

PARTICLE PHYSICS

2003

PART 2 (Lectures 2-4)

KINEMATICS

K. VARVELL

UNITS IN PARTICLE PHYSICS

SI System of units

The base quantities are mass, length and time. Units for these are chosen to be on a “human” scale.

Quantity	Dimensions	Units
Mass	[M]	1 kg
Length	[L]	1 m
Time	[T]	1 s

Other quantities are derived, e.g.

Quantity	Dimensions	Units
Velocity	[LT ⁻¹]	1 m s ⁻¹
Ang. Mom.	[ML ² T ⁻¹]	1 kg m ² s ⁻¹
Energy	[ML ² T ⁻²]	1 J

SI units are not really convenient for particle physics!

e.g.

Mass of proton $m_p = 1.67 \times 10^{-27}$ kg

“Radius” of proton $r_p \approx 0.8 \times 10^{-15}$ m

Natural Units

Particle physics relies heavily on Special Relativity and Quantum Mechanics. Formulae in Special Relativity are littered with the quantity “speed of light in vacuum”

$$c = 3.0 \times 10^8 \text{ms}^{-1}$$

and those in Quantum Mechanics with Planck’s Constant h , or

$$\hbar = h/2\pi = 1.055 \times 10^{-34} \text{Js}$$

3

In natural units we set

$$\boxed{\hbar \equiv c \equiv 1}$$

in all equations, e.g.

$$E^2 = p^2 c^2 + m^2 c^4$$

becomes

$$E^2 = p^2 + m^2$$

What does this really mean?

One way to view it is to regard energy, velocity and angular momentum as base quantities rather than mass, length and time.

4

Quantity	Dimensions	Units
Energy	[E]	1 GeV = 1.602×10^{-10} J
Velocity	[V]	1 “natural velocity unit c ”
Ang. Mom.	[W]	1 “natural ang. mom. unit \hbar ”

Now mass, length, time etc. are derived quantities

Quantity	Dimensions	Units
Mass	[E V ⁻²]	1 GeV (unit of c) ⁻²
Length	[V W E ⁻¹]	1 (unit of c) (unit of \hbar) GeV ⁻¹
Time	[W E ⁻¹]	1 (unit of \hbar) GeV ⁻¹

We don't bother to write “(unit of c)” or “(unit of \hbar)”, and when we substitute for c or \hbar in formulae using these units we just write “1” since e.g. c is “1 unit of c ”.

With this convention, the natural unit of mass is the GeV, and that of length and time GeV⁻¹ (equivalent to 0.1975×10^{-15} m and 0.659×10^{-24} s).

5

In SI units, a proton at rest has energy

$$\begin{aligned}
 E &= m_p c^2 = 1.67 \times 10^{-27} \text{ kg} \times (3.0 \times 10^8 \text{ ms}^{-1})^2 \\
 &= \frac{1.503 \times 10^{-10} \text{ J}}{1.602 \times 10^{-10} \text{ J GeV}^{-1}} = 0.938 \text{ GeV}
 \end{aligned}$$

In natural units we just write

$$E = m_p \quad \text{or} \quad m_p = E = 0.938 \text{ GeV}$$

(sometimes this is written as $0.938 \text{ GeV}/c^2$ to explicitly show it is a mass)

Often we will continue to express time in seconds, and length in *fermis* or *femtometres* (1 fm = 10^{-15} m) since this is a conveniently sized unit for particle physics

$$\text{“Radius” of proton} \quad r_p \approx 0.8 \text{ fm}$$

Having performed calculations using natural unit formulae, to get numerical results we must put back \hbar and c factors. A useful relation for calculations is

$$\hbar c = 0.1975 \text{ GeV fm}$$

6

SPECIAL RELATIVITY AND 4-VECTORS

The particles we deal with in particle physics are usually relativistic, so our theories must be consistent with (special) relativity.

Special relativity mixes up the spatial and time components specifying the coordinates of a particle. Thus we can define a 4-vector in which the zeroth component is time, and the other three components are the three spatial coordinates

$$X = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

which in our “natural” units becomes

$$X = (x^0, x^1, x^2, x^3) = (t, x, y, z)$$

The *length* of a 4-vector is defined as the square root of the dot product

$$X \cdot X = t^2 - x^2 - y^2 - z^2$$

and this is an *invariant, scalar* quantity in special relativity.

7

More formally, X above is an example of a contravariant 4-vector, written

$$X^\mu = (x^0, x^1, x^2, x^3) = (t, \tilde{x}) \quad \text{where } \mu = 0, 1, 2, 3$$

and we can also define a covariant 4-vector

$$X_\mu = (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3) = (t, -\tilde{x})$$

and a *metric tensor*

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

such that

$$X_\mu = g_{\mu\nu} X^\nu \quad X^\mu = g^{\mu\nu} X_\nu$$

We use the Einstein summation convention of summing over repeated indices, in this case ν , with $\nu = 0, 1, 2, 3$. $g^{\mu\nu}$ has the same matrix form as $g_{\mu\nu}$.

8

The dot product of two 4-vectors A and B is thus

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3 = a_0 b^0 - \underset{\sim}{a} \cdot \underset{\sim}{b}$$

4-vectors are not confined to expressing the time and spatial components of a particle. The 4-vector we will meet most in this course is the 4-vector relating the energy and momentum of a particle

$$P^\mu = (E, \underset{\sim}{p}) = (E, p_x, p_y, p_z)$$

Remembering that the dot product of two 4-vectors is invariant,

$$P_\mu P^\mu = EE - \underset{\sim}{p} \cdot \underset{\sim}{p} = E^2 - p^2 \equiv m^2$$

where $p = |\underset{\sim}{p}|$ and m is the *rest mass* of the particle.

In non-natural units, you may have met this equation as

$$E^2 = p^2 c^2 + m^2 c^4$$

9

RELATIVISTIC TRANSFORMATIONS

For an observer in a frame of reference S' moving along the z axis with velocity β with respect to frame S , the energy and momentum in S and S' are related through the transformations

$$\begin{array}{ll} E' &= \gamma(E - \beta p_z) & E &= \gamma(E' + \beta p'_z) \\ p'_x &= p_x & p_x &= p'_x \\ p'_y &= p_y & p_y &= p'_y \\ p'_z &= \gamma(p_z - \beta E) & p_z &= \gamma(p'_z + \beta E') \end{array}$$

where

$$\gamma = \sqrt{\frac{1}{1 - \beta^2}}$$

[Note that the above formulae in SI units can be obtained by making the replacements $E \rightarrow E/c$, $E' \rightarrow E'/c$, and $\beta = v/c$. β is in fact the fractional velocity of the moving frame with respect to the speed of light c]

10

The above *Lorentz transformations* can also be written in a neat form using hyperbolic functions. If we put $\gamma = \cosh \theta$ then we get

$$\beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \frac{\sqrt{\cosh^2 \theta - 1}}{\cosh \theta} = \frac{\sinh \theta}{\cosh \theta} = \tanh \theta$$

$$\beta\gamma = \sinh \theta$$

and

$$\begin{aligned} E' &= E \cosh \theta - p_z \sinh \theta & E &= E' \cosh \theta + p'_z \sinh \theta \\ p'_x &= p_x & p_x &= p_x \\ p'_y &= p_y & p_y &= p_y \\ p'_z &= -E \sinh \theta + p_z \cosh \theta & p_z &= E' \sinh \theta + p'_z \cosh \theta \end{aligned}$$

Note the similarity of form (but not equivalence!) to a rotation about an axis in three dimensions.

11

Some notes:

- ① the following are useful relationships between a particle's energy, magnitude of momentum and mass in a frame in which it has velocity β

$$E = \gamma m \quad p = \gamma \beta m \quad p = \beta E$$

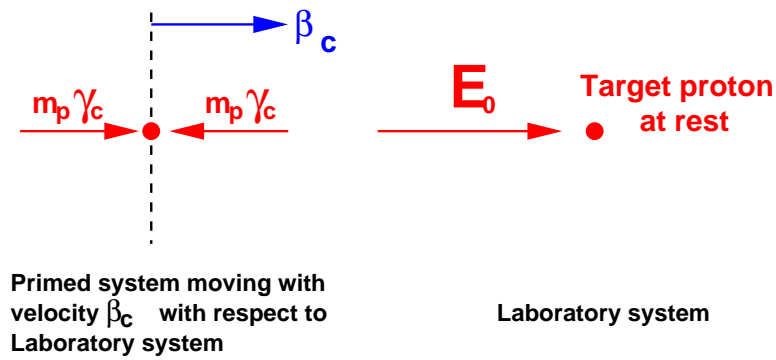
(generally in particle physics when we say “mass” we mean “rest mass”).

- ② A Lorentz transformation along an arbitrary direction in space to another frame with parallel axes is often called a *boost*.
- ③ Components of a 4-vector transverse to the boost direction do not change under a Lorentz transformation. Sometimes we will use the notation p_T and p_L to refer to the *transverse* and *longitudinal* components. Thus our previous formulae become

$$\begin{aligned} E' &= \gamma(E - \beta p_L) & p_T &= \sqrt{p_x^2 + p_y^2} \\ p'_T &= p_T & p_L &= p_z \\ p'_L &= \gamma(p_L - \beta E) \end{aligned}$$

12

Example 1 : Colliding Beams of Protons



We have

$$E' = 2m_p \gamma_c \quad p'_L = 0$$

and

$$E = \gamma_c (E' + \beta_c p'_L)$$

$$E_0 + m_p = \gamma_c (2m_p \gamma_c)$$

$$E_0 = 2m_p \gamma_c^2 - m_p$$

$$E_0 = \frac{2(m_p \gamma_c)^2}{m_p} - m_p$$

13

For the CERN LHC (Large Hadron Collider), which is due to produce beams and data in 2007, beams of 7 TeV (7000 GeV) protons will collide head-on.

Hence

$$E_0 = \left[\frac{2(7 \times 10^3)^2}{0.938} - 0.938 \right] \text{ GeV}$$

$$= 1.04 \times 10^8 \text{ GeV}$$

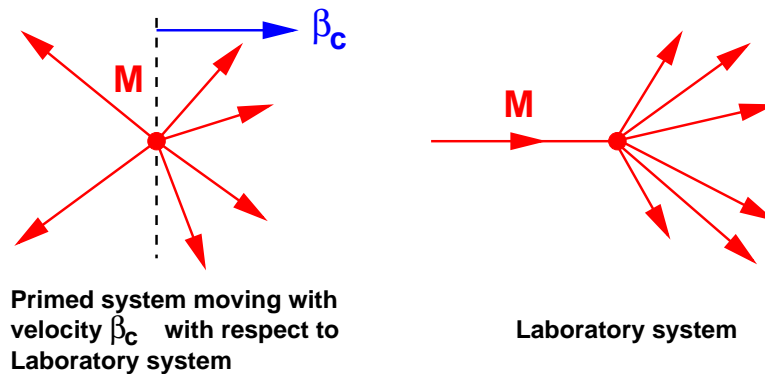
$$\approx 1.0 \times 10^{17} \text{ eV}$$

This would be the energy necessary for a fixed target machine to achieve the same thresholds. Note however that fixed target machines are capable of much higher *luminosity* - an important feature when searching for very rare events.

14

Example 2 : Resonance Production

Consider the decay of a *resonance* (extremely short-lived particle) of mass M into product particles where the decay is so fast that the resonance could never be observed or measured directly.



$$E = \gamma_c(E' + \beta_c p'_L)$$

with

$$\sum E' = M \text{ and } \sum p'_L = 0 \text{ and } \sum p'_T = 0$$

15

The sum is over all observed final state particles.

We have

$$\begin{aligned} \sum E &= \gamma_c \left(\sum E' + \beta_c \sum p'_L \right) = \gamma_c M \\ \sum p_L &= \gamma_c \left(\sum p'_L + \beta_c \sum E' \right) = \gamma_c \beta_c M \\ \sum p_T &= \sum p'_T = 0 \end{aligned}$$

and so

$$\left(\sum E \right)^2 - \left(\sum p_L \right)^2 - \left(\sum p_T \right)^2 = M^2 (\gamma_c^2 - \gamma_c^2 \beta_c^2 - 0) = M^2$$

Hence we can work out the mass of the decaying particle. It is the length (*invariant mass*) of the total energy-momentum 4-vector.

The technique has been used to identify, for example, the ρ , ω , ϕ and J/ψ resonances. It is also routinely used to identify π^0 mesons which decay in 10^{-16} s to two photons.

16

FIELDS

The underlying theories of particle physics are *Quantum Field Theories*. Whilst we will not go into these theories in mathematical rigour in this course, we will use some of the terminology.

A *field* is a mathematical object defined at every point in space-time.

Fields can transform in different ways under a Lorentz transformation. We can have for example *scalar*, *vector* and *tensor* fields. Scalar fields are left invariant under a Lorentz transformation, vector fields transform as 4-vectors, etc.

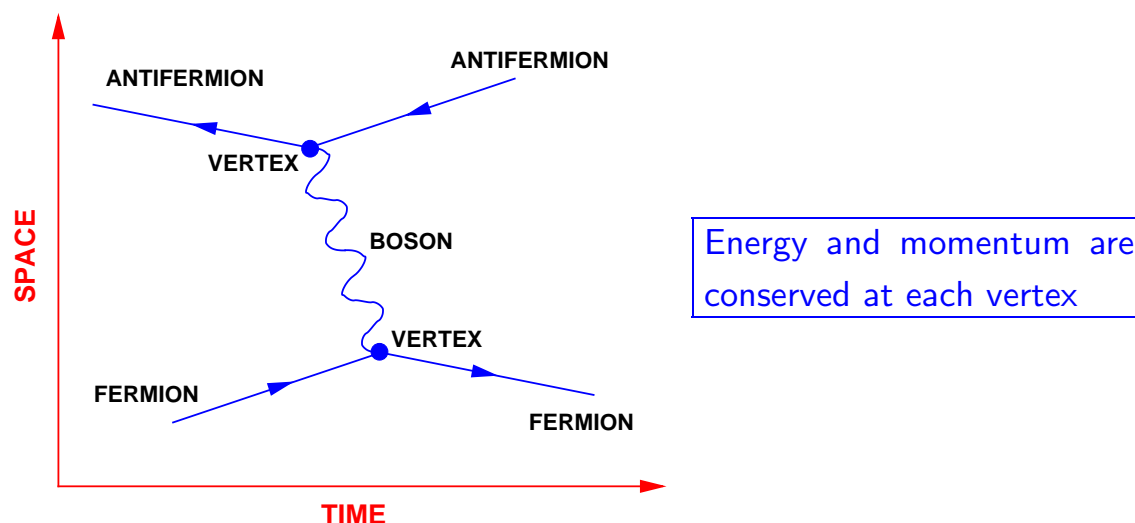
Each type of particle has a *field* associated with it.

The full theory is written down in terms of a mathematical object known as a *Lagrangian density*, which is a function of the fields and their space-time derivatives. This contains terms corresponding to the kinetic energy of each field in the theory, and terms specifying the interactions between the fields.

17

FEYNMAN DIAGRAMS

We use these as a means of visualising fundamental processes in space-time. They have a much more rigorous use in field theory.



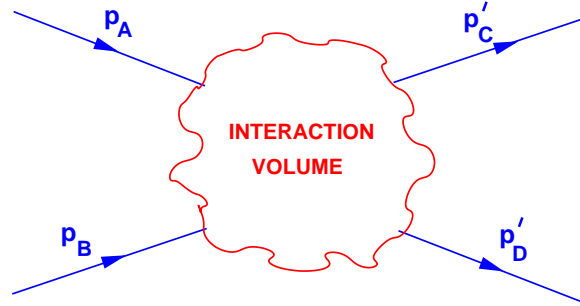
The above example could describe the scattering of a fermion (e.g. e^-) from an antifermion (e.g. e^+), through exchange of an intermediate boson (e.g. photon)

18

MANDELSTAM VARIABLES

It is convenient to define variables s , t , and u for an interaction in terms of dot products of the relevant four vectors.

These variables are therefore independent of the Lorentz frame.



Take the simplifying case of two incoming particles A and B scattering to produce two outgoing particles C and D

19

s is the **square of the total energy in the *Centre of Mass System***.

$$s = (p_A + p_B)^2 = (E_A + E_B)^2 - (\underset{\sim}{p}_A + \underset{\sim}{p}_B)^2 = (\underset{\sim}{p}'_C + \underset{\sim}{p}'_D)^2$$

In the CMS $\underset{\sim}{p}_A + \underset{\sim}{p}_B = 0$ and so $s = (E_A + E_B)^2$

Note that s is POSITIVE

t is the **square of the four momentum transfer**

$$t = (p'_C - p_A)^2 = (E'_C - E_A)^2 - (\underset{\sim}{p}'_C - \underset{\sim}{p}_A)^2$$

For *elastic scattering*, $E'_C = E_A$ and the first term disappears.

$$t = -\underset{\sim}{p}'_C{}^2 - \underset{\sim}{p}_A{}^2 + 2\underset{\sim}{p}'_C \cdot \underset{\sim}{p}_A$$

But $|\underset{\sim}{p}'_C| = |\underset{\sim}{p}_A| = p$ for elastic scattering, and so

$$t = -2p^2 + 2p^2 \cos \theta = -2p^2 (1 - \cos \theta)$$

where θ is the scattering angle. Note that t is NEGATIVE.

20

u is the **crossed four momentum transfer squared**

$$u = (p'_D - p_A)^2 = (E'_D - E_A)^2 - (\underline{p}'_D - \underline{p}_A)^2$$

u is not independent of the other two Mandelstam variables.

It is possible to show that s , t and u are linked by the relationship

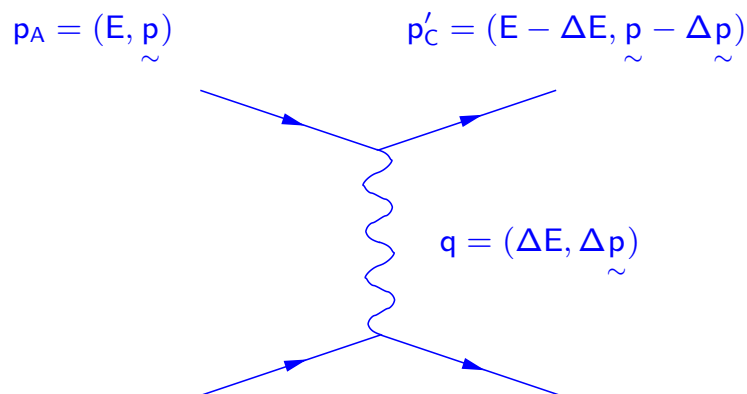
$$s + t + u = \sum m_i^2 = m_A^2 + m_B^2 + m_C^2 + m_D^2$$

and hence it is sufficient to specify s and t .

We will examine two extreme types of interaction in terms of the 4-momentum of the exchanged particle.

21

Elastic Scattering



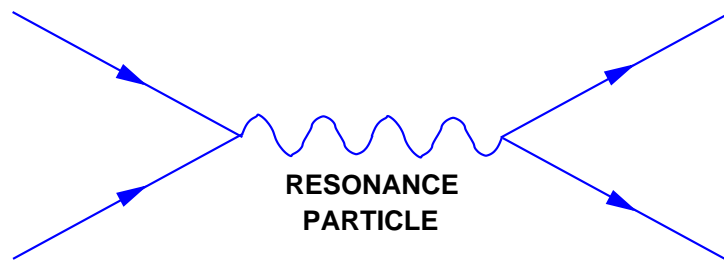
We have $t = q \cdot q = \Delta E^2 - \Delta \underline{p}^2$

For the case where there is no energy loss in the CMS, t is negative. If the exchanged particle were a real physical particle, t would represent its mass-squared (which must be positive). Therefore the exchanged particle is **VIRTUAL** or “off the mass shell”.

This type of event for which $t < 0$ is called **SPACELIKE**.

22

Resonance Production



In this case the momentum of the exchanged particle is low (zero in the CMS) and the energy is high.

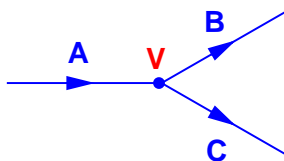
The square of the four momentum of the transferred particle is positive and this type of interaction is called **TIMELIKE**.

In this case the exchanged particle may be real or “on the mass shell”.

23

INTERACTIONS : DECAYS

Consider a simple point vertex representing a decay $A \rightarrow B + C$



[In this picture the solid lines do not have to represent fermions (they couldn't - why?), but can be any particle]

Fermi's Golden Rule gives the decay rate (probability per unit time)

$$W = \frac{1}{\tau} = \frac{0.693}{T_{1/2}} = \frac{2\pi}{\hbar} \rho(E_f) |\langle f|V|i \rangle|^2$$

$\rho(E_f) = (dn/dE)_{E=E_f}$ is the density of final states around the final energy.

V is the potential acting at the vertex. Because the interaction occurs at a point, we can write $V = M_A \cdot g \cdot \delta(\underset{\sim}{r}_A - \underset{\sim}{r}_B) \cdot \delta(\underset{\sim}{r}_A - \underset{\sim}{r}_C)$

$\langle f|V|i \rangle$ is known as the **amplitude** for the decay. It depends on what occurs at the vertex, and gets squared when calculating the decay probability.

24

The decay probability is proportional to both $\rho(E_f)$ and g^2 .

Effect of $\rho(E_f)$: We can work out $\rho(E_f)$ for the above decay. In the CMS, B and C have the same magnitude of momentum p (in opposite directions)

$$\begin{aligned}
 E_f &= E_B + E_C = (M_B^2 + p^2)^{1/2} + (M_C^2 + p^2)^{1/2} \\
 \frac{dE_f}{dp} &= \frac{1}{2}(M_B^2 + p^2)^{-1/2}(2p) + \frac{1}{2}(M_C^2 + p^2)^{-1/2}(2p) \\
 &= \frac{p}{E_B} + \frac{p}{E_C} \\
 &= \frac{p(E_C + E_B)}{E_B E_C} \\
 &= p \frac{E_A}{E_B E_C}
 \end{aligned}$$

Phase space is the six-dimensional space spanning three spatial and three momentum coordinates.

25

We need a volume h^3 of phase space for each quantum state - in natural units

$$h^3 = (2\pi\hbar)^3 = (2\pi)^3$$

So the number of states available to a final state particle enclosed in physical volume \mathcal{V} , with momentum between p and $p + dp$ is

$$dn = \frac{\mathcal{V}}{(2\pi)^3} 4\pi p^2 dp$$

and so

$$\frac{dn}{dp} = \frac{\mathcal{V} 4\pi p^2}{(2\pi)^3}$$

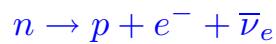
and

$$\rho(E_f) = \frac{dn}{dE_f} = \frac{dn}{dp} \frac{dp}{dE_f} = \frac{\mathcal{V} 4\pi p^2}{(2\pi)^3} \frac{E_B E_C}{p E_A} = \frac{\mathcal{V}}{(2\pi)^3} 4\pi p \frac{E_B E_C}{E_A}$$

One can see that phase space considerations can restrict the decay probability if the momentum of the final state particles is small, since $\rho(E_f) \propto p$.

26

An example occurs in beta decay



Since $m_n = 939.55$ MeV, $m_p = 938.26$ MeV, $m_e = 0.511$ MeV and $m_\nu \approx 0$

$$\Delta m = m_n - m_p - m_e - m_\nu = 0.78 \text{ MeV}$$

and the decay rate is low (beta decay is slow)

Effect of g : g is dimensionless and called a **coupling constant**. It depends on the type of interaction - differences in strengths of interactions (size of coupling constants) account for great variations in decay and reaction rates.

INTERACTION	SQUARE OF COUPLING CONSTANT
STRONG	$g_s^2 \approx 4\pi$
ELECTROMAGNETIC	$e^2 = 4\pi/137$
WEAK	$g_{eff}^2 \approx 10^{-5}$
GRAVITATION	$\approx 10^{-39}$

27

DECAY WIDTH

The decay width of a particle is defined as $\Gamma = \hbar/\tau$, where τ is its mean lifetime. In natural units

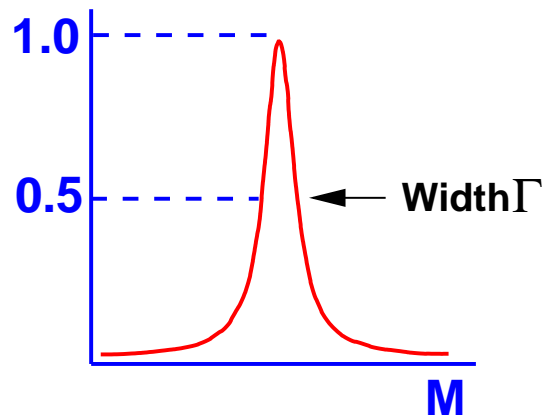
$$\Gamma = \frac{1}{\tau}$$

and the units for Γ are energy units, GeV.

The Uncertainty Principle tells us that $\Delta E \Delta t \gtrsim \hbar$. As a consequence, if we know to some accuracy how long a particle lives, there is a limit to how well we know its energy.

It follows that if we reconstruct the energy of a decaying particle from its products (or equivalently, determine its invariant mass, as we saw how to do for a resonance), and do this for a large number of these particles, we will get a distribution of invariant masses.

28



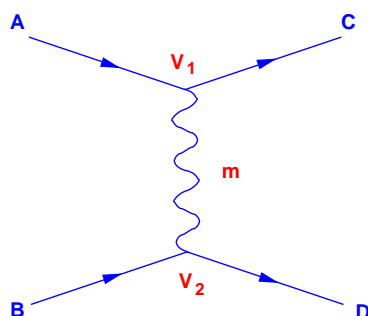
Note that

- The position of the maximum of the peak is what we quote as the “mass” of the particle
- Particles with very short lifetimes (such as resonances) have large widths
- Particles with longer lifetimes have smaller widths (and stable particles have zero width)
- Except for strongly decaying resonances, we cannot usually experimentally resolve the width of particles

29

INTERACTIONS : SCATTERING

Scattering processes are described by the exchange of an intermediate “virtual” particle



$$A = \frac{\langle \psi_C | V_1 | \psi_A \rangle \langle \psi_D | V_2 | \psi_B \rangle}{t - m^2}$$

In this case the amplitude has three pieces

- A term from the top vertex, $\langle \psi_C | V_1 | \psi_A \rangle$, proportional to the coupling strength at that vertex
- A similar term from the bottom vertex $\langle \psi_D | V_2 | \psi_B \rangle$
- A “propagator” term $\frac{1}{t - m^2}$

30

Here m is the mass of the corresponding real particle to the virtual particle exchanged.

t is the square of the four momentum transfer q , and for this type of scattering interaction we have seen that $t = q^2 < 0$. If we define the positive real quantity $Q^2 = -q^2 = -t > 0$ then

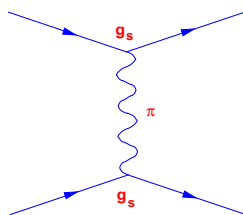
$$A \propto \frac{1}{m^2 + Q^2}$$

Remember that

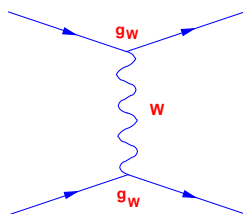
- the **rate** of the interaction, i.e. the probability of scattering, is determined by the **square** of the amplitude (and phase space factors, as in the decay case).
- the amplitude may be a complex number (since the overlap of wave functions in quantum mechanics is in general complex), but the square will be real (as it must be - the interaction rate is an observable)

31

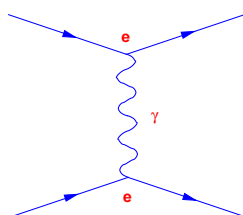
Examples of Scattering Amplitudes



One pion exchange force $A \propto \frac{g_s^2}{m_\pi^2 + Q^2} = F(Q^2)$



Weak force $A \propto \frac{g_w^2}{m_W^2 + Q^2}$



Electromagnetic force $A \propto \frac{e^2}{Q^2}$

32

MORE ON AMPLITUDES

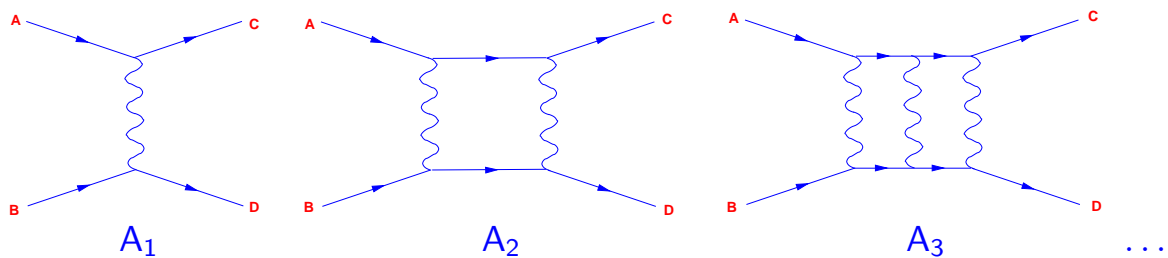
The rate of an interaction is related to the **square** of the amplitude.

If there are many possible diagrams linking **identical** initial and final states, the rate is determined by the **coherent** sum of the individual amplitudes

$$\text{Rate} \propto |A_1 + A_2 + A_3 + \dots A_n|^2$$

with possible quantum mechanical interference between the diagrams

e.g.



33

The series of diagrams is in fact generally infinite, but if the coupling strength at each vertex is small (for electromagnetic interactions, $e^2 = 4\pi/137 \approx 0.09$), then diagrams with larger numbers of vertices will contribute less to the sum.

So calculations are done using a *perturbation series*, retaining the first few diagrams only.

If the final states are **not** all identical, then the total rate is calculated using an **incoherent** sum.

$$\text{Rate} \propto |A_1|^2 + |A_2|^2 + |A_3|^2 + \dots |A_n|^2$$

An example would be in calculating the total decay rate for a particle which decays to several different possible final states

$$A \rightarrow B + C \quad \text{Amplitude } A_1$$

$$A \rightarrow D + E + F \quad \text{Amplitude } A_2$$

etc.

34

REAL SPACE AND MOMENTUM SPACE

There is a fundamental link between the behaviour of an amplitude $F(Q^2)$ in momentum (Q^2) space and the behaviour of the potential $V(r)$ in real space. They are Fourier transforms of each other.

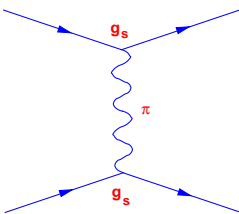
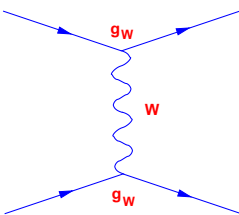
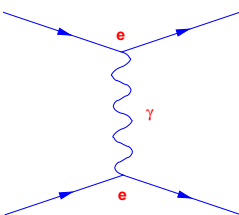
$$F(Q^2) = \int V(r) e^{i\vec{Q}\cdot\vec{r}} d\vec{r}$$

$$V(r) = \frac{1}{(2\pi)^3} \int F(Q^2) e^{-i\vec{Q}\cdot\vec{r}} d\vec{Q}$$

See the supplementary notes for a worked example.

We can re-examine our previous examples

35

	$F(Q^2)$	$V(r)$	Comment
	$\frac{g_s^2}{m_\pi^2 + Q^2}$	$\frac{g_s^2}{4\pi} \frac{e^{-m_\pi r}}{r}$	Yukawa one pion exchange force
	$\frac{g_w^2}{m_W^2 + Q^2}$	$\frac{g_w^2}{4\pi} \frac{e^{-m_W r}}{r}$	Very short range weak force
	$\frac{e^2}{Q^2}$	$\frac{e^2}{4\pi} \frac{1}{r}$	Normal Coulomb force

36