Uncertainty

1. Introduction

The notion of uncertainty is a central feature of quantum mechanics. Here we consider the 'chopped beam' problem where a wave packet occupies a region of space and we generate an uncertainty relationship between the spatial and momentum distributions.

2. Plotting Commands

A command for making plots is shown below with some new features demonstrated. The plot is enclosed in a frame and the labeling includes parameters used in the plot.

```
In[ ]:= f[k_] := Sin[k]^2;
L0 = 10;
k0 = 5;
p1 = Plot[f[k], {k, 0, 10},
  PlotRange → {0, 1.8},
  Frame → True,
  FrameLabel → {"k", "|b(k)|^2", StringForm["L=`` k0=``", L0, k0]}
 ]
```

3. The Uncertainty Principle.

Now consider a case where the potential is *V=0* and a particle is initially confined to the region $-L/2 \leq x \leq L/2$ where the size of the box remains finite. For the wave packet defined by the following function.

$$\psi(x) = \frac{1}{\sqrt{L}} e^{ik_0 x} \qquad -L/2 \leq x \leq L/2$$

$$\psi(x) = 0 \qquad \text{otherwise}$$

1. What is the spectral distribution (i.e. the spectrum of wave numbers) necessary to produce such a wave packet? Use the standard eigenfunctions of the free particle, $\phi(x) = e^{ikx} / \sqrt{2\pi}$. Your answer should be in symbolic form.

2. Plot the spectral distribution for the values of *L=10 angstroms* and $k_0 = 5$ *inverse angstroms*. Does

the distribution go to zero on either side of the central peak? If not, you have a mistake somewhere.

3. Get a symbolic expression for the values of k of the minima on either side of the central peak using the result of section 3.1. What is the separation between these minima? Treat this separation as the width Δk of the distribution.

3. Get an expression/formula for the width Δx of the probability distribution $|\psi(x)|^2$ using the result of Section 3.1. Recall that, in general, the width Δx is the standard deviation Δx so you can calculate

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

where the brackets imply averaging over all x.

4. Generate an uncertainty principle appropriate for this wave packet.

5. What happens to the position (x) and wave number (k) distributions and your uncertainty principle if you set k_0 to zero?

Include purpose, calculations, figures, and a discussion in your report.

1. Purpose: Generate an uncertainty principle for the 'chopped beam' problem.

2. Uncertainty Principle

$$\psi(x) = \frac{e^{ik_0 x}}{\sqrt{L}} \qquad -L/2 < x < L/2 \qquad \begin{matrix} L = 10\,\text{Å} \\ k_0 = 5\,\text{Å}^{-1} \end{matrix}$$

$$= 0 \qquad \text{otherwise}$$

$$b(k) = \langle \varphi(k) | \psi \rangle = \int_{-L}^{L} \frac{e^{-ikx}}{\sqrt{2\pi}} \frac{e^{ik_0 x}}{\sqrt{L}}\, dx$$

$$= \frac{1}{\sqrt{2\pi L}} \int_{-L}^{+L} e^{i(k_0 - k)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi L}} \frac{e^{i(k_0-k)L/2} - e^{-i(k_0-k)L/2}}{i(k-k_0)}$$

let $\Delta k = k_0 - k$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$b(k) = \frac{1}{\sqrt{2\pi L}} \frac{\cancel{2i}\sin\frac{\Delta k L}{2}}{\cancel{i}\Delta k} \frac{\frac{L}{2}}{\frac{L}{2}}$$

$$= \sqrt{\frac{L}{2\pi}} \frac{\sin \Delta k L/2}{\Delta k L/2}$$

$$\text{Probability Density (Spectral Dist)} = b^2(k) = \frac{L}{2\pi} \frac{\sin^2\left(\frac{\Delta k L}{2}\right)}{\left(\frac{\Delta k L}{2}\right)^2}$$

See Fig 1. It shows the probability density peaked k_0 and going to zero at minimum to either side of the peak.

Minima in the $|b(k)|^2$ distribution occur when

$$\sin \frac{\Delta k L}{2} = 0 \quad \text{or} \quad \frac{\Delta k L}{2} = \pm n\pi$$
$$n = 0, 1, 2, \ldots$$

The first minima correspond to $n = \pm 1$

$$\Delta k = \pm \frac{2\pi}{L}$$

$$k_0 - k = \pm \frac{2\pi}{L}$$

$$k = k_0 \mp \frac{2\pi}{L}$$

$$\Delta k = k_+ - k_-$$
$$= k_0 + \frac{2\pi}{L} - \left(k_0 - \frac{2\pi}{L} \right)$$

$$\boxed{\Delta k = \frac{4\pi}{L}} \xrightarrow[\text{half-width}]{\text{take the}} \Delta k = \frac{2\pi}{L} = \frac{2\pi}{10 Å} = 0.63 Å^{-1}$$

Get Δx

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} |\psi|^2 x \, dx$$

$$= \int_{-L/2}^{+L/2} \frac{e^{-ik_0 x}}{\sqrt{L}} \frac{e^{ik_0 x}}{\sqrt{L}} x \, dx$$

$$= \frac{1}{L} \frac{x^2}{2} \Big|_{-L/2}^{L/2}$$

$$= \frac{1}{2L} \left(\frac{L^2}{4} - \frac{L^2}{4} \right)$$

$$\boxed{\langle x \rangle = 0}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} |\psi|^2 x^2 \, dx$$

$$= \int_{-L/2}^{L/2} \frac{1}{L} x^2 \, dx$$

$$\langle x^2 \rangle = \frac{1}{L} \frac{x^3}{3} \Big|_{-L/2}^{L/2}$$

$$= \frac{1}{3L}\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right]$$

$$= \frac{1}{3L} \frac{L^3}{4}$$

$$\boxed{\langle x^2 \rangle = \frac{L^2}{12}}$$

$$\therefore \; \Delta x^2 = \frac{L^2}{12}$$

$$\boxed{\Delta x = \frac{L}{\sqrt{12}}}$$

Uncertainty relationship

$$\Delta x \, \Delta k = \frac{L}{\sqrt{12}} \cdot \frac{2\pi}{L} = \frac{2\pi}{\sqrt{12}}$$

$p = \hbar k$ so $\Delta p = \hbar \Delta k$

$$\Delta x \Delta p = \Delta x (\hbar \Delta k)$$

$$= \hbar \, \Delta x \, \Delta k$$

$$= \frac{2\pi \hbar}{\sqrt{12}}$$

$1.814 \hbar$

$1.9128 \times 10^{-34} \; \frac{m^2 kg}{s}$

Out[ ]=