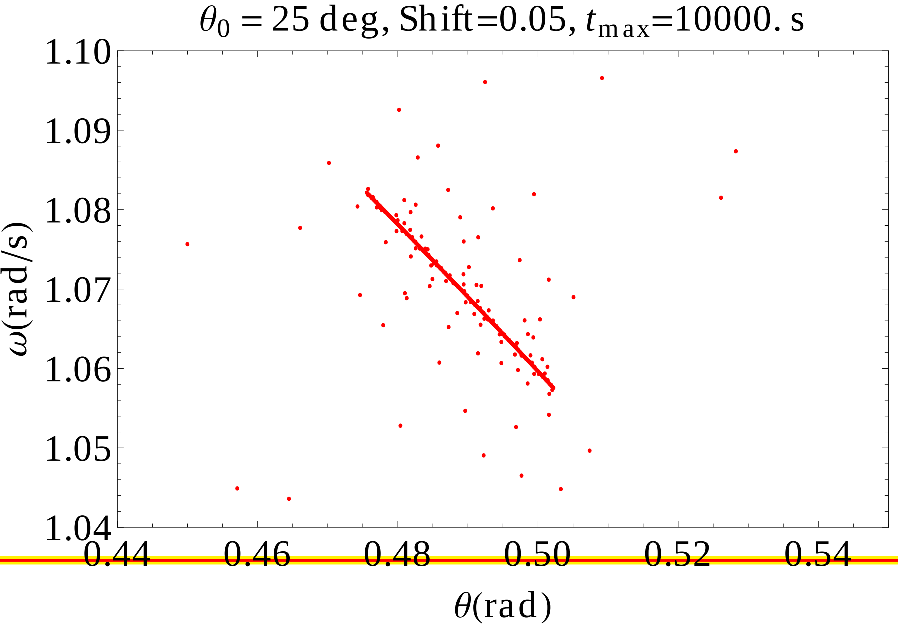




Chaos Lab 2 Results







Chaos Lab 2 Results