Why Does Uranium Alpha Decay?

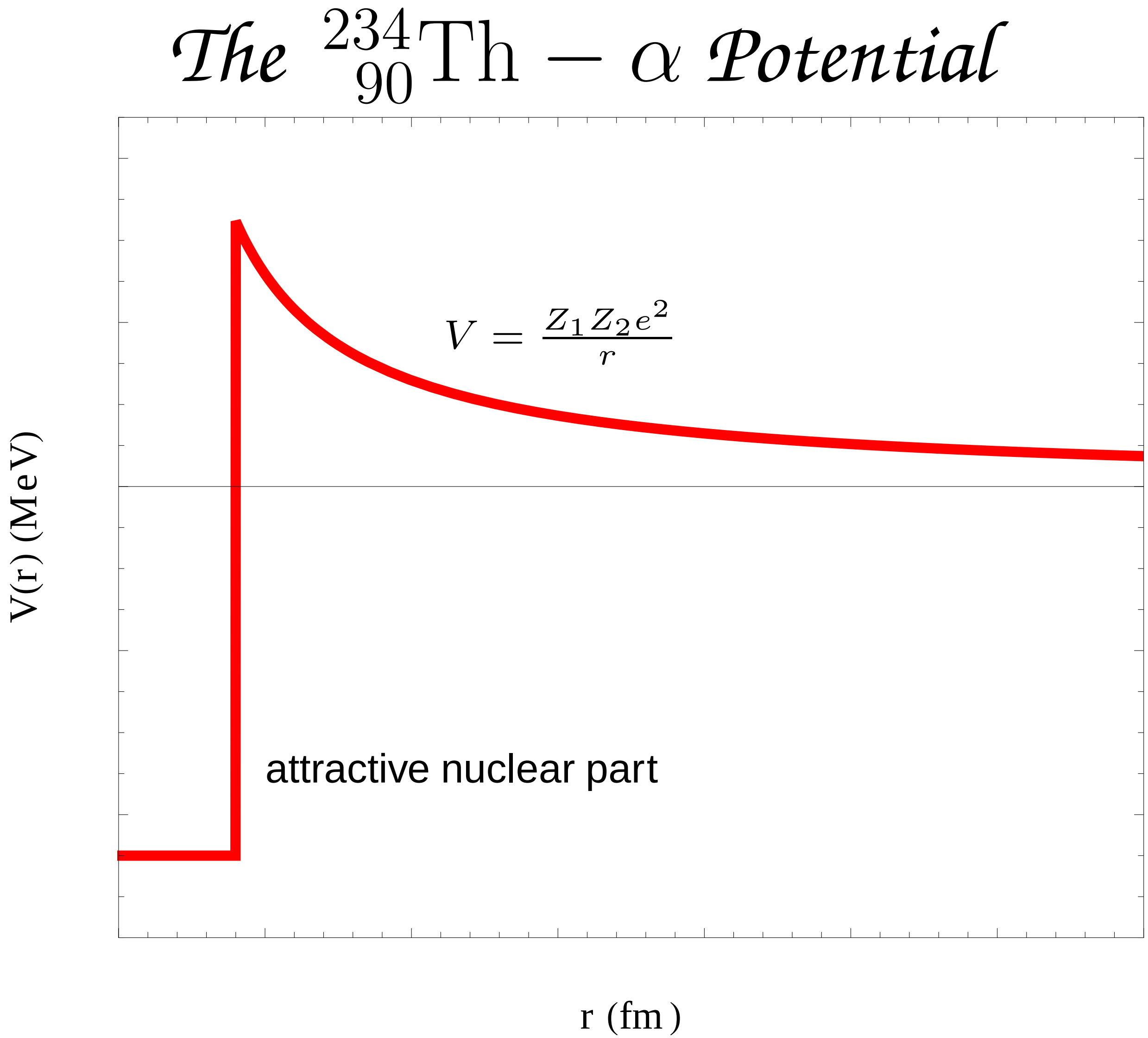
Consider the alpha decay shown below where a uranium nucleus spontaneously breaks apart into a $^4\mathrm{He}$ or alpha particle and $^{234}_{90}\mathrm{Th}$.

$$^{238}_{92}\mathrm{U} \rightarrow \ ^4\mathrm{He} \ + \ ^{234}_{90}\mathrm{Th} \qquad \mathrm{E}(^4\mathrm{He}) = 4.2 \ \mathrm{MeV}$$

To study this reaction we first map out the $^4\mathrm{He} -^{234}_{90} \mathrm{Th}$ potential energy. We reverse the decay above and use a beam of $^4\mathrm{He}$ nuclei striking a $^{234}_{90}\mathrm{Th}$ target. The $^4\mathrm{He}$ nuclei come from the radioactive decay of another nucleus $^{210}_{84}\mathrm{Po}$.

1. What is the distance of closest approach of the $^4\mathrm{He}$ to the $^{234}_{90}\mathrm{Th}$ target if the Coulomb force is the only one that matters?
2. Is the Coulomb force the only one that matters?
3. What is the lifetime of the $^{238}_{92}\mathrm{U}$?

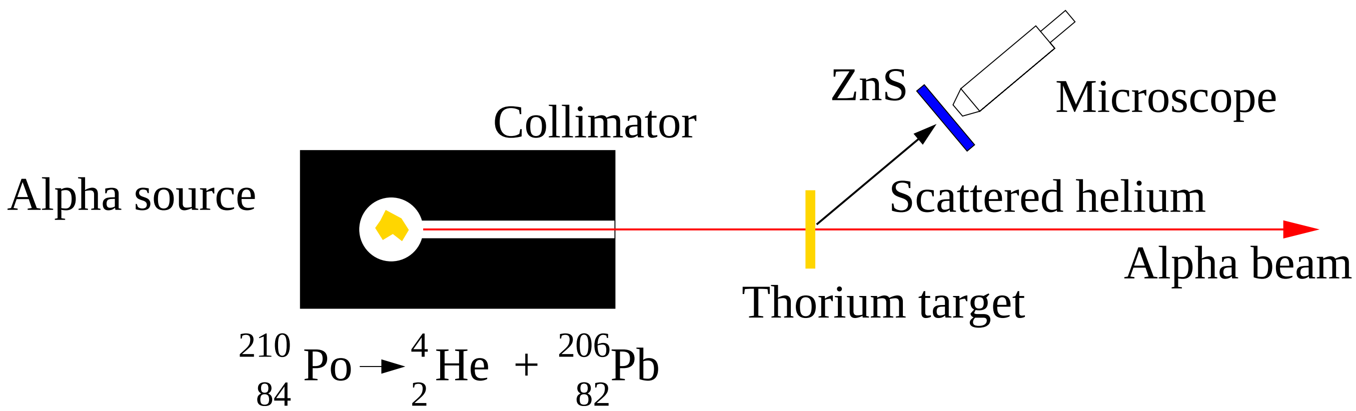
What Do We Know?



Mapping the Potential Energy

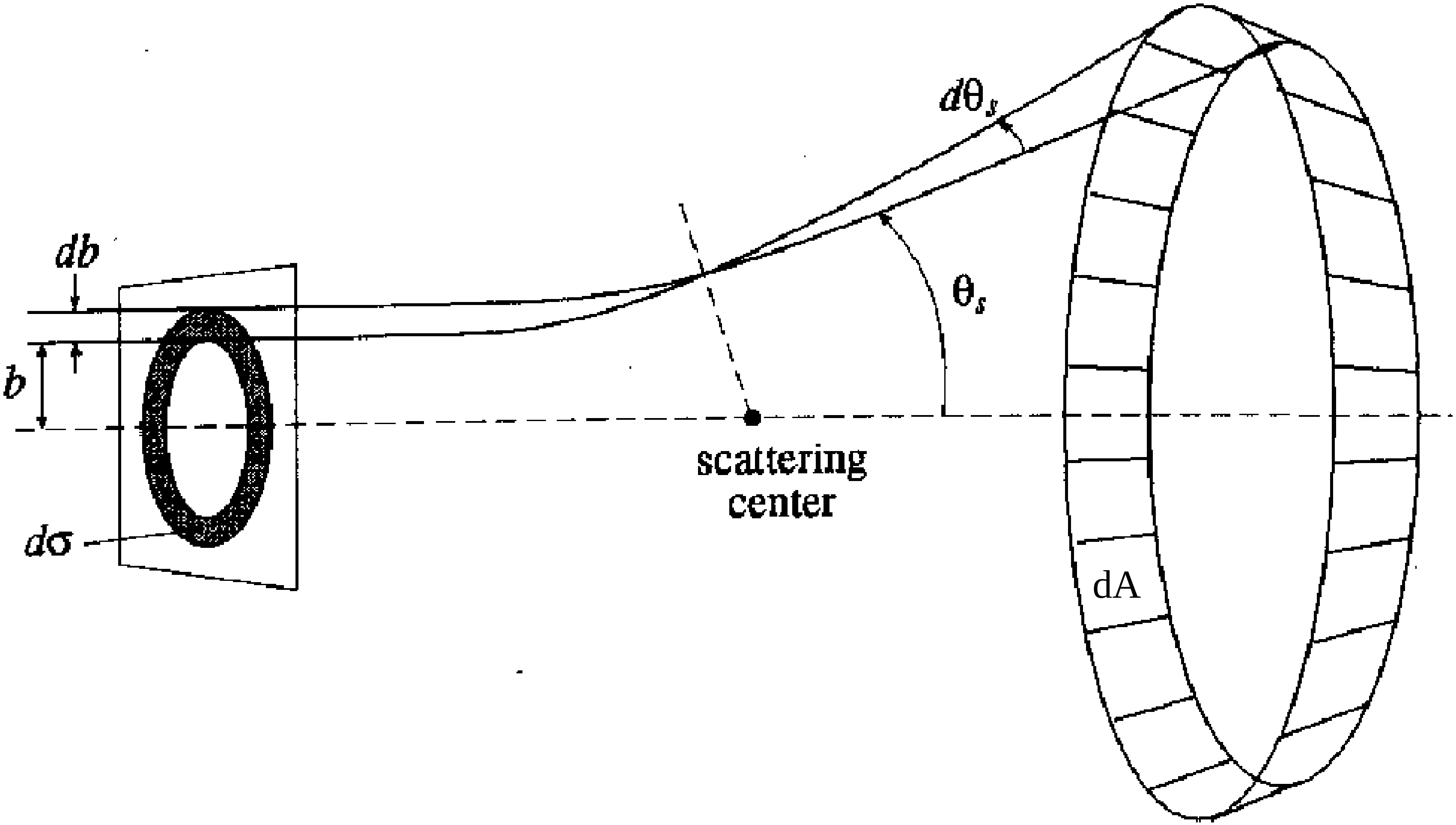
Rutherford Scattering

What is the distance of closest approach of the $^4\mathrm{He}$ to the $^{234}_{90}\mathrm{Th}$ target if only the Coulomb force is active? Is the Coulomb force the only one active? The energy of the $^4\mathrm{He}$ emitted by the $^{210}_{84}\mathrm{Po}$ is $\mathrm{E}(^4\mathrm{He}) = 5.407\ \mathrm{MeV}$.

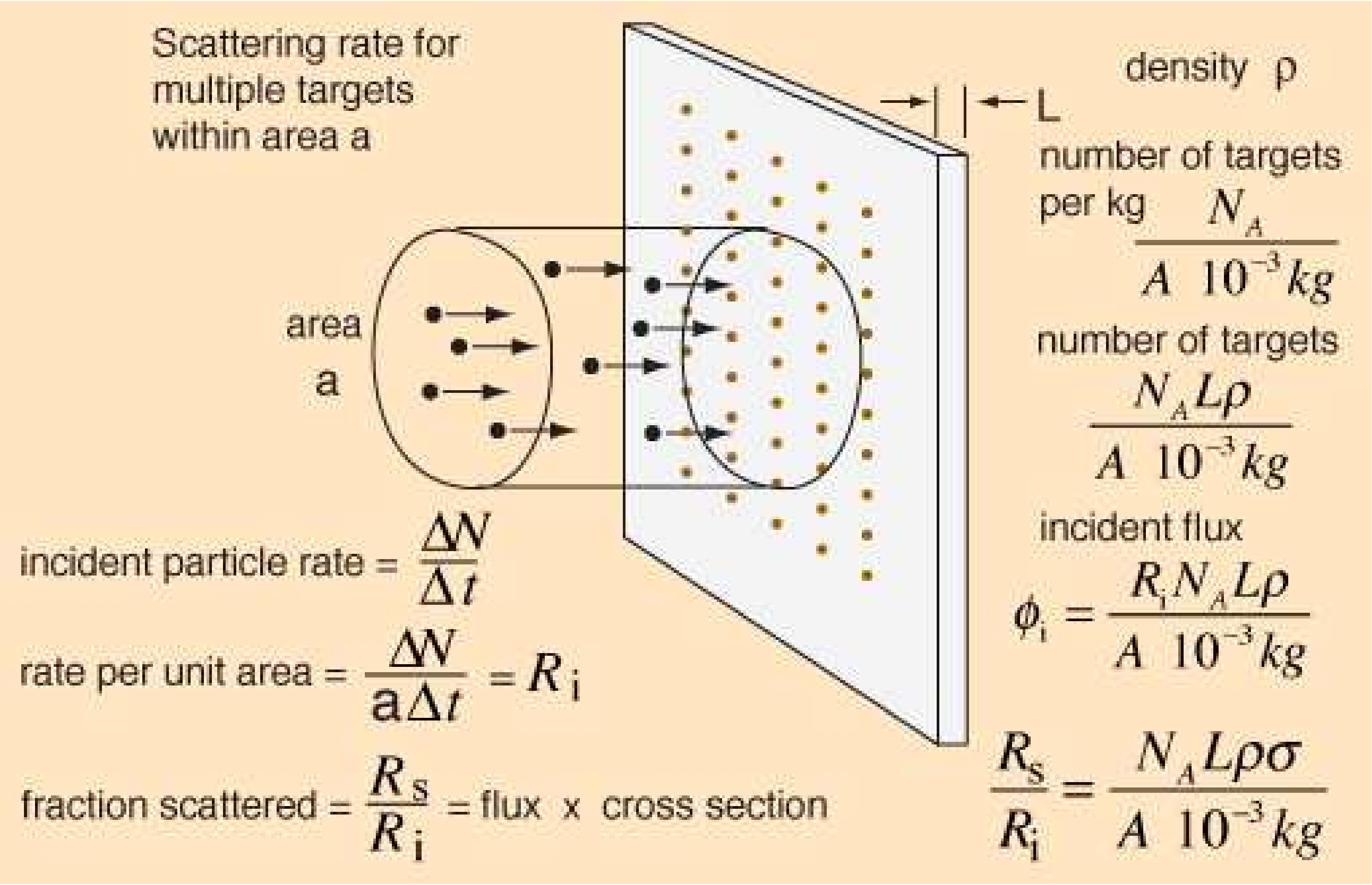


Mapping the Potential Energy

The Total Cross Section

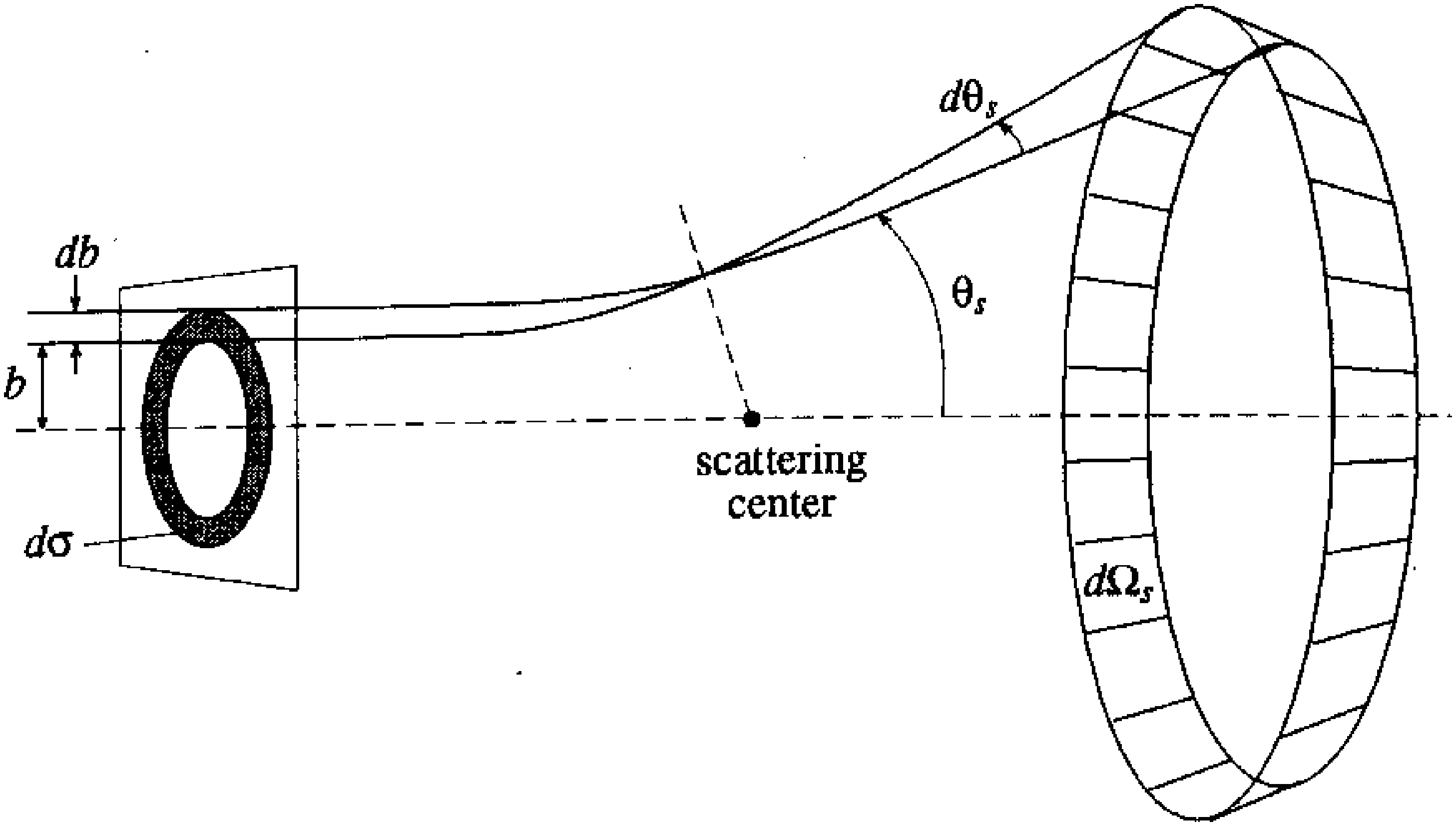


Areal or Surface Density of Nuclear Targets

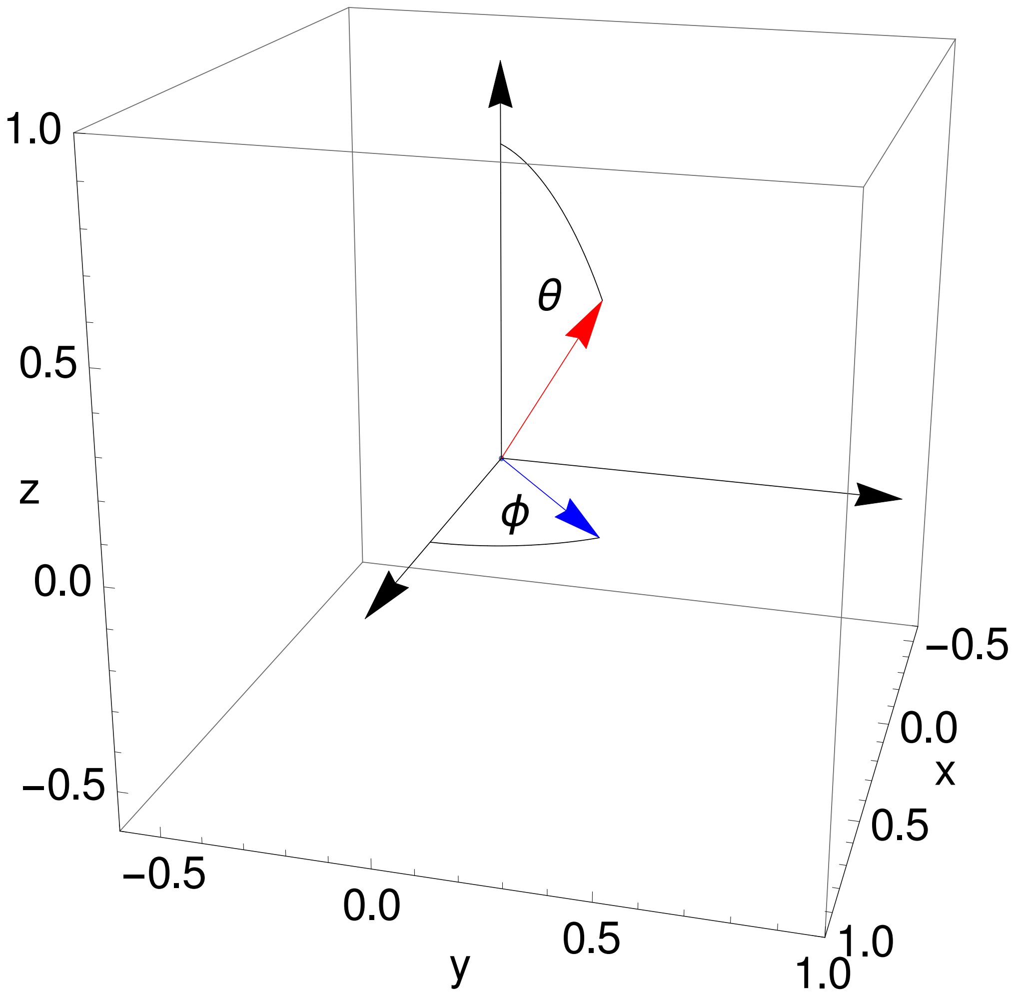


Mapping the Potential Energy

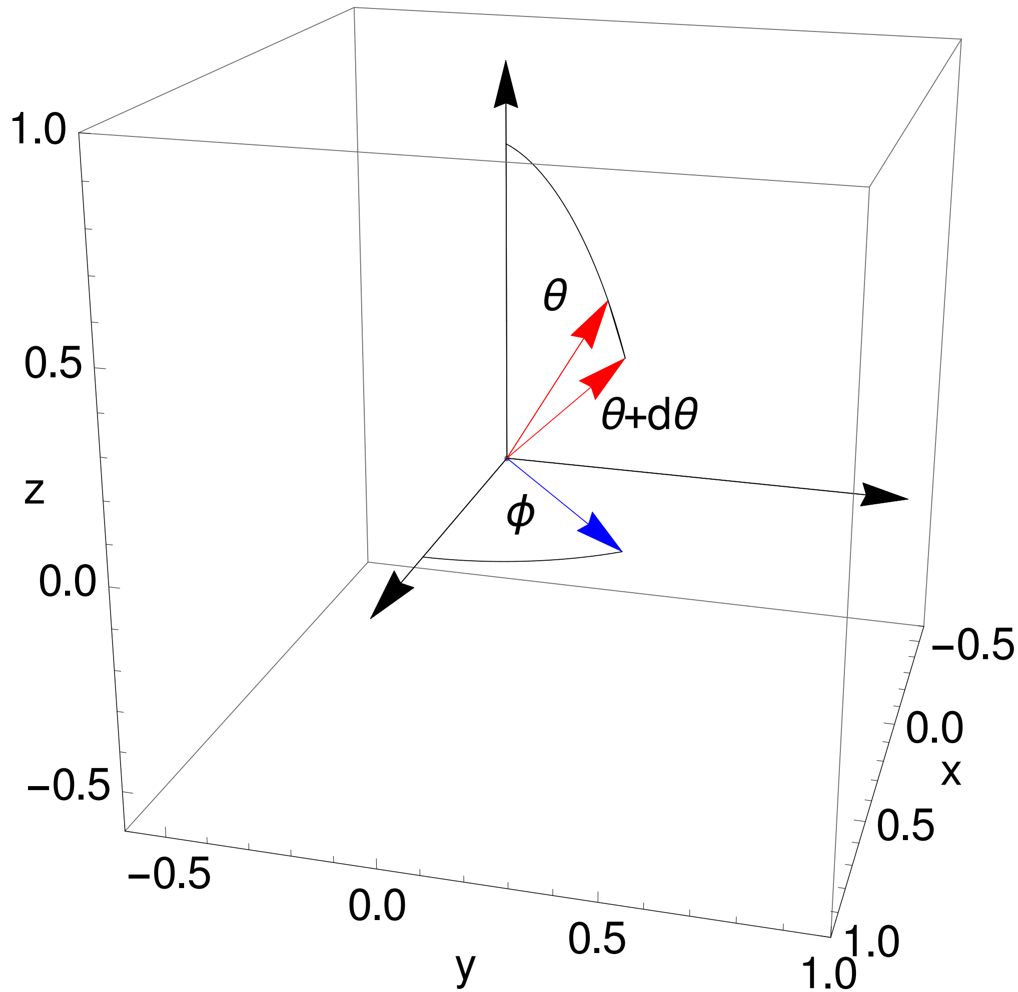
The Differential Cross Section



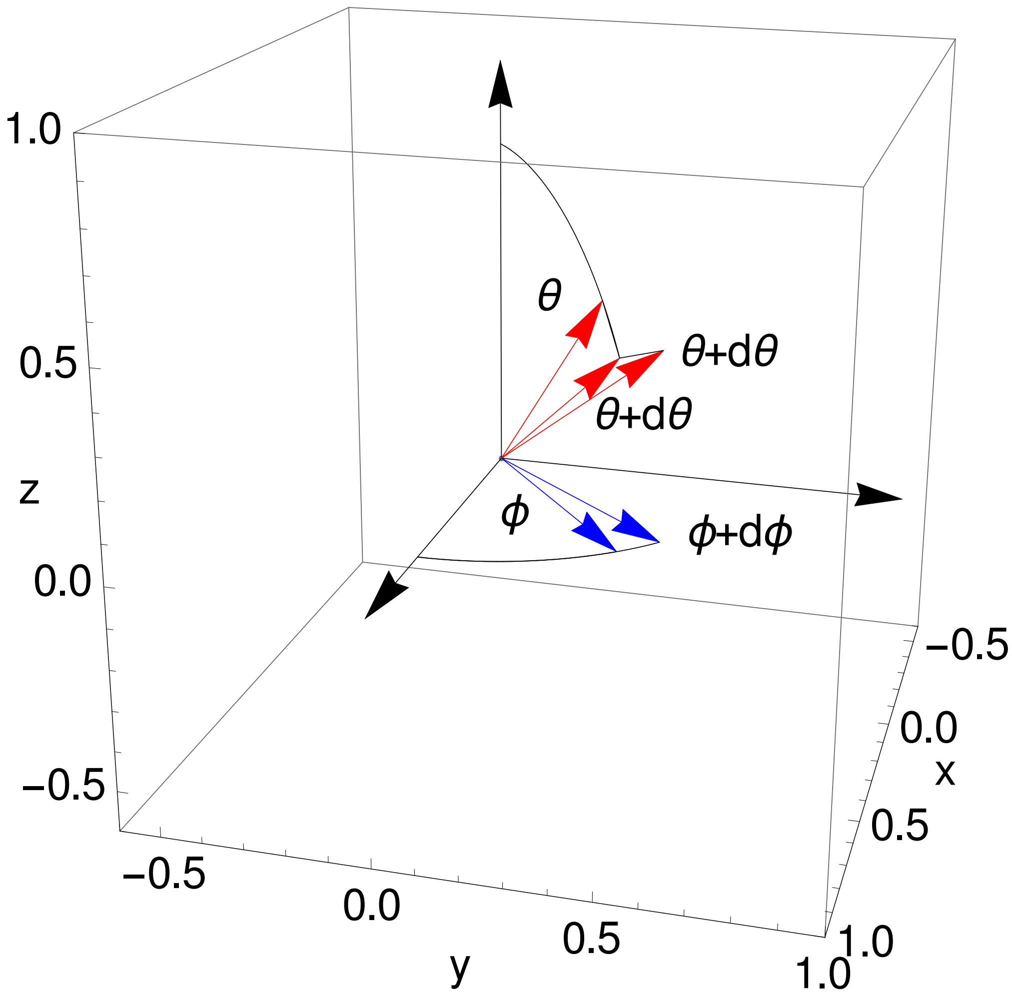
Solid Angle



Solid Angle



Solid Angle



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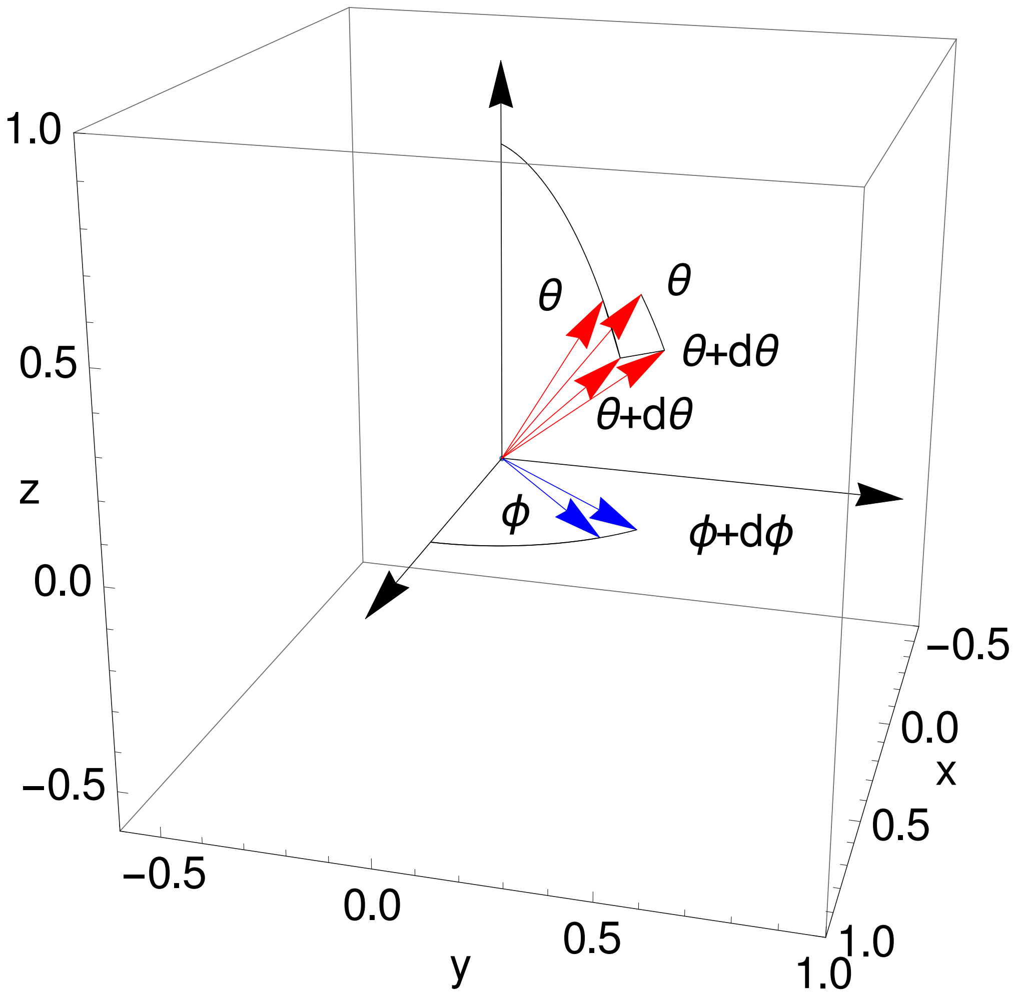

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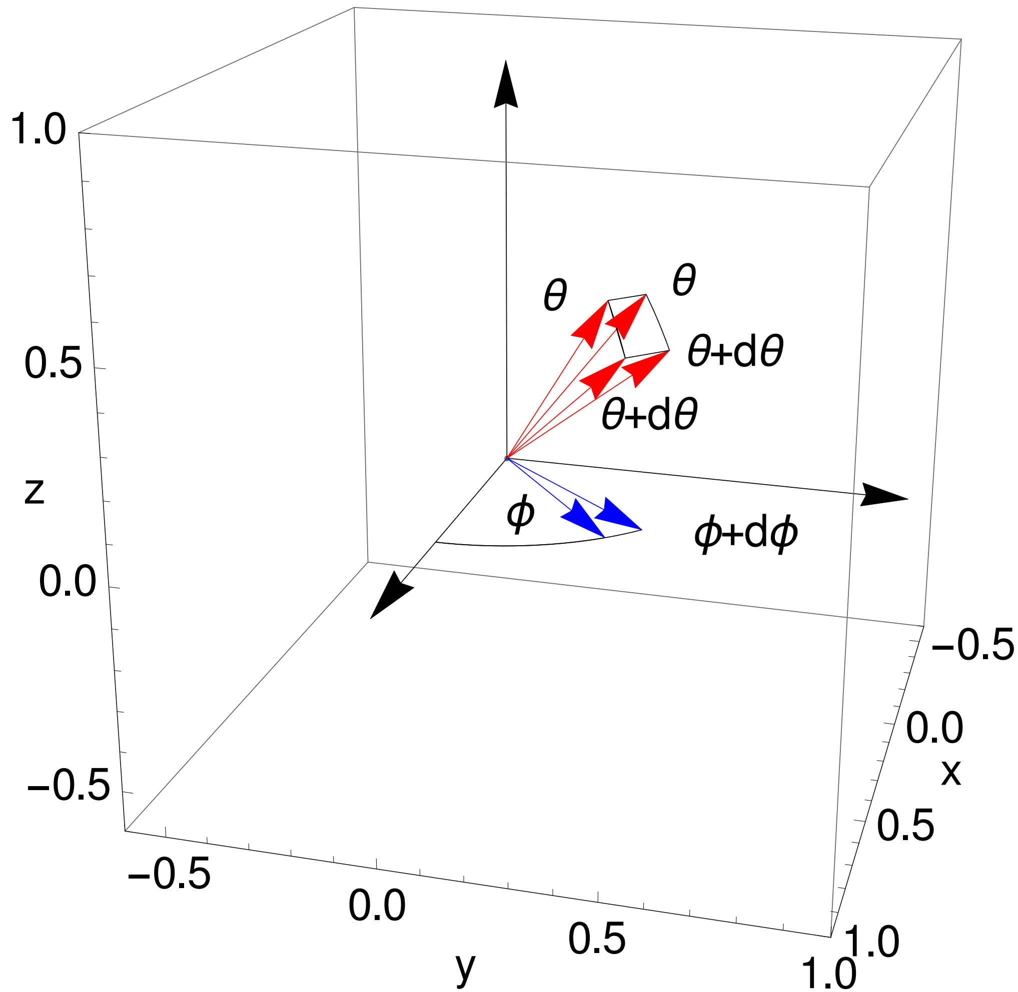

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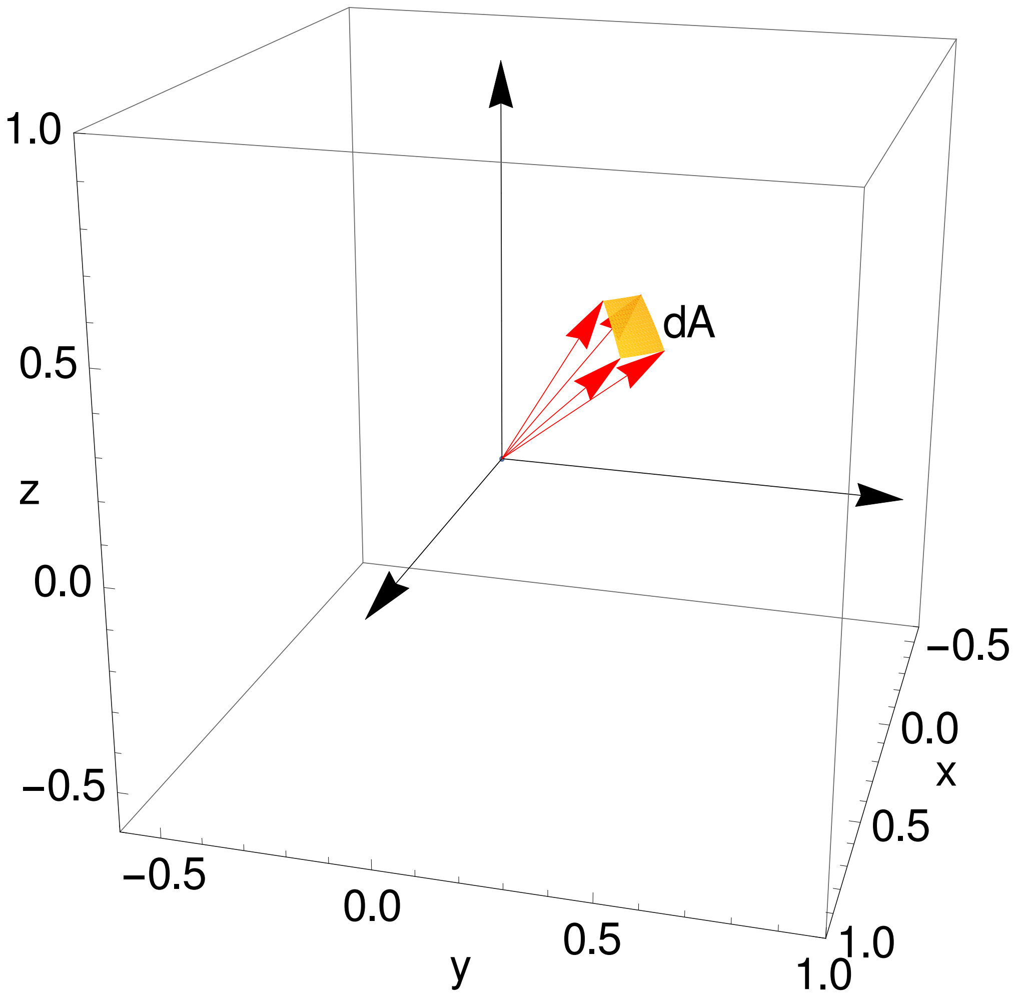
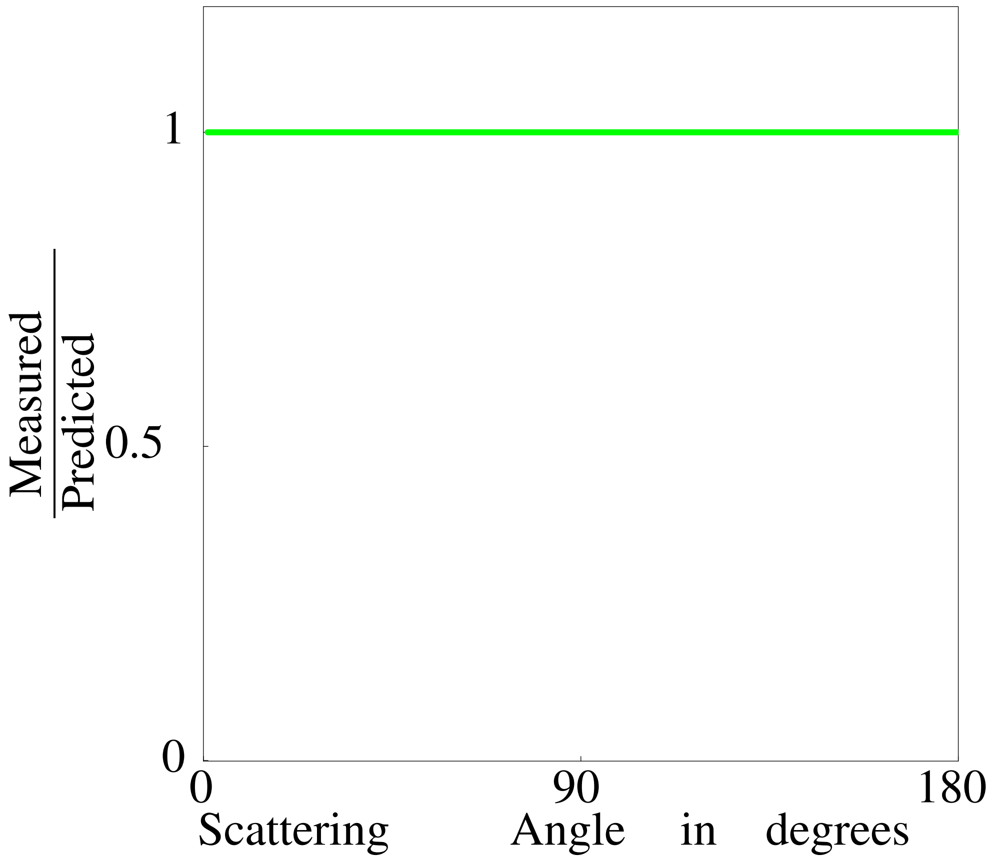
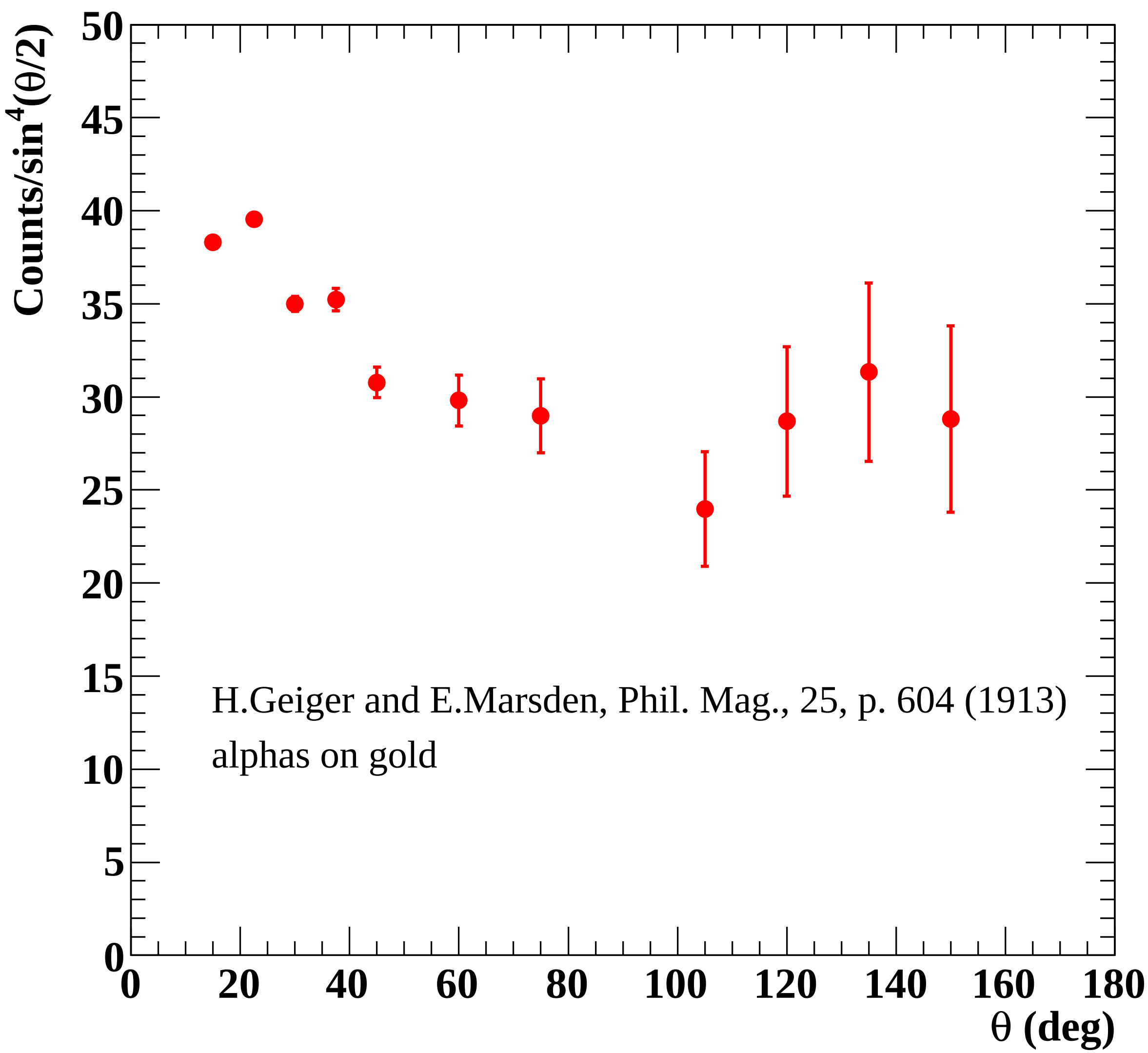


Rutherford Scattering Results

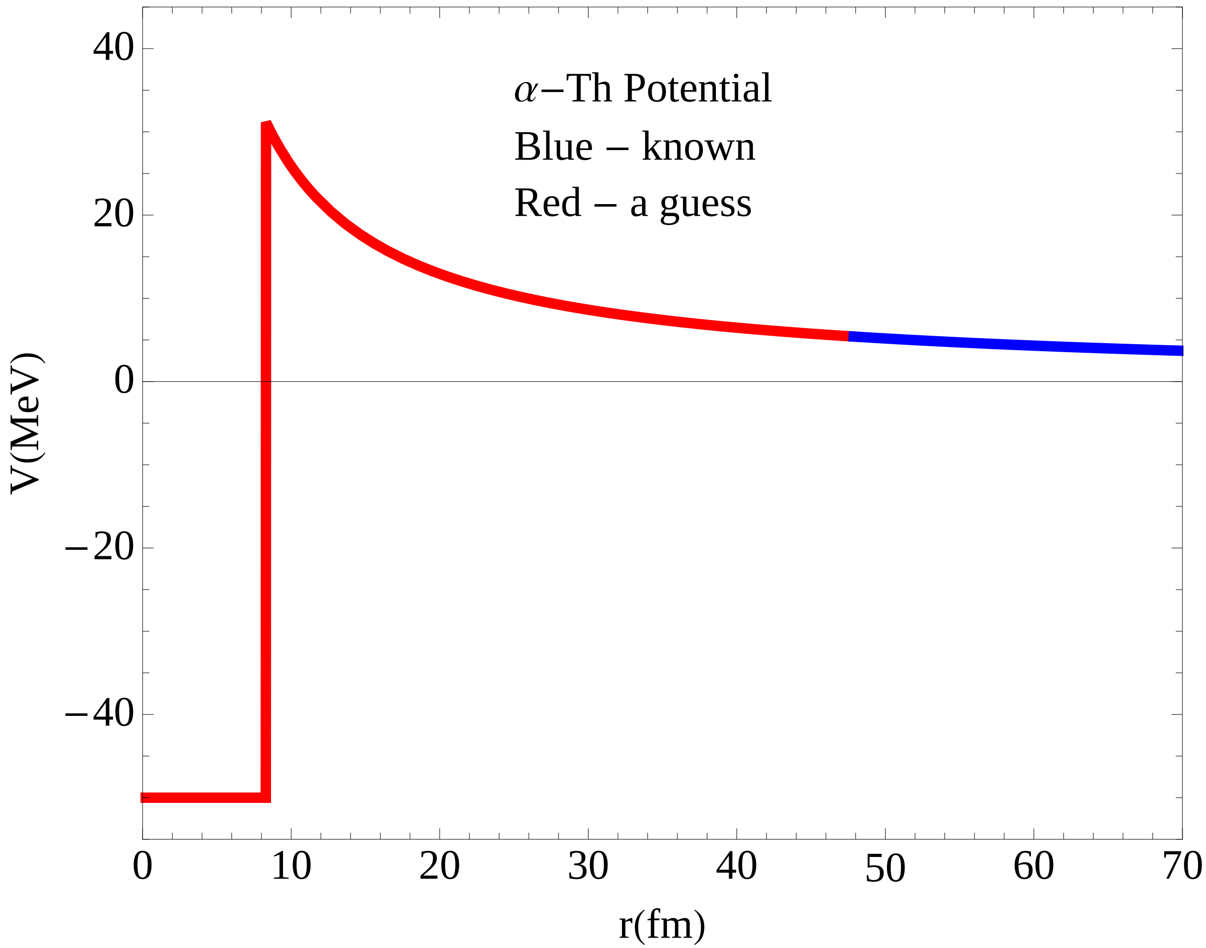


What does this say about the ${}^{4}_{2}\mathrm{He} - {}^{234}_{90}\mathrm{Th}$ potential energy?

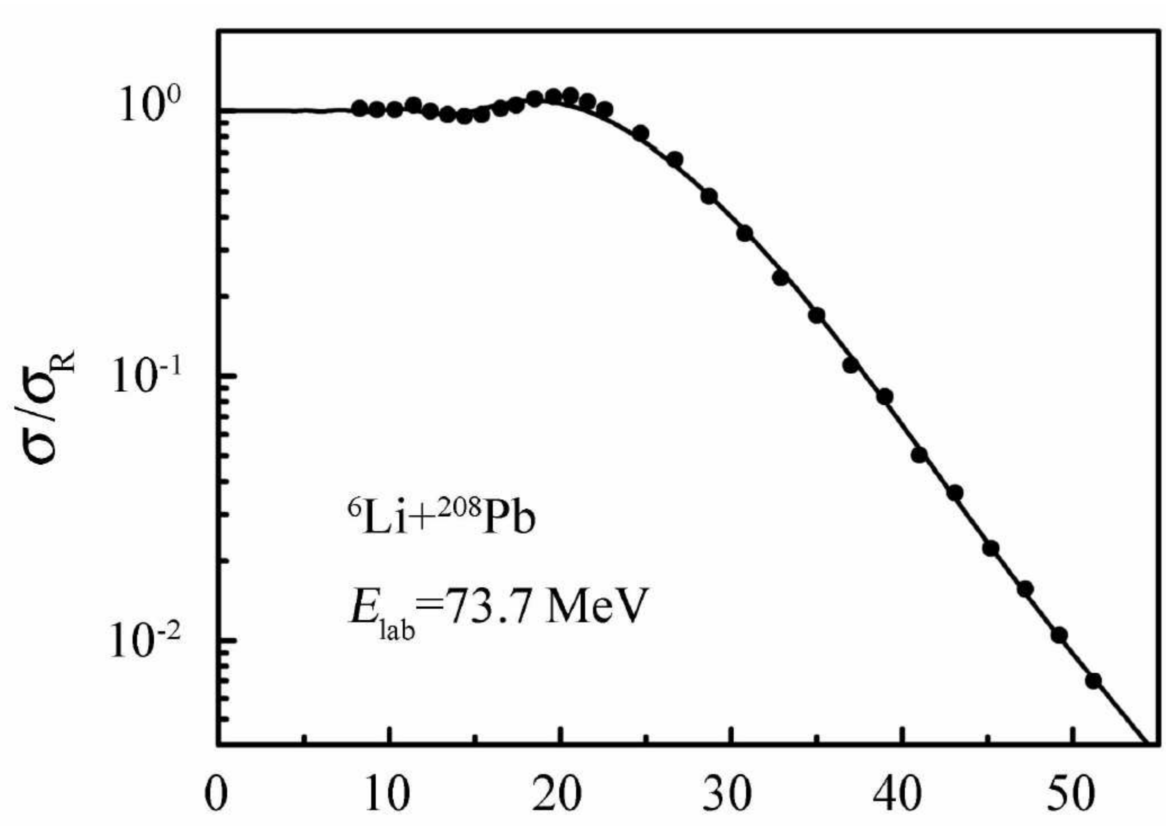
Actual Rutherford Scattering Results



The $^{4}\mathrm{He} - {}^{234}_{90}\mathrm{Th}$ Potential



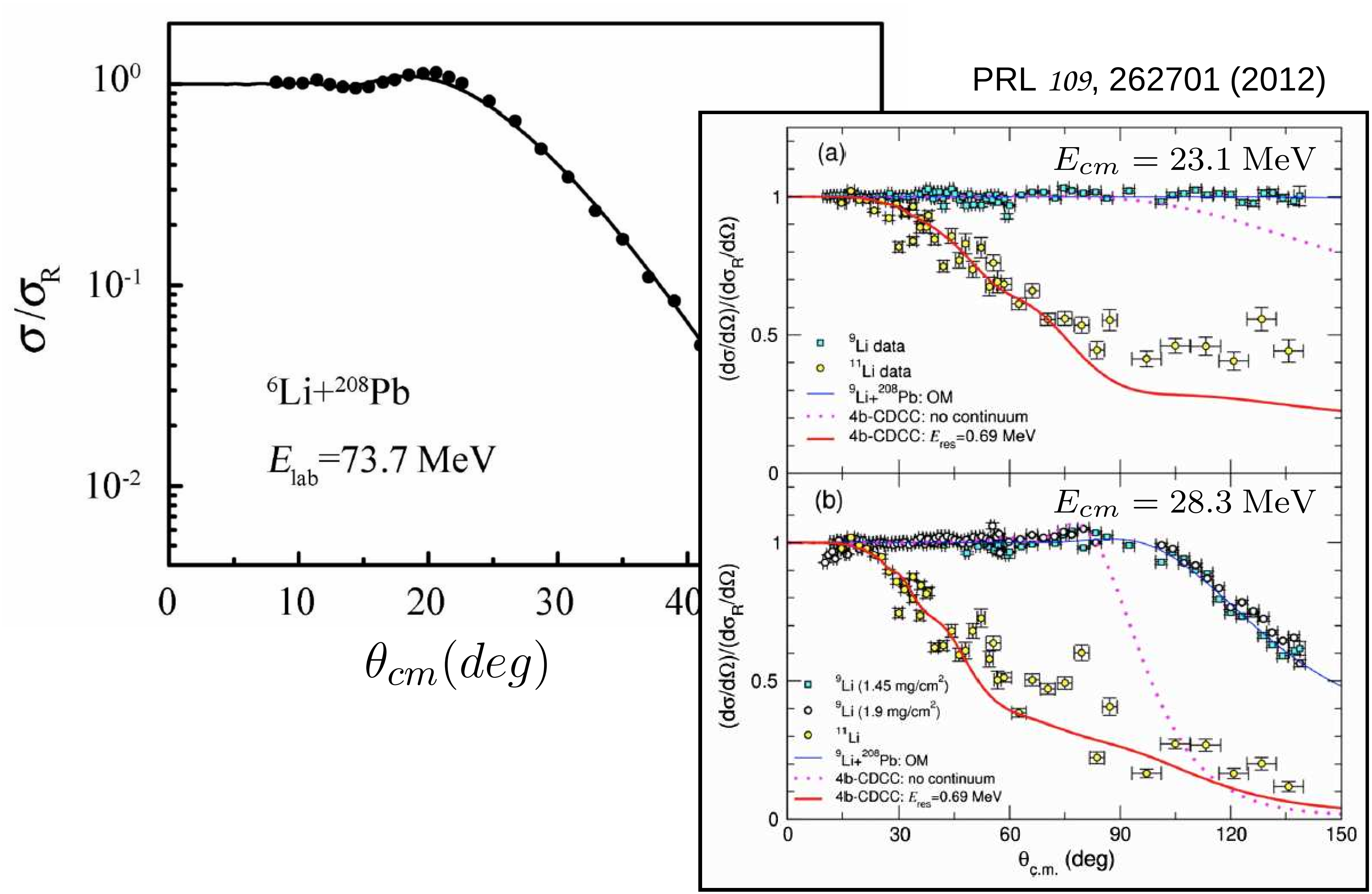
Measuring the Size of the Nucleus



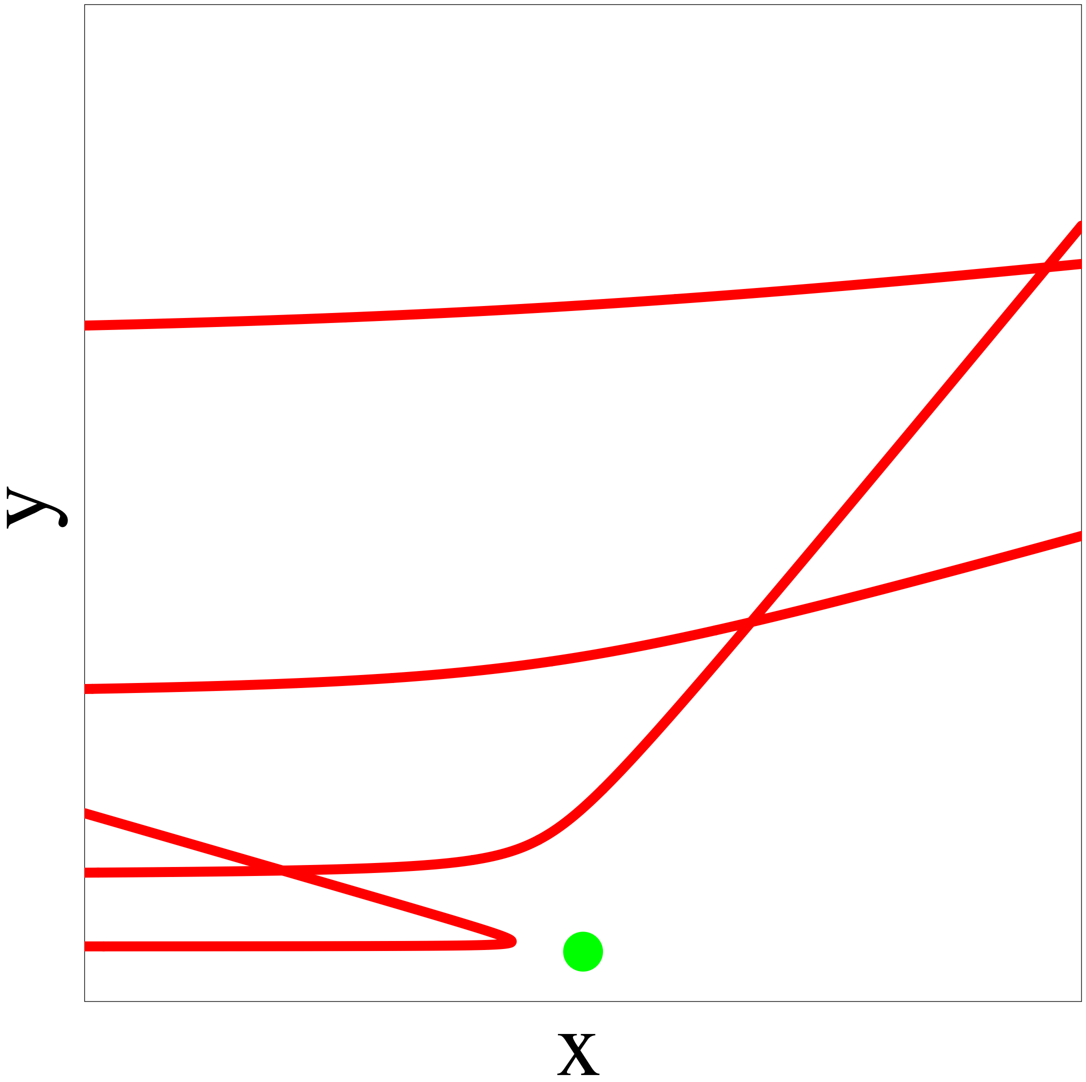
$\theta_{cm}(deg)$

Measuring the Size of the Nucleus

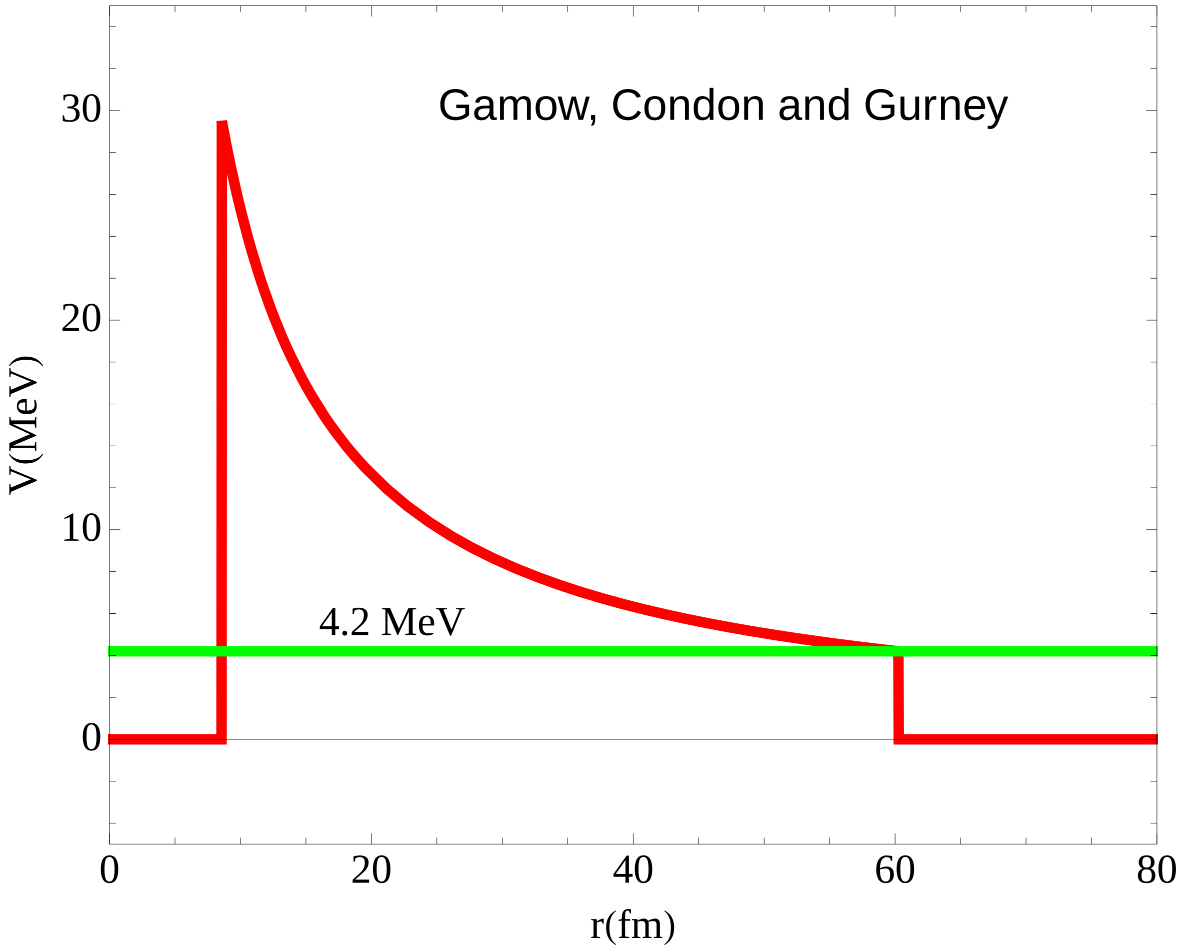
PRL *109*, 262701 (2012)



Rutherford Trajectories

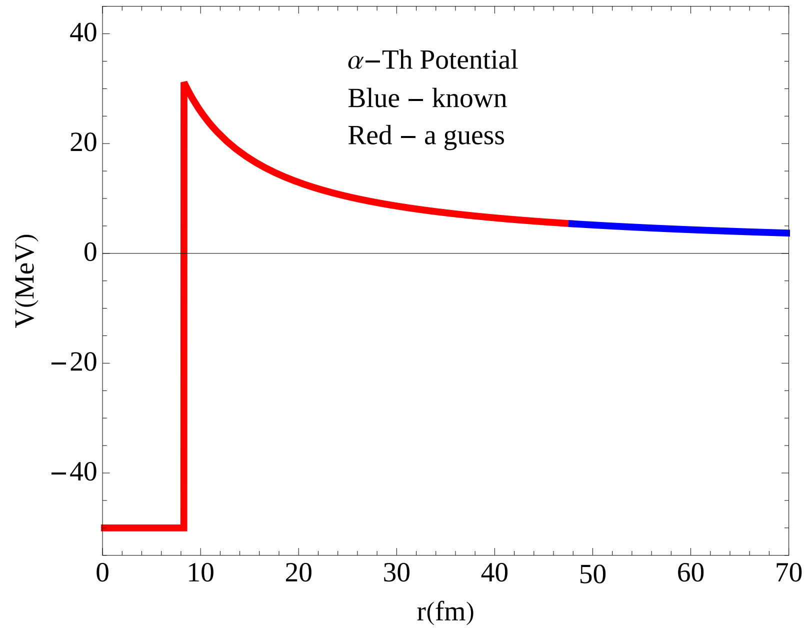


The $^{4}\mathrm{He} - {}^{234}_{90}\mathrm{Th}$ Potential



The α-Decay Puzzle

1. α-decay of uranium $^{238}\mathrm{U} \to {}^4\mathrm{He} + {}^{234}_{90}\mathrm{Th}$ ejects a 4.2-MeV $^4\mathrm{He}$.
2. Used a $5.4 \ - \mathrm{MeV}\ {}^4\mathrm{He}$ beam to probe the $^4\mathrm{He}+{}^{234}_{90}\mathrm{Th}$ force.
3. It was all Coulomb down to 48 fm.
4. The decay $^4\mathrm{He}$ with energy 4.2 MeV is, apparently, ejected at a distance of 62 fm.
5. How can that happen?



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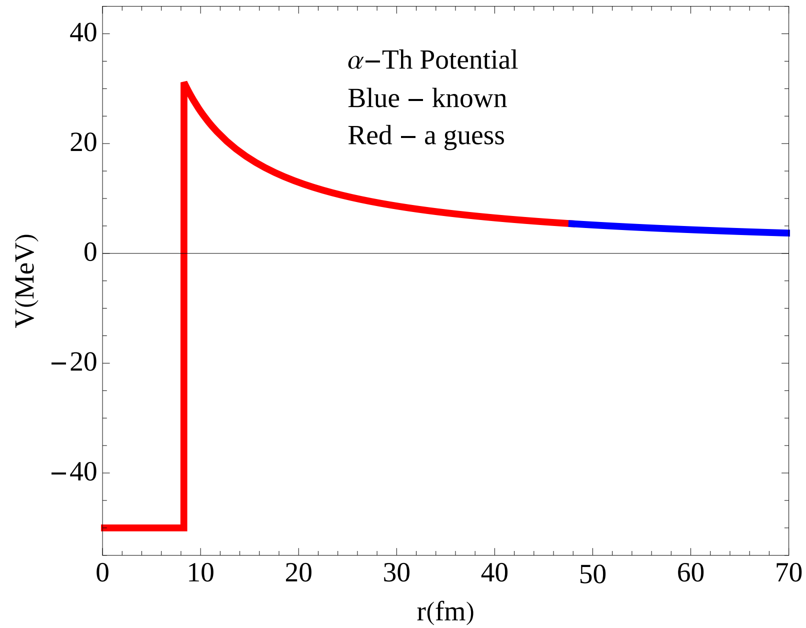
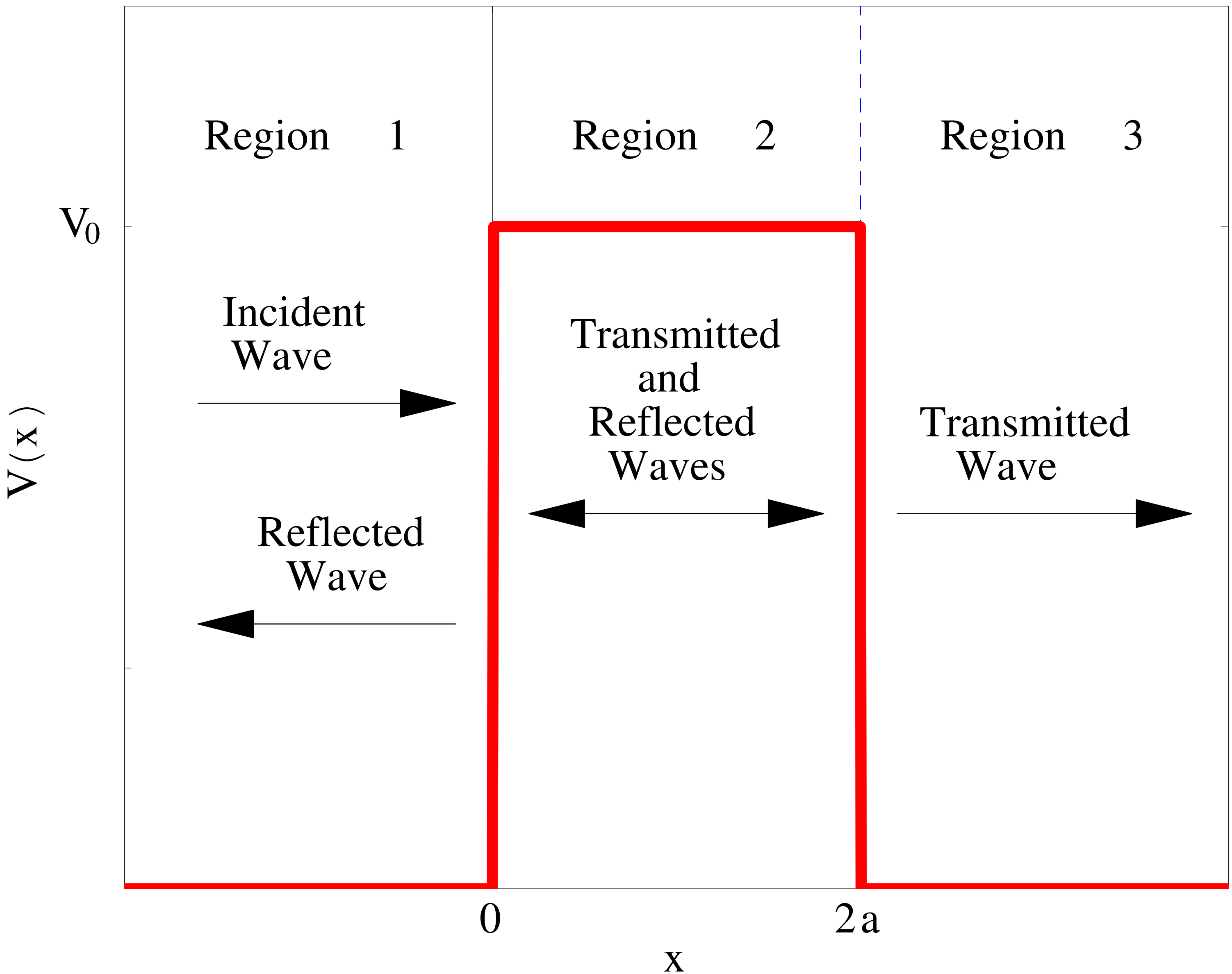


QUANTUM TUNNELING!

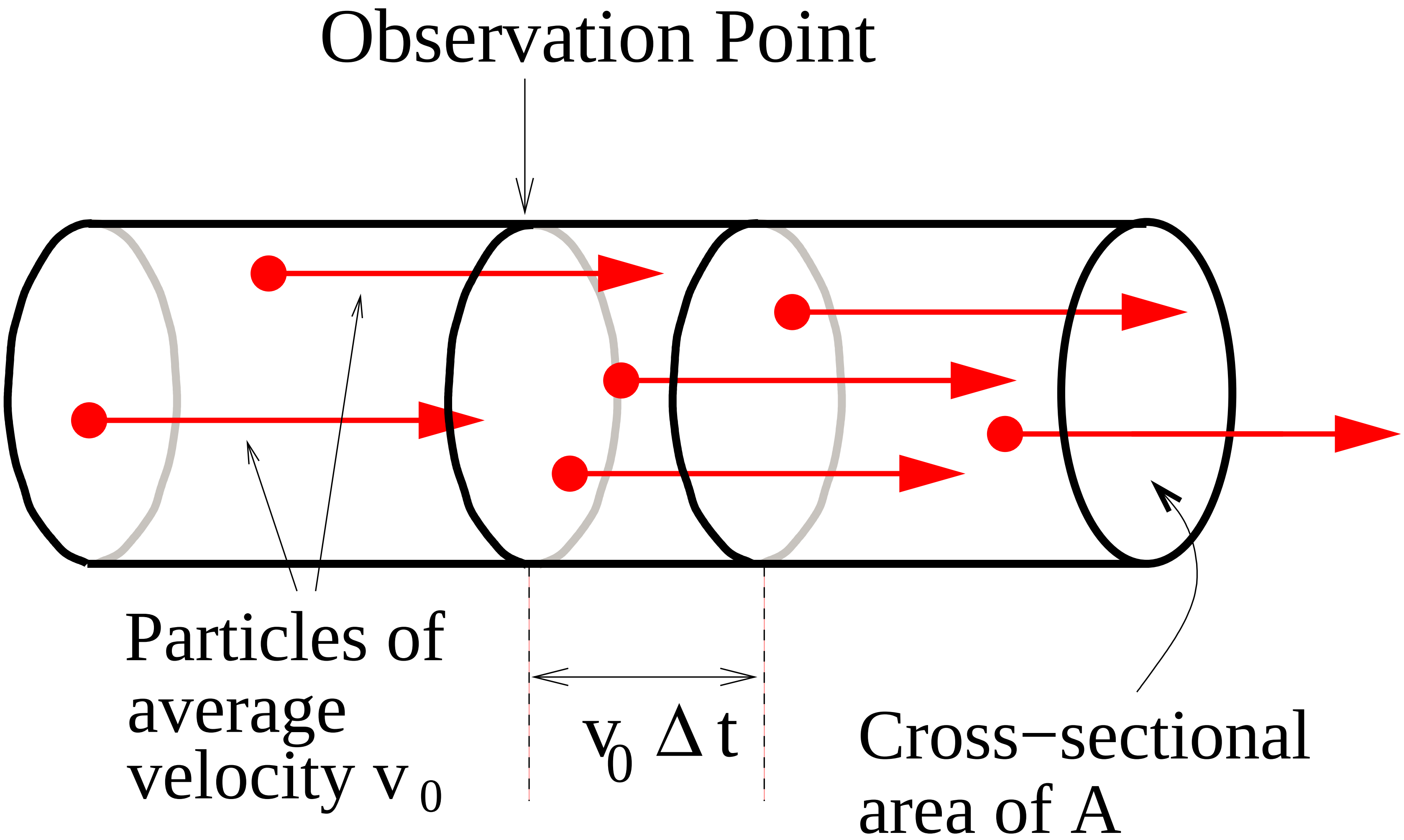
The Plan for Solving the Alpha Decay Puzzle

1. Develop the notion of particle flux or flow.
2. Solve the Schroedinger equation for the rectangular barrier potential.
3. Determine the flux penetrating the barrier.
4. Develop the transfer-matrix method using the rectangular barrier results as the starting point.
5. Build a model of what happens in a uranium nucleus and predict the lifetime for α-decay.
6. Compare with data!

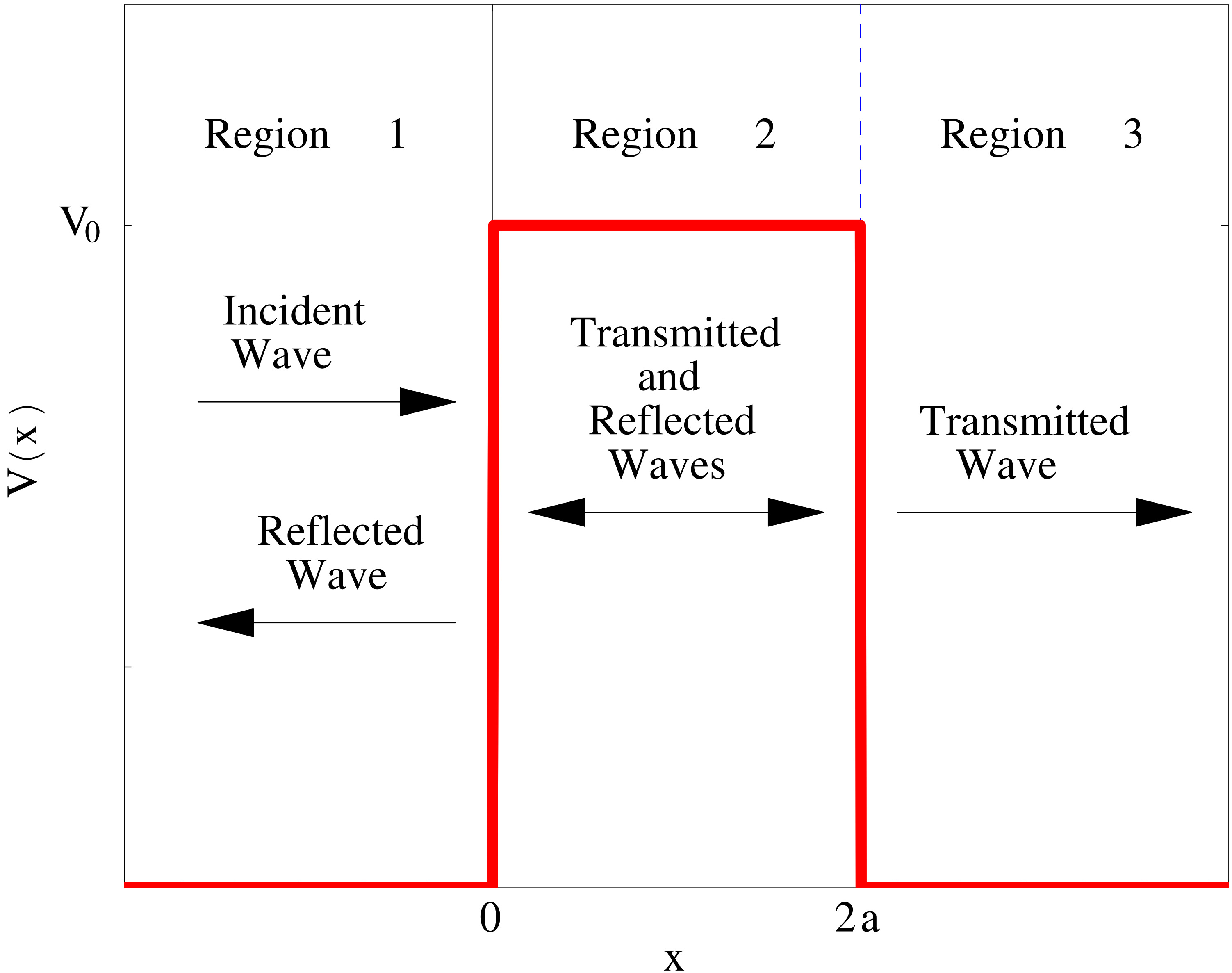
The Rectangular Barrier



Particle Flux in a Beam



The Rectangular Barrier



The Postulates

1. The state of a particle is represented by a wave function $|\psi(t)\rangle$ in a Hilbert space.
2. The independent variables x and p are represented by Hermitian operators $\hat{X}$ and $\hat{P}$ with the following matrix elements in the eigenbasis of $\hat{X}$

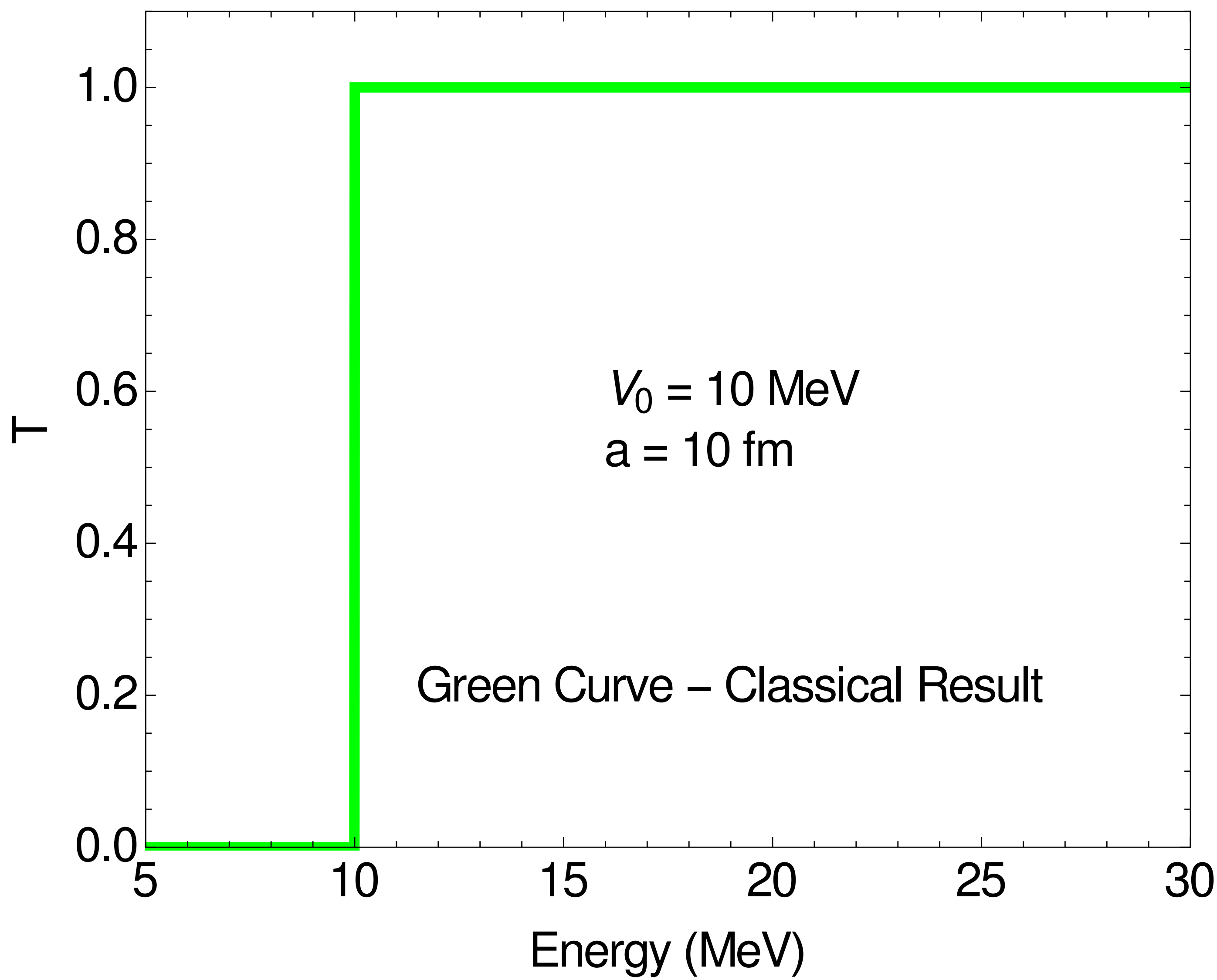
$$\langle x|\hat{X}|x'\rangle = x\delta(x-x') \qquad \langle x|\hat{P}|x'\rangle = x\delta'(x-x')$$

   The operators corresponding to dependent variables $\omega(x,p)$ are given Hermitian operators $\Omega(\hat{X},\hat{P}) = \omega(x \to \hat{X}, p \to \hat{P})$.
3. If the particle is in a state $|\psi\rangle$ measurement of the variable Ω will yield one of the eigenvalues ω with probability $P(\omega) = |\langle\omega|\psi\rangle|^2$. The state of the system will change from $|\psi\rangle$ to $|\omega\rangle$.
4. The state vector $|\psi(t)\rangle$ obeys the Schroedinger equation

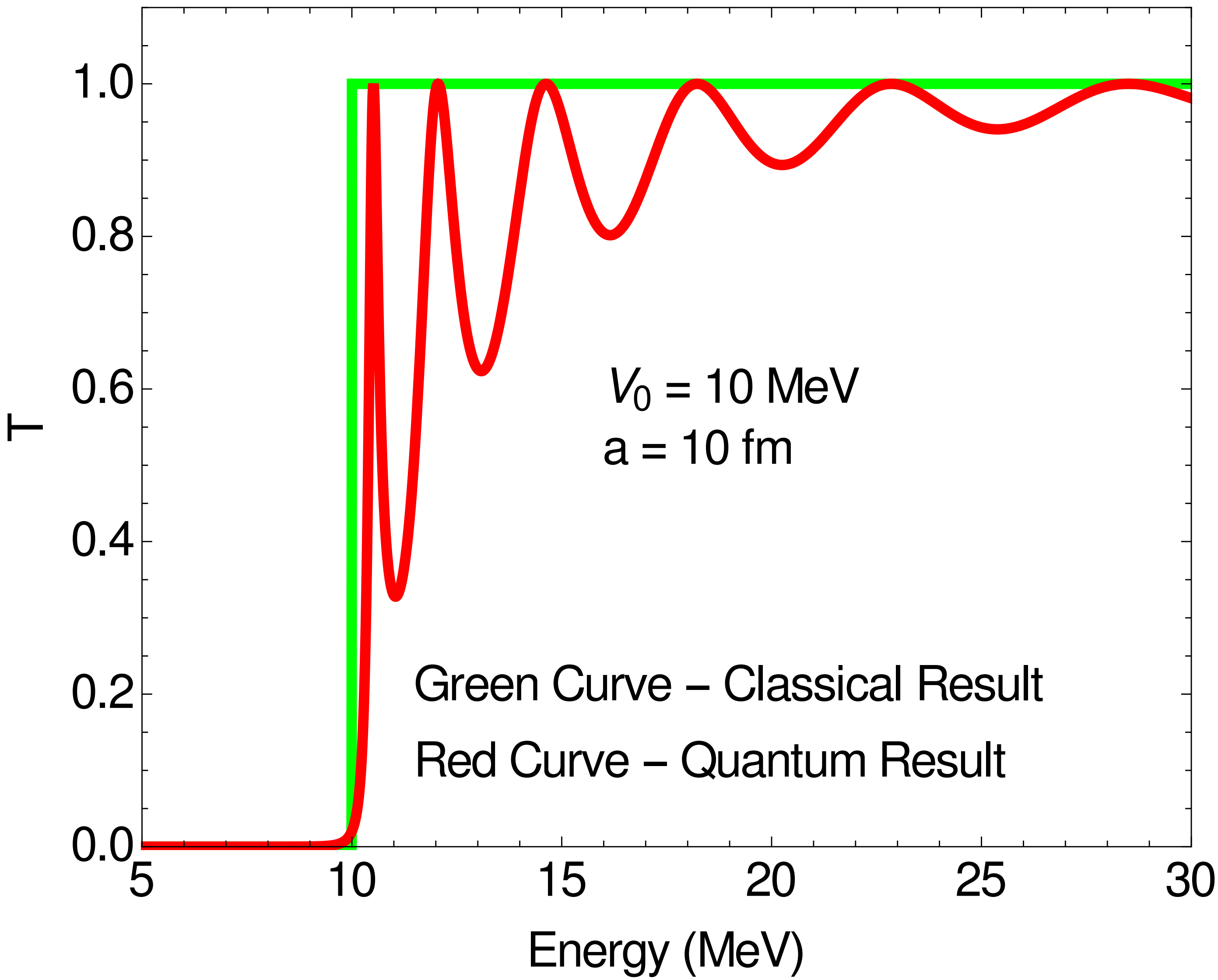
$$i\hbar\frac{d}{dt}\psi(t) = \hat{H}\,|\psi(t)\rangle$$

   where $\hat{H}(\hat{X},\hat{P}) = \mathcal{H}(x \to \hat{X}, p \to \hat{P})$ is the quantum Hamiltonian operator and $\mathcal{H}$ is the corresponding classical problem.

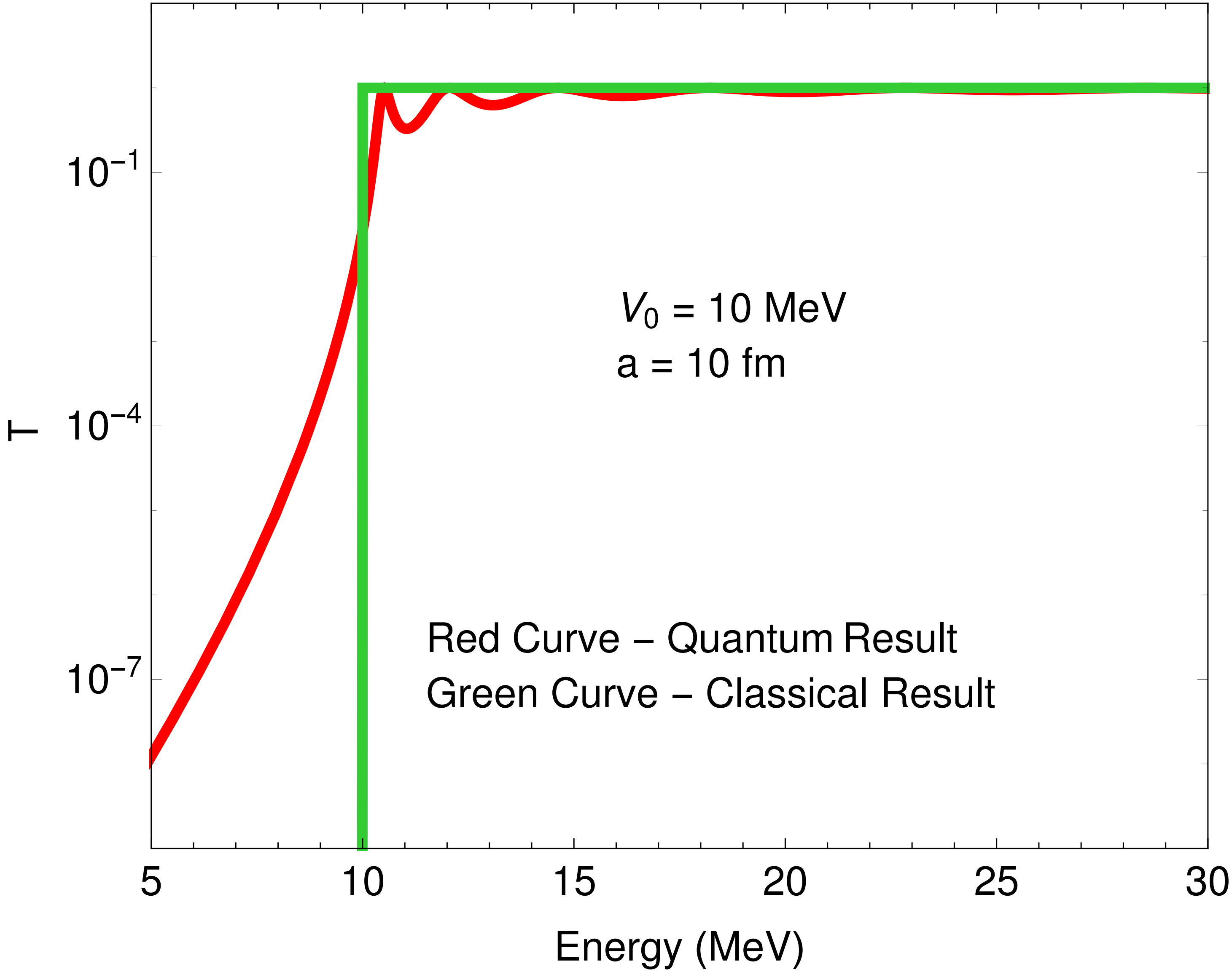
Quantum Tunneling



Quantum Tunneling



Quantum Tunneling



Hint for Shankar 5.4.2.a

Exercise 5.4.2. (a)* Calculate R and T for scattering off a potential $V(x) = V_0 a\delta(x)$. (b) Do the same for the case $V=0$ for $|x| > a$ and $V = V_0$ for $|x| < a$. Assume that the energy is positive but less than V_0.

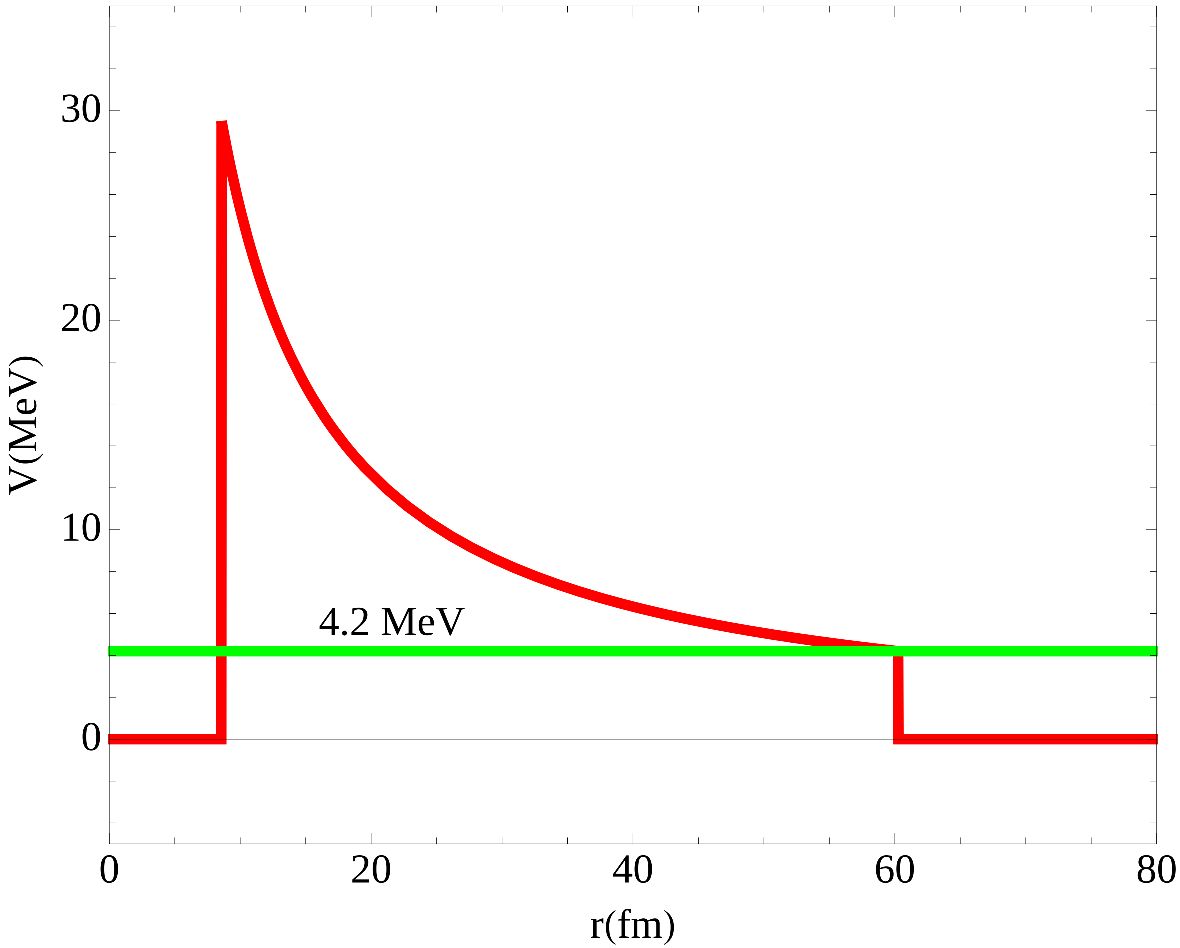
The fundamental property of the Dirac delta function is the following.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) = \int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)dx \qquad \epsilon > 0$$

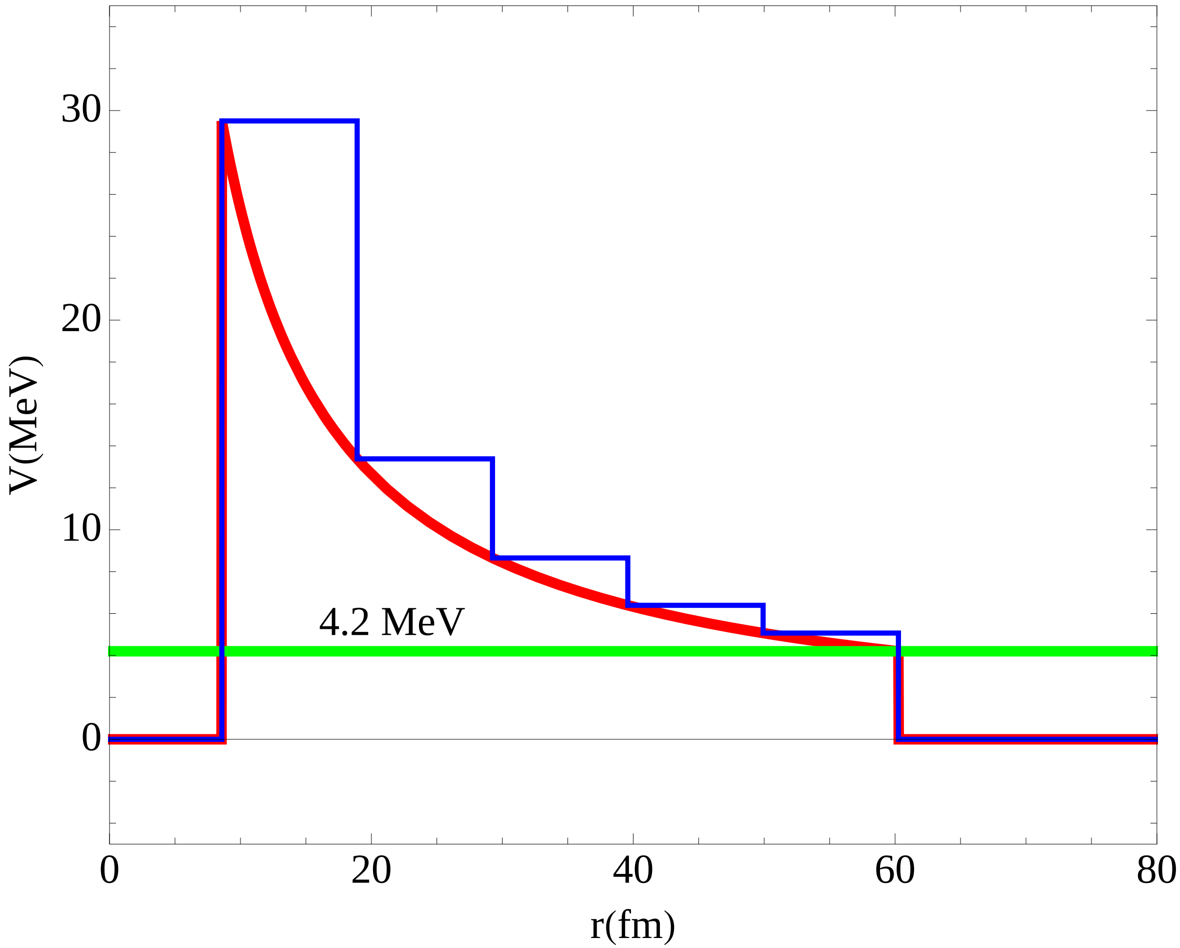
The Dirac delta function can be 'represented' by test functions that have the property defined above in the appropriate limit.

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

The Transfer-Matrix Solution



The Transfer-Matrix Solution



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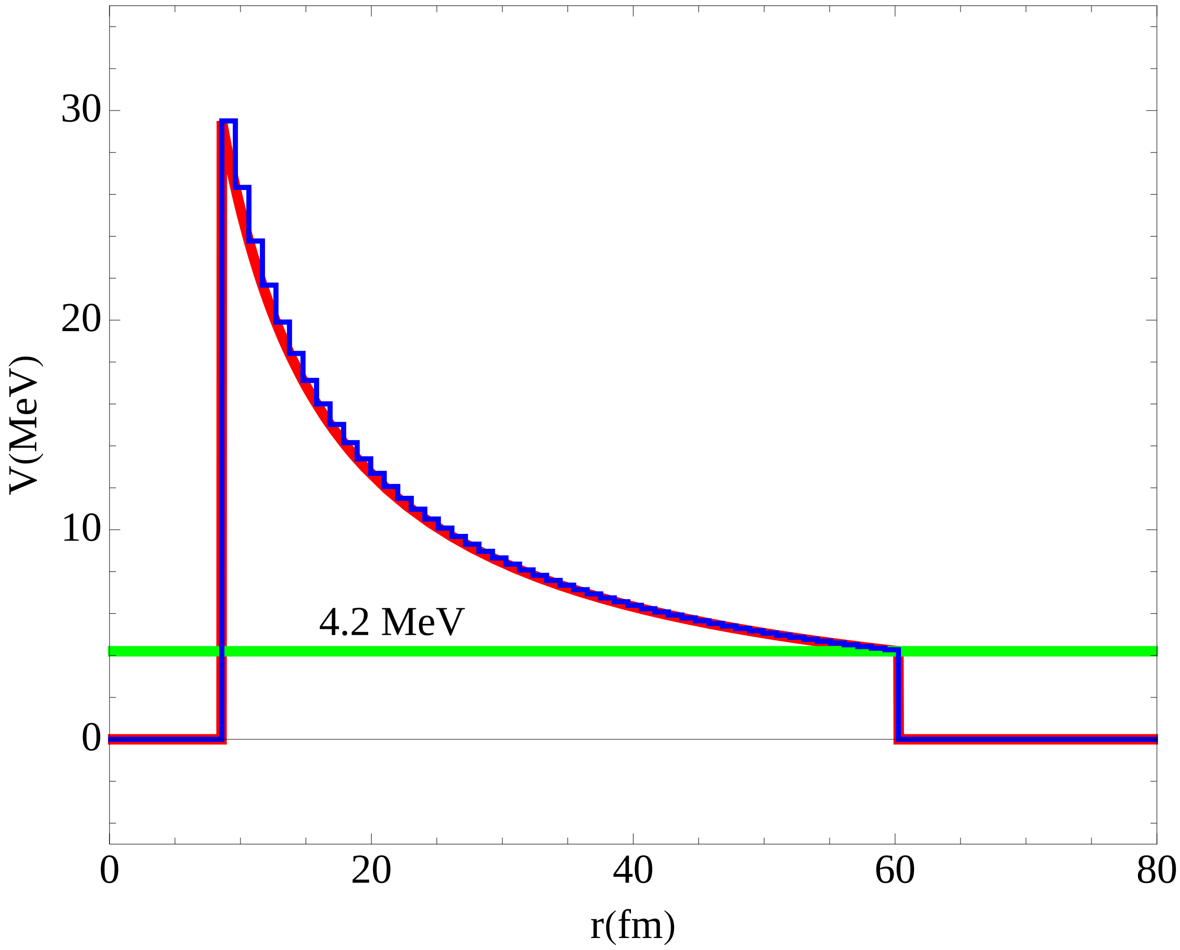
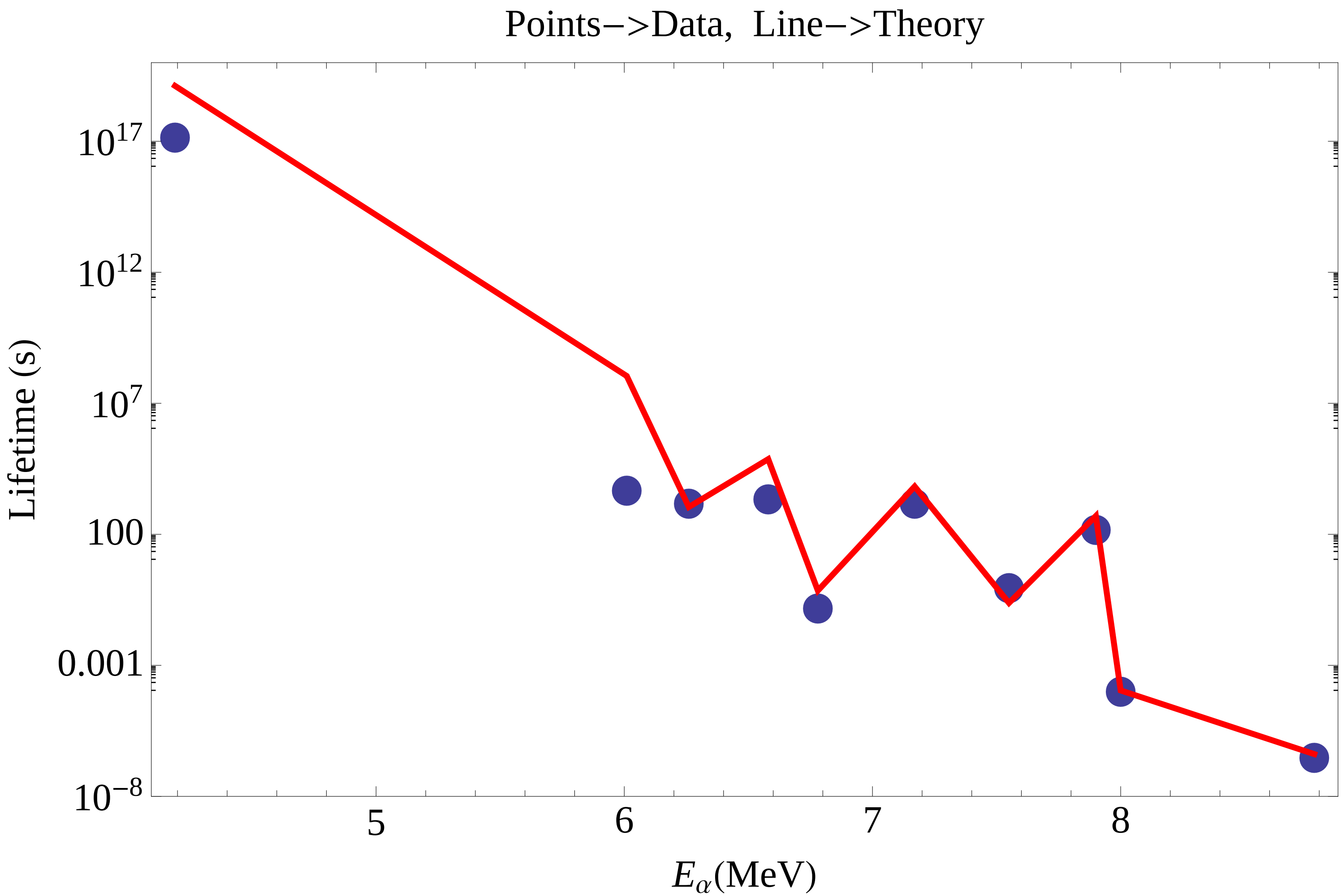
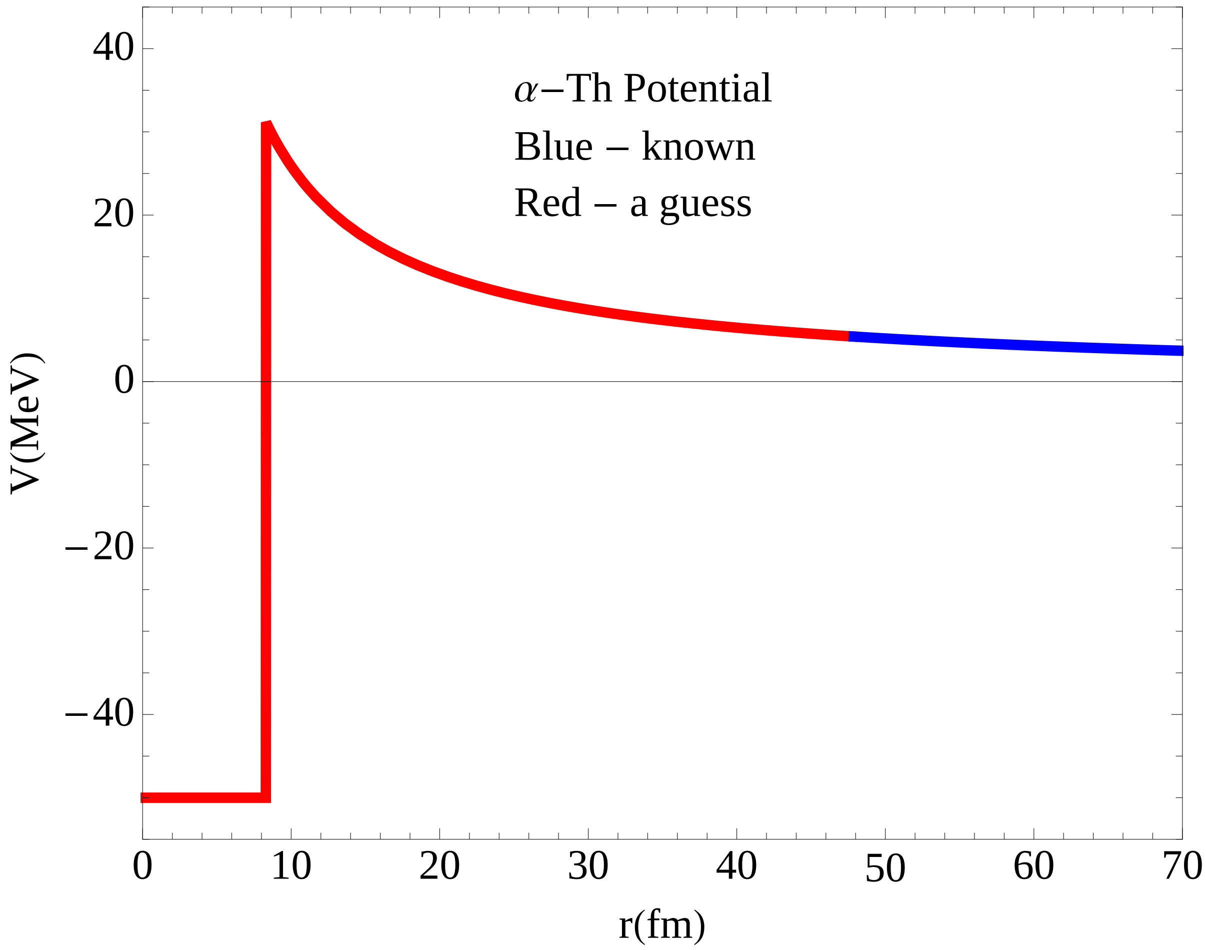


Results

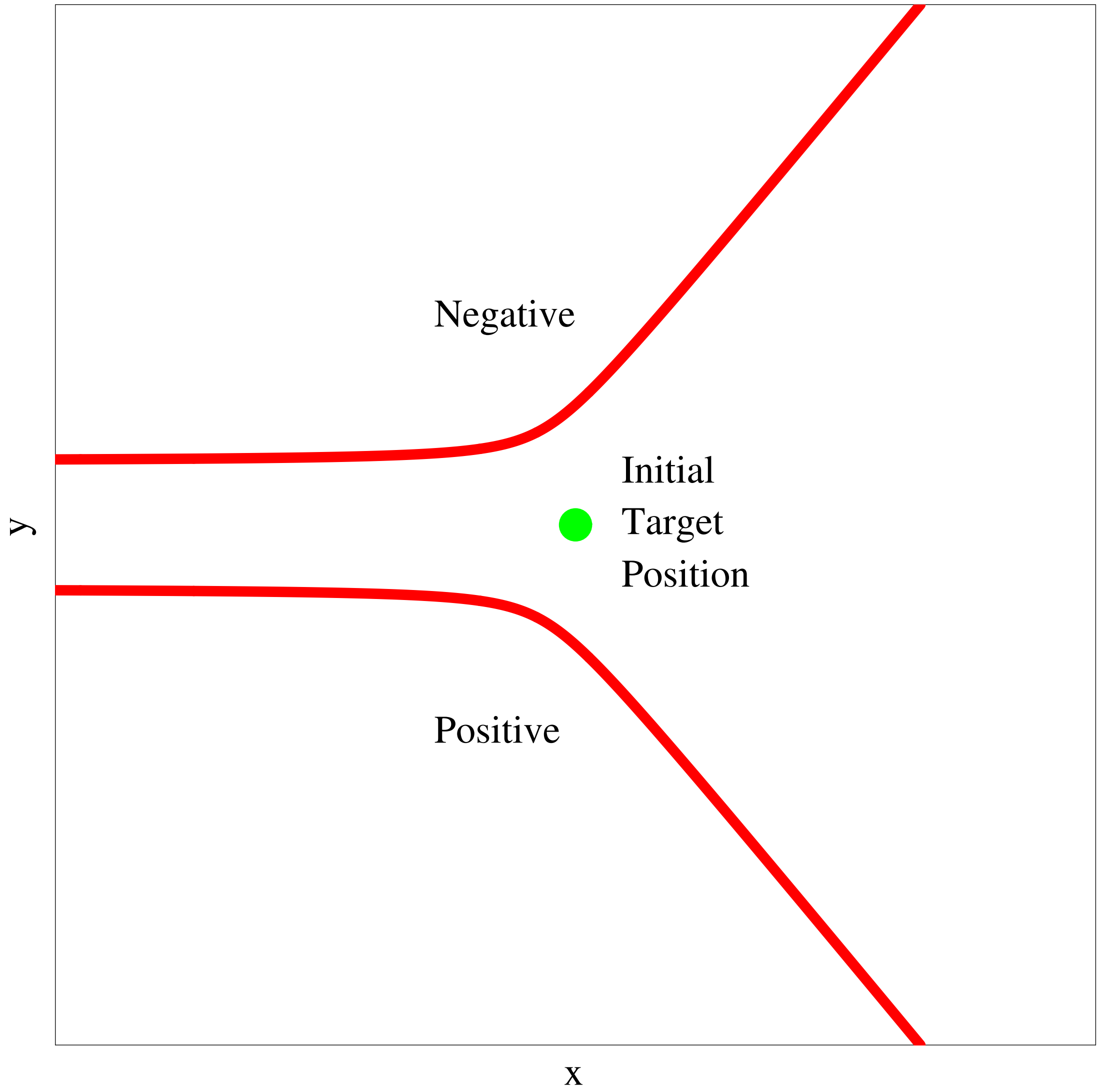


Points−>Data, Line−>Theory
Lifetime (s)
10^17
10^12
10^7
100
0.001
10^−8
5
6
7
8
E_α(MeV)

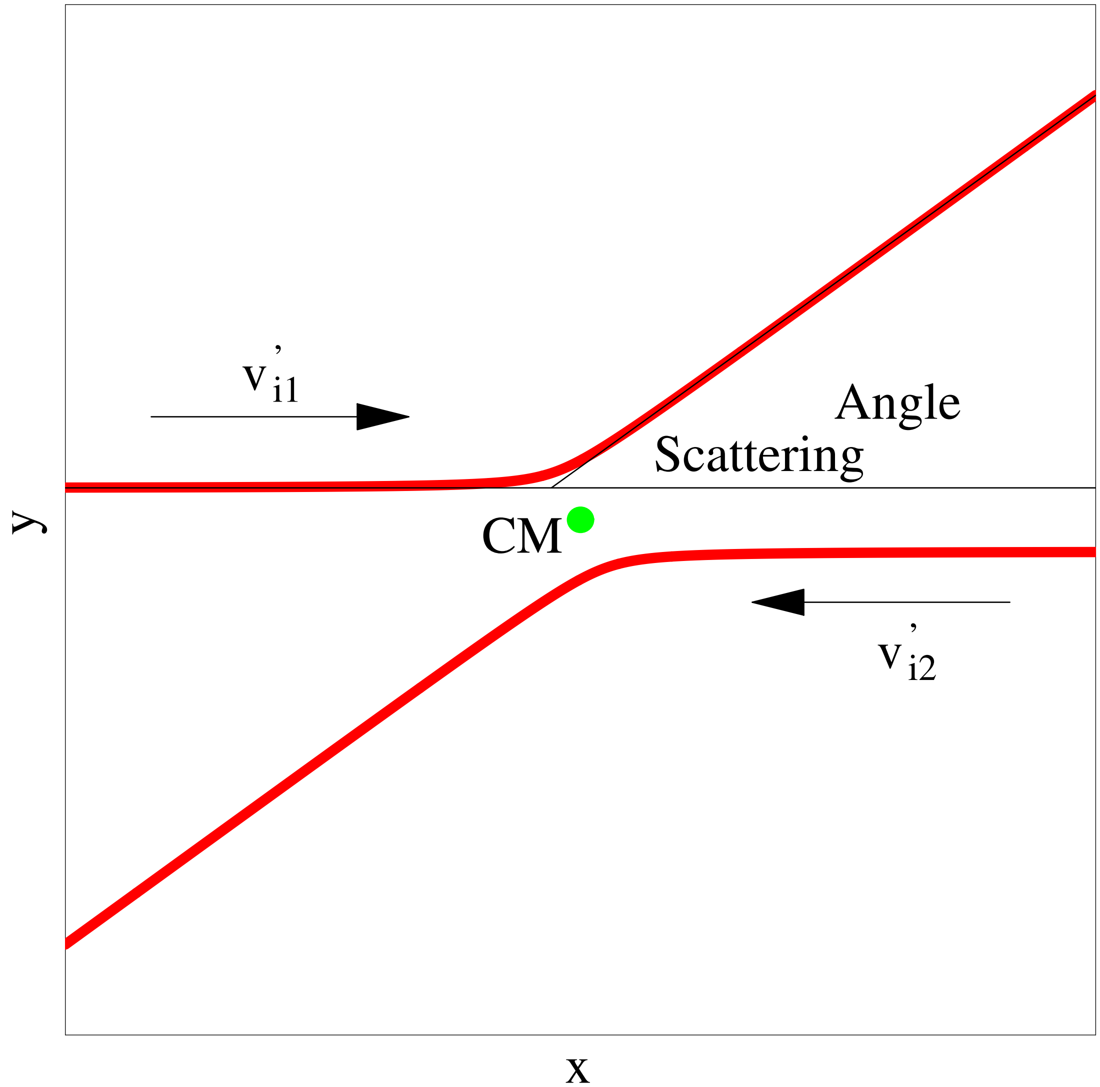
Coordinates



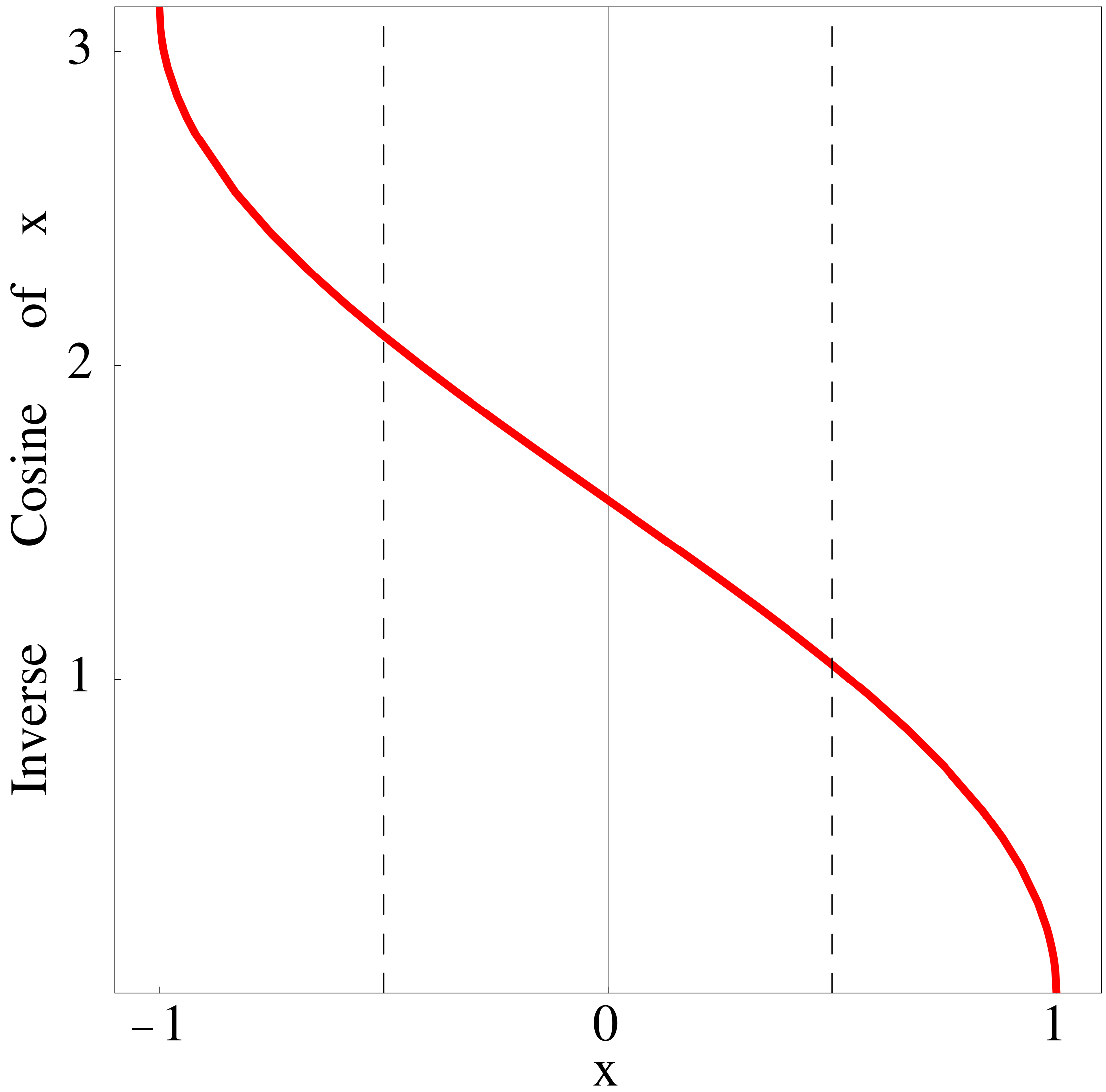
Choosing the Sign



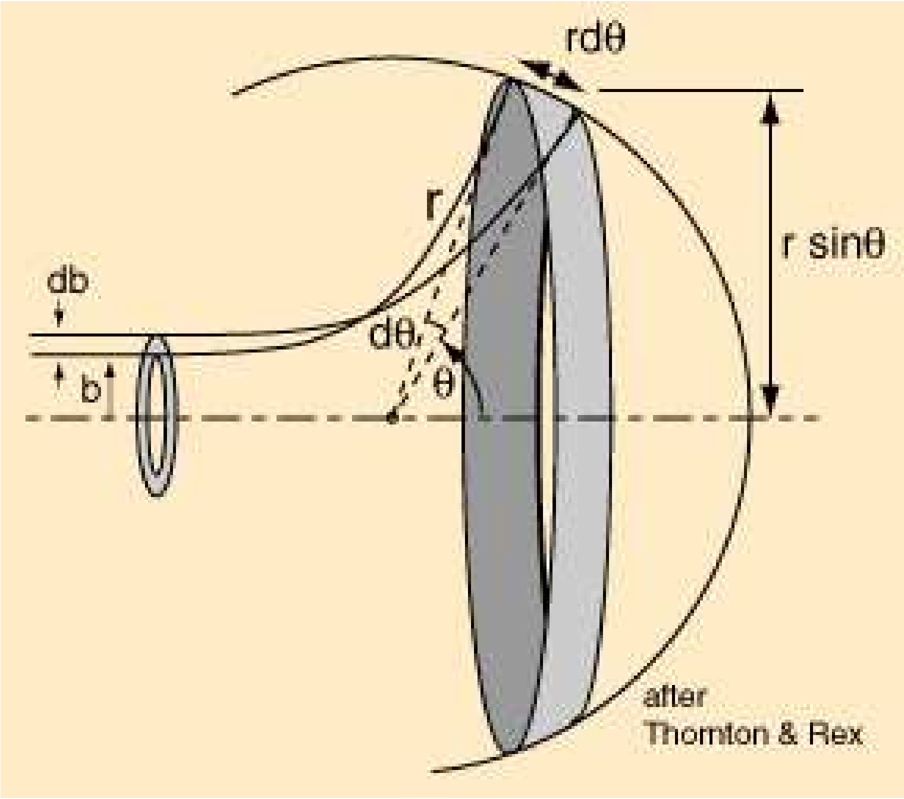
Center-of-Mass Rutherford Trajectories



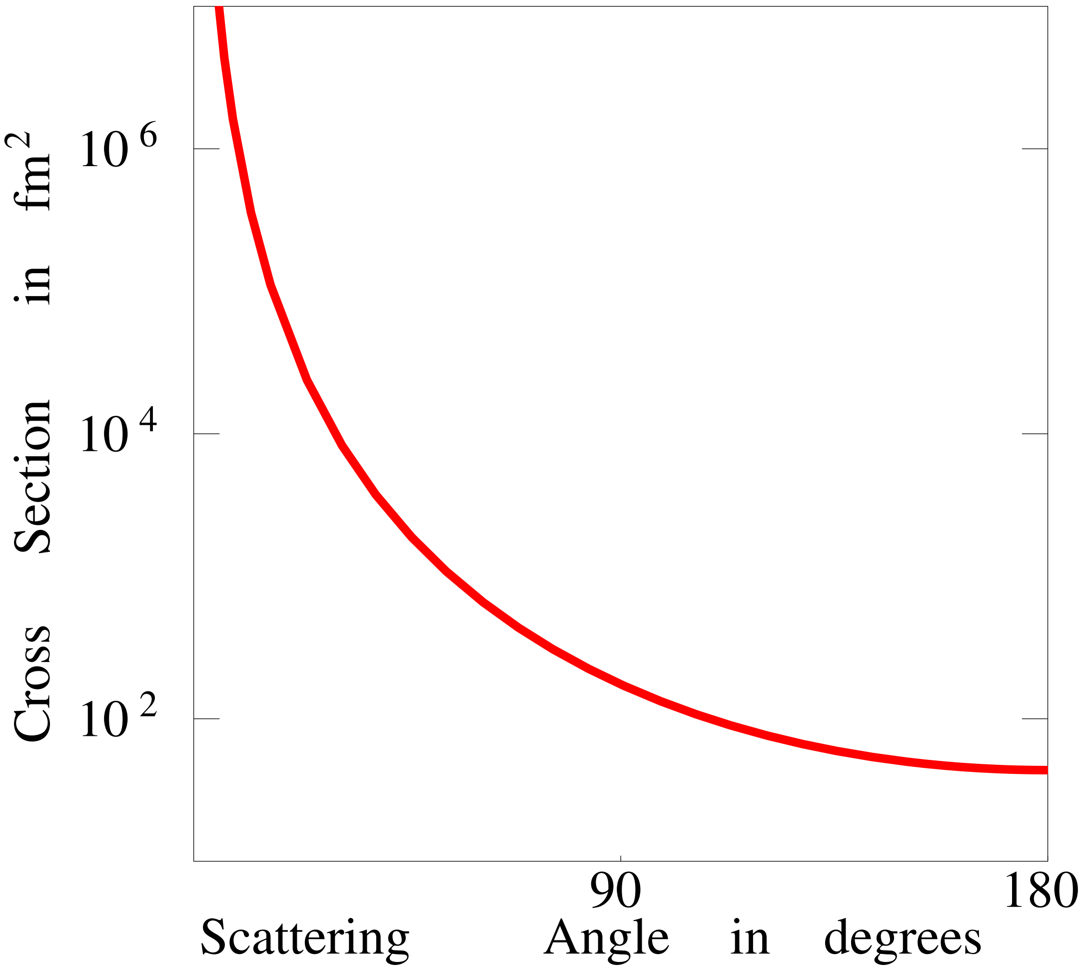
The Inverse Cosine



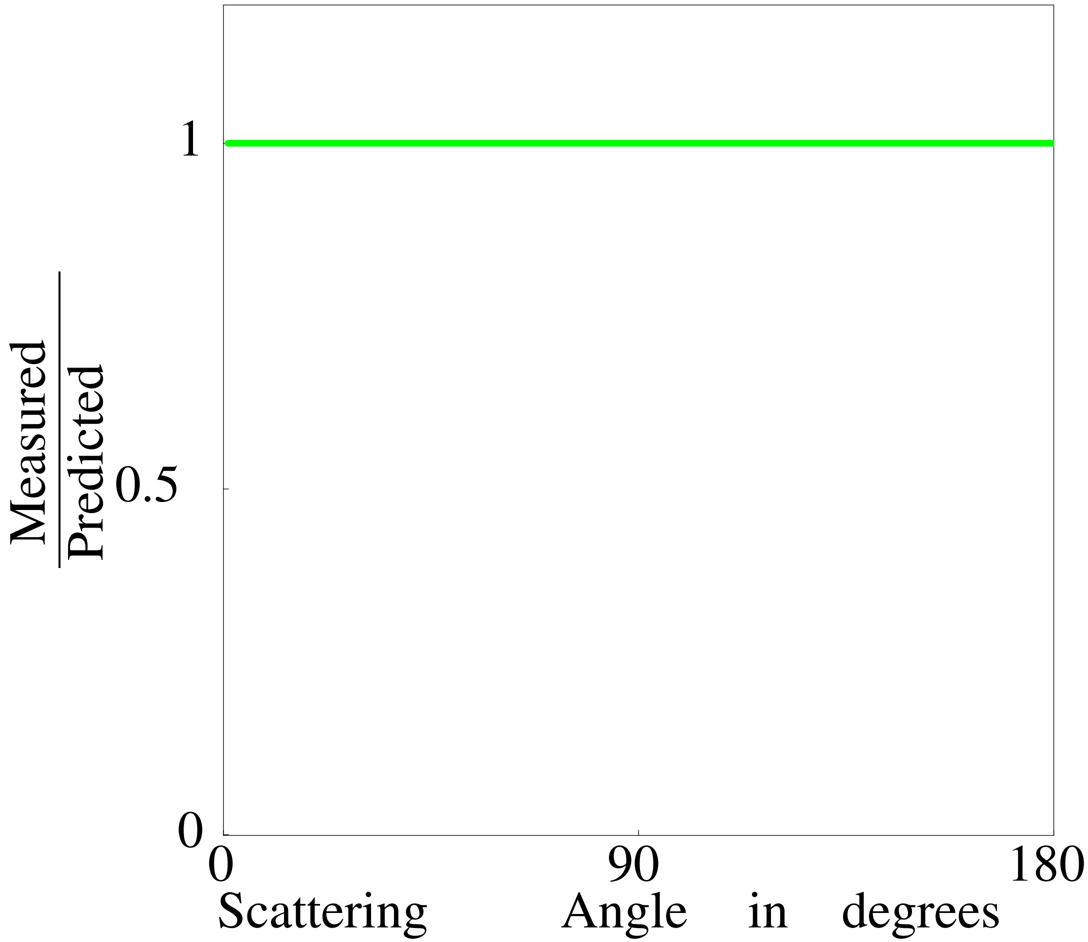
The Differential Cross Section



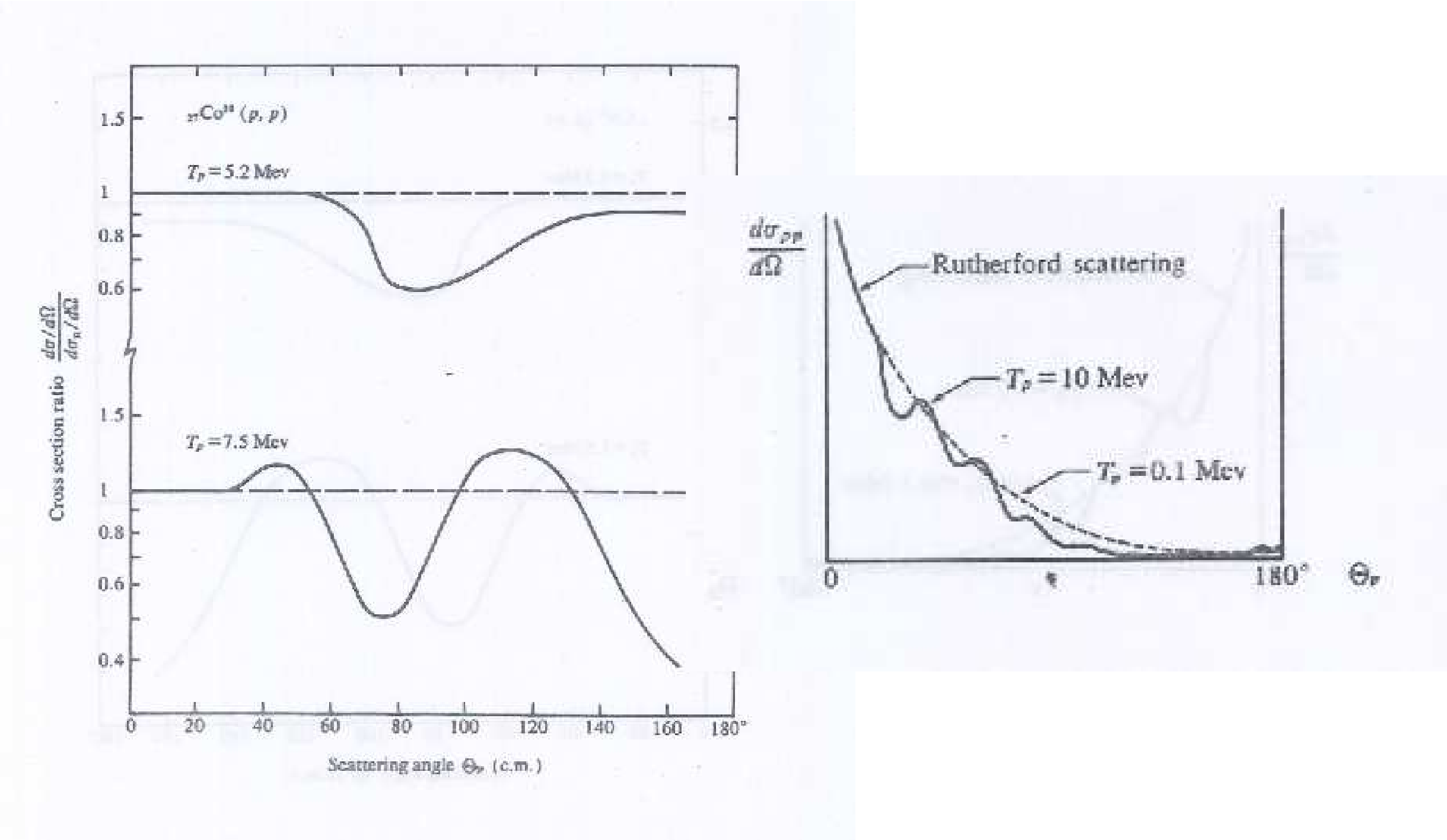
Predicted Differential Cross Section for $^4\mathrm{He} - \mathrm{Au}$



Measured Differential Cross Section for $^{4}\mathrm{He}-\mathrm{Au}$



The Evidence



The Evidence